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Stability and Stabilization of Nonlinear Systems

Foreword by E. Sontag

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To Christina, Katerina, and Olympia (I.K.)

To His Family and Friends (Z.P.J.)

Foreword

Control systems and feedback loops are ubiquitous in engineering, in areas such as aerospace control, manufacturing and robotics, active damping, climate control of buildings, process control in chemical plants, electrical power systems, bioengineering, consumer products, and active suspensions, automatic braking systems, and engine timing in the automobile industry. One finds control and feedback in nature as well, for example, in the homeostatic mechanisms that allow organisms to finely tune their internal variables such as temperature, pressure, or chemical levels.

The field of *mathematical control theory* concerns itself with the basic theoretical principles underlying the analysis of feedback and the design of control systems. It differs from the more classical study of dynamical systems in its emphasis on inputs (or controls) and outputs (or measurements). Linearized analysis of systems is the basic foundation of most practical control engineering and has been phenomenally successful. Nonetheless, linearization techniques can only deal with "small" deviations from desired behavior. Thus, the development of tools appropriate to the "global" study of systems with inputs and outputs has been the focus of a major research effort since at least the early 1970s.

In the late 1980s, there emerged a novel paradigm for nonlinear system analysis, based on the notions of "input-to-state stability" and several variants, which allow a seamless integration of classical Lyapunov-like stability theory with input/output operator approaches. The program of research that ensued has as its ultimate goal a complete formulation of the foundations of nonlinear behavior in two dual ways: the analysis of given systems in terms of these notions and the use of these notions in the form of systematic design tools which assign desirable properties to feedback systems. This duality between analysis and design is well summarized by the authors' maxim: "for every method of proving global stability, there is a corresponding method of nonlinear feedback design," and the book systematically applies that principle.

An original and very valuable aspect of this monograph is its treatment not only of ordinary differential equation systems, but also of delay and more general functional differential equations, as well as allowing a study of time-varying systems through "nonuniform" stability notions. Another original feature of the book is that

ISS and its variants are extended to various important classes of interconnected systems using small-gain theorems. This duality between ISS (for single systems) and small-gain (for coupled systems) plays a key role in addressing the problems of robust stability and stabilization.

The field is still active and full of open problems, and this monograph should encourage much further research and further development of applications, many of which are discussed here. The authors of this volume have been two of the main contributors to the program, and in this book they provide a detailed and rigorous introduction, suitable for beginning graduate students as well as for more experienced researchers who wish to transition to the field.

Piscataway, NJ Eduardo Sontag

Preface

Phenomenal progress in nonlinear systems theory has been made during the last decades. It has been reflected in two aspects. On the one hand, internal and external global stability notions have been studied intensely for uncertain nonlinear systems. On the other hand, the applications of these advanced stability results to control engineering systems have led to numerous novel methodologies for the design of nonlinear feedback controllers. It is fair to say that input-to-state stability (ISS), a notion invented by E.D. Sontag in the late 1980s, plays an influential role in the work of many researchers including the authors of this book. ISS has bridged the gap which previously existed between the input–output and the state-space methods, two popular approaches within the control systems community. Roughly speaking, the importance of ISS for the study of nonlinear systems is reflected by the intriguing fact that it captures two main stability notions: Lyapunov stability (i.e., the behavior of the zero-input response with respect to nonzero initial conditions) and input–output stability (i.e., the behavior of the zero-state response with respect to nonzero external inputs).

Nonlinear systems are encountered frequently in almost all branches of science and engineering. In fact, in engineering, physics, economics, and biology, nonlinearity is the rule, and linear systems are rare (which almost exclusively exist only in our computer programs). Despite the importance of nonlinear system theory, graduate students or researchers in mathematics, engineering, physics, economics, and biology often have difficulties in taking advantage of recent advances in mathematical systems and control theories. There are several excellent textbooks that provide nice introductions to nonlinear systems theory, but many recent stability results are scattered in the vast literature. Motivated by this observation, we set our hands to write this monograph about a year and half ago. The specific objectives of this book are described in the following.

The first aim of the book is to provide the basic knowledge needed for a graduate student in order to be able to understand the current research in nonlinear stability theory and nonlinear control theory. A relatively high level of mathematical background is assumed: the reader is required of having basic knowledge in differential equations, calculus, and real analysis. Measure theory is not needed (although measurable functions are met even in the first pages of the book): the reader can replace “measurable” functions by “piecewise continuous” functions. The book is self-contained in the sense that all results are proved in detail by using basic mathematical knowledge and other results presented in the book. Only global stability notions are studied in the present book. It should be mentioned that there are many important results about local or regional stability in the literature.

The second aim of the book is to give a perspective of nonlinear stability theory and nonlinear control theory that is not frequently encountered in the literature. The idea can be stated in the following (informal) way:

“*for every method of proving global stability, there is a corresponding method of nonlinear feedback design*.”

Therefore, the book is designed to help the reader to understand this one-to-one correspondence. In the first five chapters the reader is introduced to internal and external stability notions and characterizations. Necessary and sufficient conditions for each stability notion are provided, and a description of the various methods for proving stability is presented. Finally, in Chaps. 6 and 7 of the book the reader is introduced to the various methods of nonlinear feedback design. Each method aims to design a feedback law such that the resulting closed-loop system is “stable.” The proof of the stability properties of the closed-loop system is performed by using one particular method of proving stability. We believe that this perspective can help the reader to understand the proposed feedback design methodologies and can inspire the reader to suggest new ones. However, for want of space, some methods of proving stability and feedback design are only briefly mentioned (e.g., the method of using Matrosov’s theorem).

The third aim of the book is to show that the same mathematical tools, up to minor modifications, can be used for all kinds of systems. Working within an abstract system-theoretic framework, one can see that systems described by Ordinary Differential Equations (ODEs), systems described by retarded functional differential equations (RFDEs), systems described by coupled retarded functional difference equations and retarded functional differential equations, and sampled-data systems can be analyzed and studied using (almost) the same tools. We believe that this feature is important: many recent contributions to nonlinear systems theory are developed for complex dynamical systems other than those described by ODEs. Furthermore, it has been recognized that feedback laws of new kinds (e.g., feedback laws with delays, hybrid/switching feedback laws) can give rise to features for the closed-loop system that cannot be encountered in systems described by ODEs. In such cases, the closed-loop system in question becomes a system with different mathematical description from the original open-loop system. Time-varying systems are not excluded: the proposed system-theoretic framework can capture all features of time-varying systems (e.g., nonuniform stability phenomena).

As the fourth and last aim of the book, it is the authors’ view that nonlinear systems theory has reached a level of maturity which can provide interesting contributions to other areas of applied mathematics. There are many results and examples in the book that illustrate the use of nonlinear systems theory to game theory, fixed-point theory, numerical analysis, and (only superficially) mathematical biology. Some open problems are listed in the last Chap. 8 with a unique objective to entice the reader, in particular graduate students, to develop their novel ideas and techniques, which will contribute to the further development of modern mathematical control theory.

The list of people who must be acknowledged for their support and help to the authors' research is too long to be given. I.K. would like to thank Professor Panagiotis Christofides for his help and good advice that he gave to him when he was a student. Professor John Tsinias has been the academic mentor for Iasson Karafyllis; he owes major gratitude to him that cannot be expressed in a few words. Professors Stelios Kotsios, Eduardo D. Sontag, and Costas Kravaris helped I.K. very much in the first steps of his academic career (each one in a different way). I.K. also owes gratitude to Professor Zhong-Ping Jiang for being one of the first persons who believed in him (and does not mind if it is not right for one author to write acknowledgments for a coauthor!). Z.P.J. would like to thank all his coauthors for contributing to this book, directly or indirectly. Special thanks go to his Ph.D. advisor Laurent Praly for having introduced him to the field of nonlinear control and his two Australian mentors David Hill and Iven Mareels for their long-lasting influence in his career. Z.P.J. also would like to thank Professors A. Isidori, P.V. Kokotović, and E.D. Sontag, and President Jerry Hultin of the Polytechnic Institute of New York University, for the high expectations they set on him, explicitly or implicitly. Yes, it is also a pleasure to thank his nonlinear control friends Jie Huang, Miroslav Krstić, Andy Teel, and Yuan Wang for the moral support over the past years. Finally, the authors would like to thank their students, in particular Yu Jiang, for helping with the typesetting of the book and the drawing of some figures. In the end, all possible errors and typos in the book remain to be the sole responsibility of the authors.

Last but not least, both authors would like to thank their families (Christina, Katerina, Olympia and Xiaoming, Jenny, Jack) for their understanding, support, and patience during the long period when the book was written.

Chania, Greece — Iasson Karafyllis
New York, USA — Zhong-Ping Jiang

Contents

Notations

$\Re$	The set of real numbers
$\Re^+$	The set of nonnegative real numbers
$\Re^n_+$	The set of nth-order vectors of which each component is nonnegative
$\mathbb{N}$	The set of natural numbers
Z	The set of integers
Z^+	The set of nonnegative integers
$\|x\|$	Euclidean norm of a vector $x \in \Re^n$
x'	The transpose of a vector $x \in \Re^n$
x_{-i}	The vector obtained by deleting the ith component of a vector $x \in \Re^n$, with $n \geq 2$
$\Re^{n \times m}$	The space of real matrices with dimensions $n \times m$
A'	The transpose of a matrix $A \in \Re^{n \times m}$
A^{-1}	The inverse of an invertible square matrix $A \in \Re^{n \times n}$
I	Generic notation of an identity matrix
$\text{int}(A)$	The interior of a set $A \subseteq \Re^n$
K^+	The class of positive continuous functions on $\Re^+$
$\mathcal{N}$	A function $a : \Re^+ \to \Re^+$ is of class $\mathcal{N}$ if a is continuous and nondecreasing with $a(0) = 0$
$\text{Pr}_U(x)$	The projection of $x \in \Re^n$ on a convex set $U \subseteq \Re^n$, i.e., the unique vector y such that $\|x - y\| = \min_{u \in U} \|x - u\|$
p.d.	A function $f : A \to \Re^+$, where $A \subseteq \Re^n$ with $0 \in A$, is positive definite (p.d.) if $f(0) = 0$ and $f(x) > 0$ for all $x \in A \backslash \{0\}$
r.u.	A function $f : A \to \Re^+$, where $A \subseteq \Re^n$ is a nonempty and unbounded set, is said to be radially unbounded (r.u.) if, for every $a \geq 0$, the level set $\{x \in A : f(x) \leq a\}$ is bounded
$\text{supp}(f)$	The support of a function $f : A \to \Re$, where $A \subseteq \Re^n$
$x \leq y$	For a pair of vectors $x, y \in \Re^n$, we say that $x \leq y$ if and only if $(y - x) \in \Re^n_+$
$\mathcal{N}_n$	A function $\rho : \Re^n_+ \to \Re^+$ is of class $\mathcal{N}_n$ if ρ is continuous with $\rho(0) = 0$ and such that $\rho(x) \leq \rho(y)$ for all $x, y \in \Re^n_+$ with $x \leq y$

$[V]_{[t_0,t]}$	For $t \geq t_0 \geq 0$, let $[t_0,t] \ni \tau \to V(\tau) = (V_1(\tau), \ldots, V_n(\tau))' \in \Re^n$ be a bounded map. We define $[V]_{[t_0,t]} := (\sup_{\tau \in [t_0,t]} V_1(\tau), \ldots, \sup_{\tau \in [t_0,t]} V_n(\tau))'$
$\Gamma^{(k)}$	We say that $\Gamma : \Re^n_+ \to \Re^m_+$ is nondecreasing if $\Gamma(x) \leq \Gamma(y)$ for all $x, y \in \Re^n_+$ with $x \leq y$. For any positive integer k, we define $\Gamma^{(k)}(x) = \underbrace{\Gamma \circ \Gamma \circ \cdots \circ \Gamma}_{k \text{ times}}(x)$. By convention, we set $\Gamma^{(0)}(x) = x$, for all $x \in \Re^n_+$
1	We define $\mathbf{1} = (1, 1, \ldots, 1)' \in \Re^n$. If $u, v \in \Re$ and $u \leq v$, then $\mathbf{1}u \leq \mathbf{1}v$
K	A function $\alpha : \Re^+ \to \Re^+$ is of class K if α is continuous and increasing with $\alpha(0) = 0$
K_∞	A function $\alpha : \Re^+ \to \Re^+$ is of class K_∞ if it is of class K and satisfies $\lim_{s \to +\infty} \alpha(s) = +\infty$
KL	A function $\beta : \Re^+ \times \Re^+ \to \Re^+$ is of class KL if, for each $t \geq 0$, the mapping $\sigma(\cdot, t)$ is of class K, and, for each $s \geq 0$, the mapping $\beta(s, \cdot)$ is nonincreasing with $\lim_{t \to +\infty} \beta(s,t) = 0$
$\mathcal{E}$	The set of nonnegative continuous functions $\mu : \Re^+ \to \Re^+$ with $\lim_{t \to +\infty} \mu(t) = 0$ and $\int_0^{+\infty} \mu(t)\,dt < +\infty$
$\\|\cdot\\|_{\mathcal{X}}$	The norm of the normed linear space $\mathcal{X}$
$B_U[0,r]$	The intersection of $U \subseteq \mathcal{X}$ with the closed sphere of radius $r \geq 0$, centered at $0 \in U$, i.e., $B_U[0,r] := \{u \in U;\ \\|u\\|_{\mathcal{X}} \leq r\}$
$\\|(x,y)\\|_C$	Unless specified otherwise, the linear space $C = \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces, is endowed with norm $\\|(x,y)\\|_C = \sqrt{\\|x\\|^2_{\mathcal{X}} + \\|y\\|^2_{\mathcal{Y}}}$, for all $(x,y) \in C$
$\mathcal{M}(U)$	The set of all functions $u : \Re^+ \to U$
u_0	The identically zero input, i.e., $u_0(t) = 0 \in U$ for all $t \geq 0$
$\mathcal{L}^\infty(I;A)$	The set of Lebesgue measurable and essentially bounded functions $u : I \to A$ for an interval $I \subseteq \Re$ and a set $A \subseteq \Re^n$
$\mathcal{L}^\infty_{\text{loc}}(I;A)$	The set of Lebesgue measurable and locally essentially bounded functions $u : I \to A$ for an interval $I \subseteq \Re$ and a set $A \subseteq \Re^n$
$\{T_i\}_{i=0}^\infty$	A partition $\pi = \{T_i\}_{i=0}^\infty$ of $\Re^+$ that is an increasing sequence of times with $T_0 = 0$ and $T_i \to +\infty$. Its diameter is defined as $\sup\{T_{i+1} - T_i;\ i = 0, 1, 2, \ldots\}$
$q_\pi(t)$	For every partition $\pi = \{T_i\}_{i=0}^\infty$ of $\Re^+$, $q_\pi(t) := \min\{T \in \pi;\ t < T\}$
$C^0(A;\Omega)$	The class of continuous functions on a subset A of a normed linear space $\mathcal{X}$, taking values in a subset Ω of a normed linear space $\mathcal{Y}$
$T_r(t)x$	The "r-history" of a function $x : [a-r, b) \to \Re^n$ at time $t \in [a,b)$, for constants $b > a > -\infty$ and $r > 0$. That is, $T_r(t)x$ maps $\theta \in [-r, 0]$ to $x(t+\theta)$
$\\|x\\|_r$	For $x : [-r, 0] \to \Re^n$, $\\|x\\|_r := \sup_{\theta \in [-r,0]} \|x(\theta)\|$
q.c.	A function $f : (x,u) \in A \times U \to f(x,u) \in \Re$, where $A \subseteq \mathcal{X}$, $\mathcal{X}$ is a normed linear space and $U \subseteq \Re^m$ is a convex set, is said to be quasi-convex (q.c.) with respect to $u \in U$ if the inequality $f(x, \alpha u_1 + (1-\alpha)u_2) \leq \max\{f(x,u_1), f(x,u_2)\}$ holds for all $x \in A$, $u_1, u_2 \in U$, and $\alpha \in [0,1]$

c.c. A mapping $f : (t, x) \in \mathcal{T} \times A \to f(t, x) \in W$, where $\mathcal{T} = Z^+$ or $\mathcal{T} = \Re^+$, $A \subseteq \mathcal{X}$, and $\mathcal{X}$ and W are normed linear spaces, is said to be completely continuous (c.c.) with respect to $x \in A$, written as $f \in CU(\mathcal{T} \times A; W)$, if for every pair of bounded sets $I \subset \mathcal{T}$ and $S \subseteq A$, f maps $I \times S$ into a bounded set and, additionally, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(t, x) - f(t, y)\|_W < \varepsilon$ for all $t \in I$ and $x, y \in S$ with $\|x - y\|_{\mathcal{X}} < \delta$

c.l.l. A mapping $f : (x, d) \in \mathcal{X} \times A \to f(x, d) \in W$, where $A \subseteq \mathcal{Y}$, and $\mathcal{X}$, $\mathcal{Y}$, and W are normed linear spaces, is said to be completely locally Lipschitz (c.l.l.) with respect to $x \in \mathcal{X}$ if, for every pair of bounded sets $S \subset \mathcal{X}$ and $G \subseteq A$, there exists $L \geq 0$ such that $\|f(x, d) - f(y, d)\|_W \leq L\|x - y\|_{\mathcal{X}}$ for all $x, y \in S$ and $d \in G$

Abbreviations

BIC	Boundedness Implies Continuation
RFC	Robust Forward Completeness
ODEs	Ordinary Differential Equations
RFDEs	Retarded Functional Differential Equations
FDEs	Functional Difference Equations
PDEs	Partial Differential Equations
RGAOS	Robust Global Asymptotic Output Stability
URGAOS	Uniform Robust Global Asymptotic Output Stability
RGAS	Robust Global Asymptotic Stability
URGAS	Uniform Robust Global Asymptotic Stability
WIOS	Weighted Input to Output Stability
UWIOS	Uniform Weighted Input to Output Stability
IOS	Input-to-Output Stability
IOpS	Input-to-Output practical Stability
UIOS	Uniform Input-to-Output Stability
WISS	Weighted Input-to-State Stability
UWISS	Uniform Weighted Input to State Stability
ISS	Input-to-State Stability
ISpS	Input-to-State practical Stability
UISS	Uniform Input-to-State Stability
ORCLF	Output Robust Control Lyapunov Functional or Function
SRCLF	State Robust Control Lyapunov Functional or Function
CLF	Control Lyapunov Functional or Function
KSE	Keep Supply at Equilibrium

Chapter 1
Introduction to Control Systems

1.1 Introduction

This chapter is devoted to the analysis of certain crucial notions in mathematical control theory:

- the notion of a deterministic control system
- the notion of a robust equilibrium point
- the notion of the feedback connection of control systems
- the notion of the transformation of control systems.

The above notions are presented in a system-theoretic framework which allows the study of various classes of systems that arise in physics, biology, economics, and engineering. Then, it is shown that, under mild assumptions, the following important classes of control systems:

- control systems described by ordinary differential equations
- control systems described by retarded functional differential equations
- control systems described by coupled retarded functional differential equations and functional difference equations
- control systems described by functional difference equations
- control systems with variable sampling partition

satisfy our requirements of a deterministic control system with the Boundedness-Implies-Continuation property and a robust equilibrium point. Finally, it is shown that discrete-time systems can be cast in the proposed system-theoretic framework.

1.2 Examples of Control Systems

We next present some examples of control systems. The examples are qualitatively different because the mathematical formulation of the model of each system is different. However, all examples show that there are common points, which can be used in order to treat them in the same way. We begin with the first class of continuous-time control systems described by ordinary differential equations.

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1.2.1 Control Systems Described by Ordinary Differential Equations (ODEs)

Let $U \subseteq \Re^m$, $D \subseteq \Re^l$, with $0 \in U$. Consider two locally bounded mappings $f : \Re^+ \times \Re^n \times U \times D \to \Re^n$ and $H : \Re^+ \times \Re^n \times U \to \Re^k$ with $H(t,0,0) = 0$ and $f(t,0,0,d) = 0$ for all $(t,d) \in \Re^+ \times D$ that satisfy the following hypotheses:

(H1) The mapping $(x,u,d) \to f(t,x,u,d)$ is continuous for each fixed $t \geq 0$, and there exists a symmetric positive definite matrix $P \in \Re^{n\times n}$ such that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset \Re^n \times U$, there exists a constant $L \geq 0$ such that

$$(x-y)'P\big(f(t,x,u,d) - f(t,y,u,d)\big) \leq L|x-y|^2$$
$$\text{for all } t \in I,\ (x,u,y,u) \in S \times S, \text{ and } d \in D.$$

(H2) The mapping $t \to f(t,x,u,d)$ is Lebesgue measurable and locally essentially bounded for each fixed $(x,u,d) \in \Re^n \times U \times D$.

(H3) There exist functions $\gamma \in K^+$ and $a \in K_\infty$ such that $|f(t,x,u,d)| \leq \gamma(t)a(|x| + |u|)$ for all $(t,x,u,d) \in \Re^+ \times \Re^n \times U \times D$.

(H4) The mapping $H : \Re^+ \times \Re^n \times U \to \Re^k$ is continuous.

Remark 1.1 It is of interest to note that Hypothesis (H1) is equivalent to the existence of a continuous function $L : \Re^+ \times \Re^+ \to \Re^+$ such that, for each fixed $t \geq 0$, the mappings $L(t,\cdot)$ and $L(\cdot,t)$ are nondecreasing, satisfying the following inequality:

$$(x-y)'P\big(f(t,x,u,d) - f(t,y,u,d)\big) \leq L\big(t, |x| + |y| + |u|\big)|x-y|^2$$
$$\text{for all } (t,x,y,d,u) \in \Re^+ \times \Re^n \times \Re^n \times D \times U \tag{1.1}$$

Let M_U and M_D denote the sets of Lebesgue measurable and locally essentially bounded functions $u : \Re^+ \to U$ and $d : \Re^+ \to D$, respectively.

Clearly, by virtue of Hypotheses (H1–3) above and Lemma 1 on p. 4 of [11], for every $(d,u) \in M_D \times M_U$, the composite map $f(t,x,u(t),d(t))$ satisfies the Carathéodory conditions on $\Re^+ \times \Re^n$. Consequently, by application of Theorem 1 on p. 4 of [11], for every $(t_0,x_0,u,d) \in \Re^+ \times \Re^n \times M_U \times M_D$, there exist $h > 0$ and at least one absolutely continuous function $x : [t_0, t_0+h] \to \Re^n$ with $x(t_0) = x_0$ such that $\dot{x}(t) = f(t,x(t),u(t),d(t))$ almost everywhere on $[t_0, t_0+h]$.

Let $x : [t_0, t_0+h] \to \Re^n$ and $y : [t_0, t_0+h] \to \Re^n$ be two absolutely continuous functions with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, respectively, that satisfy $\dot{x}(t) = f(t,x(t),u(t),d(t))$ and $\dot{y}(t) = f(t,y(t),u(t),d(t))$ almost everywhere on $[t_0, t_0+h]$. Evaluating the derivative of the absolutely continuous map $z(t) = (x(t) - y(t))'P(x(t) - y(t))$ on $[t_0, t_0+h]$ in conjunction with Hypothesis (H1) above, we obtain the integral inequality

$$\big|x(t) - y(t)\big|^2 \leq \frac{K_2}{K_1}\big|x(t_0) - y(t_0)\big|^2 + \frac{2}{K_1}\int_{t_0}^{t} \tilde{L}\big|x(\tau) - y(\tau)\big|^2 d\tau$$
$$\forall t \in [t_0, t_0+h]$$

where $\tilde{L} := L(t_0+h, a(x,y,u))$, $L(\cdot,\cdot)$ is the function involved in (1.1), $K_1, K_2 > 0$ are constants that satisfy $K_1|x|^2 \le x'Px \le K_2|x|^2$ for all $x \in \Re^n$, and $a(x,y,u) := \sup_{t\in[t_0,t_0+h]}|x(t)| + \sup_{t\in[t_0,t_0+h]}|y(t)| + \sup_{t\in[t_0,t_0+h]}|u(t)|$. A direct application of the Gronwall–Bellman inequality gives

$$\left|x(t)-y(t)\right| \le \sqrt{\frac{K_2}{K_1}}|x_0-y_0|\exp\left(\frac{\tilde{L}}{K_1}(t-t_0)\right) \quad \text{for all } t \in [t_0, t_0+h] \quad (1.2)$$

Thus, we conclude that under Hypotheses (H1–3), for every $(t_0, x_0, d, u) \in \Re^+ \times \Re^n \times M_D \times M_U$, there exist $h > 0$ and exactly one absolutely continuous function $x : [t_0, t_0+h] \to \Re^n$ with $x(t_0) = x_0$ such that $\dot{x}(t) = f(t, x(t), u(t), d(t))$ almost everywhere on $[t_0, t_0+h]$. Using the fact that the mapping $f : \Re^+ \times \Re^n \times U \times D \to \Re^n$ is locally bounded, Hypotheses (H1–3) above, and Theorem 3.2 in [14], we conclude that, for every $(t_0, x_0, u, d) \in \Re^+ \times \Re^n \times M_D \times M_U$, there exists $t_{\max} \in (t_0, +\infty]$ such that the unique solution $x(t)$ of $\dot{x}(t) = f(t, x(t), u(t), d(t))$ with $x(t_0) = x_0$ is defined on $[t_0, t_{\max})$ and cannot be further continued. Moreover, if $t_{\max} < +\infty$, then we must necessarily have $\limsup_{t\to t_{\max}^-}|x(t)| = +\infty$.

Thus, for every $(t_0, x_0, u, d) \in \Re^+ \times \Re^n \times M_D \times M_U$, the unique solution $x(t)$ of $\dot{x}(t) = f(t, x(t), u(t), d(t))$ with $x(t_0) = x_0$ and the output trajectory $Y(t)$ satisfy the following equations a.e.:

$$\begin{aligned} &\dot{x}(t) = f\big(t, x(t), u(t), d(t)\big) \\ &Y(t) = H\big(t, x(t), u(t)\big) \\ &\quad x(t) \in \Re^n, Y(t) \in \Re^k, u(t) \in U, d(t) \in D \end{aligned} \quad (1.3)$$

We next provide an example of a biological system which takes the form (1.3) and satisfies Hypotheses (H1–4). The example shows that it may be necessary to make certain manipulations in order to be able to satisfy Hypotheses (H1–4).

Example 1.2.1 Consider the chemostat with one microbial species and one nutrient (see [45]):

$$\begin{aligned} &\dot{X}(t) = \big(\mu\big(s(t)\big) - D(t) - b\big)X(t) \\ &\dot{s}(t) = D(t)\big(s_{\text{in}}(t) - s(t)\big) - K\big(s(t)\big)\mu\big(s(t)\big)X(t) \\ &\quad X(t) > 0, s(t) > 0 \end{aligned} \quad (1.4)$$

The concentration of the microbial species is denoted by $X(t)$, $s(t)$ denotes the concentration of the nutrient, $s_{\text{in}}(t) \ge 0$ denotes the inlet concentration of the limiting nutrient, and $D(t) \ge 0$ denotes the dilution rate. The specific growth rate $\mu(s)$ of the microbial species is a locally Lipschitz, bounded function $\mu : [0, +\infty) \to [0, +\infty)$ with $\mu(0) = 0$ and $\mu(s) > 0$ for all $s > 0$. The constant $b \ge 0$ is the mortality rate of the microbial species, while the locally Lipschitz function $K : [0, +\infty) \to (0, +\infty)$ is the (possibly varying) yield coefficient.

System (1.4) is a system described by ODEs. However, it does not satisfy all requirements described by Hypotheses (H1–3). First of all, the set where the state variables take values is not the Euclidean space $\Re^2$. In order to be able to recast system (1.4) to the form of a system which satisfies Hypotheses (H1–3), we assume the existence of $(X^*, s^*, D^*, s_{in}^*) \in (0, +\infty) \times (0, +\infty) \times (0, +\infty) \times (0, +\infty)$ with

$$\mu(s^*) = D^* + b \quad K(s^*)\mu(s^*)X^* = D^*(s_{in}^* - s^*) \tag{1.5}$$

and we apply the transformation:

$$X = X^* \exp(x_1) \quad s = s^* \exp(x_2) \qquad D = D^*(1 + u_1) \quad s_{in} = s_{in}^* + s^* u_2 \tag{1.6}$$

It follows from (1.5) and (1.6) that system (1.4) is described by the following differential equations:

$$\begin{aligned} \dot{x}_1 &= D^*\big(g(x_2) - u_1\big) \\ \dot{x}_2 &= D^* \exp(-x_2)\big[M\big(1 - p(x_2)\exp(x_1)\big) - RMp(x_2)g(x_2)\exp(x_1) \\ &\qquad + 1 - \exp(x_2) + u_2 + u_1\big(M + 1 + u_2 - \exp(x_2)\big)\big] \end{aligned} \tag{1.7}$$

where $g(x_2) := \frac{\mu(s^* \exp(x_2)) - \mu(s^*)}{D^*}$, $p(x_2) := \frac{K(s^* \exp(x_2))}{K(s^*)}$, $R := \frac{D^*}{D^*+b}$, and $M := \frac{s_{in}^* - s^*}{s^*}$. Notice that by virtue of (1.5) and previous definitions, the transformed system (1.7) now satisfies Hypotheses (H1–3) with $x = (x_1, x_2)' \in \Re^2$, $u = (u_1, u_2)' \in U = [-1, +\infty) \times [-M, +\infty)$.

In this case, $d(t)$ is irrelevant. Moreover, we define the output map $Y = (x_1, x_2)' = x$ to be the identity map.

Let $\phi : A_\phi \to \Re^n$ be the mapping $\phi(t, t_0, x_0, u, d) := x(t)$ which for each $(t_0, x_0, d, u) \in \Re^+ \times \Re^n \times M_D \times M_U$ and $t \in [t_0, t_{max})$ gives the value $x(t) \in \Re^n$ of the unique solution of $\dot{x}(t) = f(t, x(t), u(t), d(t))$ with $x(t_0) = x_0$. The mapping $\phi : A_\phi \to \Re^n$ is defined on the set $A_\phi = \bigcup_{(t_0, x_0, u, d) \in \Re^+ \times \Re^n \times M_U \times M_D} [t_0, t_{max}) \times \{(t_0, x_0, u, d)\}$. It should be clear from all the above that the mapping $\phi : A_\phi \to \Re^n$ is well defined and satisfies the following properties:

(1) *Existence*: For each $(t_0, x_0, d, u) \in \Re^+ \times \Re^n \times M_D \times M_U$, there exists $t > t_0$ such that $[t_0, t] \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$.
(2) *Identity Property*: For each $(t_0, x_0, d, u) \in \Re^+ \times \Re^n \times M_D \times M_U$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.
(3) *Causality*: For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ for all $\tau \in [t_0, t]$, it holds that $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ and $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.
(4) *The "Boundedness-Implies-Continuation" (BIC) Property*: For each quadruplet $(t_0, x_0, u, d) \in \Re^+ \times \Re^n \times M_U \times M_D$, there exists a maximal existence time, i.e., there exists $t_{max} \in (t_0, +\infty]$ such that $[t_0, t_{max}) \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$, and for all $t \geq t_{max}$, it holds that $(t, t_0, x_0, u, d) \notin A_\phi$. In addition, if $t_{max} < +\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{max})$ with $|\phi(t, t_0, x_0, u, d)| > M$.
(5) *The Classical Semigroup Property*: For each $t \in [t_0, t_{max})$, it holds that, for all $\tau \in [t_0, t]$, $\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$.

1.2.2 Control Systems Described by Retarded Functional Differential Equations (RFDEs)

Let $D \subseteq \Re^l$ be a nonempty set, and $U \subseteq \Re^m$ a nonempty set with $0 \in U$. Also consider locally bounded mappings $f : \Re^+ \times C^0([-r,0];\Re^n) \times U \times D \to \Re^n$, $H : \Re^+ \times C^0([-r,0];\Re^n) \times U \to \mathcal{Y}$, where $\mathcal{Y}$ is a normed linear space, that satisfy $f(t,0,0,d)=0$, $H(t,0,0)=0$ for all $(t,d) \in \Re^+ \times D$ and the following hypotheses:

(S1) The mapping $(x,u,d) \to f(t,x,u,d)$ is continuous for each fixed $t \geq 0$, and there is a symmetric positive definite matrix $P \in \Re^{n\times n}$ such that for every bounded $I \subseteq \Re^+$ and every bounded $S \subset C^0([-r,0];\Re^n) \times U$, there exists a constant $L \geq 0$ such that

$$\big(x(0)-y(0)\big)' P\big(f(t,x,u,d)-f(t,y,u,d)\big) \leq L \max_{\tau\in[-r,0]} \big|x(\tau)-y(\tau)\big|^2$$
for all $t \in I$, $(x,u,y,u) \in S \times S$, and $d \in D$.

(S2) There exist functions $\gamma \in K^+$ and $a \in K_\infty$ such that $|f(t,x,u,d)| \leq \gamma(t)a(\|x\|_r + |u|)$ for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$, where $\|x\|_r$ denotes the sup-norm of the space $C^0([-r,0];\Re^n)$, i.e., $\|x\|_r := \max_{\theta\in[-r,0]} |x(\theta)|$.

(S3) There exists a countable set $A \subset \Re^+$, which is either finite or $A = \{t_k; k = 1,\dots,\infty\}$ with $t_{k+1} > t_k > 0$ for all $k = 1,2,\dots$ and $\lim t_k = +\infty$, such that the mapping $(t,x,u,d) \in (\Re^+ \backslash A) \times C^0([-r,0];\Re^n) \times U \times D \to f(t,x,u,d)$ is continuous. Moreover, for each fixed $(t_0,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$, we have $\lim_{t\to t_0^+} f(t,x,u,d) = f(t_0,x,u,d)$.

(S4) The mapping $H : \Re^+ \times C^0([-r,0];\Re^n) \times U \to \mathcal{Y}$ is a continuous mapping that maps bounded sets of $\Re^+ \times C^0([-r,0];\Re^n) \times U$ into bounded sets of $\mathcal{Y}$.

Remark 1.2 It is worth noting that Hypothesis (S1) is equivalent to the existence of a continuous function $L : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $t \geq 0$, the mappings $L(t,\cdot)$ and $L(\cdot,t)$ are nondecreasing and satisfying the following inequality:

$$\big(x(0)-y(0)\big)' P\big(f(t,x,u,d)-f(t,y,u,d)\big) \leq L\big(t, \|x\|_r + \|y\|_r\big)\|x-y\|_r^2$$
$$\text{for all } (t,x,y,u,d) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \times C^0\big([-r,0];\Re^n\big) \times U \times D \tag{1.8}$$

Let M_U, M_D denote the sets of Lebesgue-measurable and locally essentially bounded functions $u : \Re^+ \to U$ and $d : \Re^+ \to D$, respectively. By virtue of Hypotheses (S1–3) above and Lemma 1 on p. 4 of [11], for every $(d,u) \in M_D \times M_U$, the composite map $f(t,x,u(t),d(t))$ satisfies the Caratheodory condition on $\Re^+ \times C^0([-r,0];\Re^n)$. Consequently, by virtue of Theorem 2.1 in [14] and its extension given in Paragraph 2.6 of the same book, for every $(t_0,x_0,d,u) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D \times M_U$, there exist $h > 0$ and at least one continuous function $x : [t_0 - r, t_0 + h] \to \Re^n$, which is absolutely continuous on $[t_0, t_0 + h]$ with $T_r(t_0)x = x_0$ and satisfies $\dot{x}(t) = f(t, T_r(t)x, u(t), d(t))$ almost everywhere on $[t_0, t_0 + h]$.

Let $x : [t_0 - r, t_0 + h] \to \Re^n$ and $y : [t_0 - r, t_0 + h] \to \Re^n$ be two continuous functions, absolutely continuous on $[t_0, t_0 + h]$ with $T_r(t_0)x = x_0$, $T_r(t_0)y = y_0$, and $\dot{x}(t) = f(t, T_r(t)x, u(t), d(t))$, $\dot{y}(t) = f(t, T_r(t)y, u(t), d(t))$ almost everywhere on $[t_0, t_0 + h]$. Evaluating the derivative of the absolutely continuous map $z(t) = (x(t) - y(t))' P(x(t) - y(t))$ on $[t_0, t_0 + h]$ in conjunction with Hypothesis (S1) above, we obtain the integral inequality:

$$\left|x(t) - y(t)\right|^2 \le \frac{K_2}{K_1}\left|x(t_0) - y(t_0)\right|^2 + \frac{2}{K_1}\int_{t_0}^{t} \tilde{L}\left\|T_r(\tau)x - T_r(\tau)y\right\|_r^2 d\tau$$
$$\forall t \in [t_0, t_0 + h]$$

where $\tilde{L} := L(t_0 + h, a(t_0 + h))$, $L(\cdot)$ is the function involved in (1.8), $a(t) := \sup_{\tau\in[t_0-r,t]} |x(\tau)| + \sup_{\tau\in[t_0-r,t]} |y(\tau)| + \sup_{\tau\in[t_0,t]} |u(\tau)|$, and K_1, $K_2 > 0$ are constants that satisfy $K_1|x|^2 \le x'Px \le K_2|x|^2$ for all $x \in \Re^n$. Consequently, we obtain

$$\left\|T_r(t)(x - y)\right\|_r^2 \le \frac{K_2}{K_1}\|x_0 - y_0\|_r^2 + \frac{2}{K_1}\int_{t_0}^{t} \tilde{L}\left\|T_r(\tau)(x - y)\right\|_r^2 d\tau$$
$$\forall t \in [t_0, t_0 + h]$$

An immediate application of the Gronwall–Bellman inequality gives:

$$\left\|T_r(t)(x - y)\right\|_r \le \sqrt{\frac{K_2}{K_1}}\|x_0 - y_0\|_r \exp\left(\frac{\tilde{L}}{K_1}(t - t_0)\right) \quad \forall t \in [t_0, t_0 + h] \tag{1.9}$$

Thus, we conclude that, under Hypotheses (S1–3), for every $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times M_D \times M_U$, there exist $h > 0$ and exactly one continuous function $x : [t_0 - r, t_0 + h] \to \Re^n$, which is absolutely continuous on $[t_0, t_0 + h]$ with $T_r(t_0)x = x_0$ and $\dot{x}(t) = f(t, T_r(t)x, u(t), d(t))$ almost everywhere on $[t_0, t_0 + h]$. Using Hypothesis (S2) above and Theorem 3.2 in [14], we conclude that, for every $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times M_D \times M_U$, there exists $t_{\max} \in (t_0, +\infty]$ such that the unique solution of $\dot{x}(t) = f(t, T_r(t)x, u(t), d(t))$ with $T_r(t_0)x = x_0$ is defined on $[t_0 - r, t_{\max})$ and cannot be further continued. Moreover, if $t_{\max} < +\infty$, then we must necessarily have $\limsup_{t\to t_{\max}^-} |x(t)| = +\infty$, which implies $\limsup_{t\to t_{\max}^-} \|T_r(t)x\|_r = +\infty$.

Thus, for every $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times M_D \times M_U$, the unique solution $x(t)$ of $\dot{x}(t) = f(t, T_r(t)x, u(t), d(t))$ with $T_r(t_0)x = x_0$ and the output trajectory $Y(t)$ satisfy the following equations a.e.:

$$\begin{aligned}
\dot{x}(t) &= f\big(t, T_r(t)x, u(t), d(t)\big)\\
Y(t) &= H\big(t, T_r(t)x, u(t)\big)\\
&\quad x(t) \in \Re^n, d(t) \in D, Y(t) \in \mathcal{Y}, u(t) \in U
\end{aligned} \tag{1.10}$$

By allowing the output to take values in abstract normed linear spaces, we can consider general outputs. More precisely, we may consider the following cases:

- outputs with no delays, e.g., $Y(t) = h(t, x(t))$ with $\mathcal{Y} = \Re^k$,
- outputs with discrete or distributed delay, e.g., $Y(t) = h(x(t), x(t-r))$ or $Y(t) = \int_{t-r}^{t} h(t, \theta, x(\theta))\, d\theta$ with $\mathcal{Y} = \Re^k$,
- functional outputs with memory, e.g., $(Y(t))(\theta) = h(t, \theta, x(t+\theta))$ for $\theta \in [-r, 0]$ or the identity output $Y(t) = T_r(t)x$ with $\mathcal{Y} = C^0([-r, 0]; \Re^k)$.

We next provide an example of a biological system, similar to the one of Example 1.2.1, which takes the form (1.10) and satisfies Hypotheses (S1–4). Again, the example shows that it may be necessary to make certain manipulations in order to be able to satisfy Hypotheses (S1–4).

Example 1.2.2 Consider the chemostat with one microbial species and one nutrient. In this case, it is assumed that the specific growth rate of the microbial species depends not only on the current value of the concentration of the nutrient $s(t)$ but on its past values as well. We thus obtain the following chemostat model with delays:

$$\begin{aligned} \dot{X}(t) &= \big(p\big(T_r(t)s\big) - D(t) - b\big)X(t) \\ \dot{s}(t) &= D(t)\big(s_{\text{in}} - s(t)\big) - K\big(s(t)\big)\mu\big(s(t)\big)X(t) \\ & X(t) \in (0, +\infty), s(t) \in (0, s_{\text{in}}), D(t) > 0 \end{aligned} \tag{1.11}$$

where $(T_r(t)s)(\theta) = s(t+\theta)$ for $\theta \in [-r, 0]$ is the r-history of s, $b \geq 0$ is the cell mortality rate, $r \geq 0$ is the maximum delay, $K(S) > 0$ is a possibly variable yield coefficient, and $p : C^0([-r, 0]; (0, s_{\text{in}})) \to (0, +\infty)$ is a continuous functional that satisfies

$$\min_{t-r \leq \tau \leq t} \mu\big(s(\tau)\big) \leq p\big(T_r(t)s\big) \leq \max_{t-r \leq \tau \leq t} \mu\big(s(\tau)\big) \tag{1.12}$$

The functions $\mu : [0, s_{\text{in}}] \to [0, \mu_{\max}]$, $K : [0, s_{\text{in}}] \to (0, +\infty)$ with $\mu(0) = 0$, $\mu(s) > 0$ for all $s > 0$ are assumed to be locally Lipschitz functions. The chemostat model (1.11) under (1.12) is very general, since it covers the following cases:

- $p(T_r(t)s) = \mu(s(t))$, which gives a chemostat model similar to the chemostat model (1.4) with no delays,
- $p(T_r(t)s) = \mu(s(t-r))$, which gives the time-delayed chemostat model studied in [45],
- $p(T_r(t)s) = \lambda \sum_{i=0}^{n} w_i \mu(s(t-r_i)) + (1-\lambda) \int_{t-r}^{t} h(\tau + r - t)\mu(s(\tau))\, d\tau$, where $\lambda \in [0, 1]$, $h \in C^0([0, r]; [0, +\infty))$ with $\int_0^r h(s)\, ds = 1$, $w_i \geq 0$, $r_i \in [0, r]$ $(i = 0, \ldots, n)$ with $\sum_{i=0}^{n} w_i = 1$.

As remarked in [45], the chemostat model (1.11) under (1.12) allows the expression of the effect of the time difference between consumption of nutrient and growth of the cells (see the discussion on pp. 238–240 in [45]). Since the mapping $p : C^0([-r, 0]; (0, s_i)) \to (0, +\infty)$ is rarely known, we will consider the uncertain chemostat model:

$$\dot{X}(t) = \Big(\min_{t-r\le\tau\le t} \mu\big(s(\tau)\big) + d(t)\Big(\max_{t-r\le\tau\le t} \mu\big(s(\tau)\big) - \min_{t-r\le\tau\le t} \mu\big(s(\tau)\big)\Big)$$
$$- D(t) - b\Big) X(t)$$
$$\dot{s}(t) = D(t)\big(s_{\text{in}} - s(t)\big) - K\big(s(t)\big)\mu\big(s(t)\big)X(t)$$
$$X(t) \in (0, +\infty), s(t) \in (0, s_{\text{in}}), D(t) > 0, d(t) \in [0, 1] \tag{1.13}$$

where $d(t) \in [0, 1]$ is the uncertainty.

System (1.13) is an uncertain system described by RFDEs. However, it does not satisfy all requirements described by Hypotheses (S1–3). First of all, the set where the state variables take values is not the Banach space $C^0([-r, 0]; \Re^2)$. In order to be able to recast system (1.13) to the form of system (1.10) which satisfies Hypotheses (S1–3), we assume the existence of $(X^*, s^*, D^*) \in (0, +\infty) \times (0, s_{\text{in}}) \times (0, +\infty)$ such that

$$\mu\big(s^*\big) = D^* + b \qquad X^* = \frac{D^*(s_{\text{in}} - s^*)}{K(s^*)(D^* + b)} \tag{1.14}$$

It should be noted that the change of coordinates:

$$X = X^* \exp(x_1) \qquad s = \frac{s_{\text{in}} \exp(x_2)}{G + \exp(x_2)} \tag{1.15}$$

where $G := \frac{s_{\text{in}}}{s^*} - 1$, and the input transformation

$$D = D^* \exp(u)$$

give the transformed control system:

$$\dot{x}_1(t) = \min_{t-r\le\tau\le t} \tilde{\mu}\big(x_2(\tau)\big) + d(t)\Big(\max_{t-r\le\tau\le t} \tilde{\mu}\big(x_2(\tau)\big) - \min_{t-r\le\tau\le t} \tilde{\mu}\big(x_2(\tau)\big)\Big)$$
$$- D^* \exp\big(u(t)\big) - b$$
$$\dot{x}_2(t) = D^*\big(G \exp\big(-x_2(t)\big) + 1\big)$$
$$\times \big[\exp\big(u(t)\big) - \big(G + \exp\big(x_2(t)\big)\big) g\big(x_2(t)\big) \exp\big(x_1(t)\big)\big]$$
$$(x_1, x_2) \in \Re^2, u(t) \in \Re, d(t) \in [0, 1] \tag{1.16}$$

where

$$\tilde{\mu}(x_2) := \mu\left(\frac{s_{\text{in}} \exp(x_2)}{G + \exp(x_2)}\right)$$
$$g(x_2) := \frac{X^*}{D^* s_{\text{in}} G} K\left(\frac{s_{\text{in}} \exp(x_2)}{G + \exp(x_2)}\right) \mu\left(\frac{s_{\text{in}} \exp(x_2)}{G + \exp(x_2)}\right) \tag{1.17}$$

Notice that by virtue of (1.14) and previous assumptions and definitions, system (1.16) satisfies Hypotheses (S1–3) with $T_r(t)x \in C^0([-r, 0]; \Re^2)$, $u(t) \in U = \Re$, and $d(t) \in D = [0, 1]$.

We finally define the output mapping $Y(t) = T_r(t)x$ to be the identity mapping with $\mathcal{Y} = C^0([-r, 0]; \Re^2)$.

Let $\phi : A_\phi \to C^0([-r,0];\Re^n)$ be the mapping $\phi(t,t_0,x_0,u,d) := T_r(t)x$, which for each $(t_0,x_0,d,u) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D \times M_U$ and $t \in [t_0,t_{\max})$ gives the "r-history" $T_r(t)x \in C^0([-r,0];\Re^n)$ of the unique solution of $\dot{x}(t) = f(t,T_r(t)x,u(t),d(t))$ with $T_r(t_0)x = x_0$. The mapping $\phi : A_\phi \to C^0([-r,0];\Re^n)$ is defined on the set $A_\phi = \bigcup_{(t_0,x_0,u,d)\in\Re^+\times C^0([-r,0];\Re^n)\times M_U\times M_D}[t_0,t_{\max}) \times \{(t_0,x_0,u,d)\}$. It should be clear from all the above that the mapping $\phi : A_\phi \to C^0([-r,0];\Re^n)$ is well defined and satisfies the following properties:

(1) *Existence*: For each $(t_0,x_0,d,u) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D \times M_U$, there exists $t > t_0$ such that $[t_0,t] \times \{(t_0,x_0,u,d)\} \subseteq A_\phi$.
(2) *Identity Property*: For each $(t_0,x_0,d,u) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D \times M_U$, it holds that $\phi(t_0,t_0,x_0,u,d) = x_0$.
(3) *Causality*: For each $(t,t_0,x_0,u,d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u},\tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau),\tilde{d}(\tau)) = (u(\tau),d(\tau))$ for all $\tau \in [t_0,t]$, it holds that $(t,t_0,x_0,\tilde{u},\tilde{d}) \in A_\phi$ and $\phi(t,t_0,x_0,u,d) = \phi(t,t_0,x_0,\tilde{u},\tilde{d})$.
(4) *The "Boundedness-Implies-Continuation" (BIC) Property*: For each quadruplet $(t_0,x_0,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_U \times M_D$, there exists a maximal existence time, i.e., there exists $t_{\max} \in (t_0,+\infty]$ such that $[t_0,t_{\max}) \times \{(t_0,x_0,u,d)\} \subseteq A_\phi$ and for all $t \ge t_{\max}$, it holds that $(t,t_0,x_0,u,d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$ then for every $M > 0$, there exists $t \in [t_0,t_{\max})$ with $\|\phi(t,t_0,x_0,u,d)\|_r > M$.
(5) *The Classical Semigroup Property*: For each $t \in [t_0,t_{\max})$, it holds that, for all $\tau \in [t_0,t]$, $\phi(t,\tau,\phi(\tau,t_0,x_0,u,d),u,d) = \phi(t,t_0,x_0,u,d)$.

1.2.3 Control Systems Described by Coupled Retarded Functional Differential Equations (RFDEs) and Functional Difference Equations (FDEs)

Let $D \subseteq \Re^l$ be a nonempty set, and $U \subseteq \Re^m$ be a nonempty set with $0 \in U$. Consider the system described by the following equations:

$$\dot{x}_1(t) = f_1\big(t,d(t),T_{r_1}(t)x_1,T_{r_2-\tau(t)}\big(t-\tau(t)\big)x_2,u(t)\big) \tag{1.18}$$

$$x_2(t) = f_2\big(t,d(t),T_{r_1}(t)x_1,T_{r_2-\tau(t)}\big(t-\tau(t)\big)x_2,u(t)\big) \tag{1.19}$$

$$Y(t) = H\big(t,T_{r_1}(t)x_1,T_{r_2}(t)x_2,u(t)\big) \in \mathcal{Y} \tag{1.20}$$

$$x_1(t) \in \Re^{n_1}, x_2(t) \in \Re^{n_2}, d(t) \in D, u(t) \in U, t \ge 0$$

where $r_1,r_2 \ge 0$, $f_i : \bigcup_{t\ge0}\{t\} \times D \times C^0([-r_1,0];\Re^{n_1}) \times \mathcal{L}^\infty([-r_2+\tau(t),0];\Re^{n_2}) \times U \to \Re^{n_i}$, $i = 1,2$, $H : \Re^+ \times C^0([-r_1,0];\Re^{n_1}) \times \mathcal{L}^\infty([-r_2,0];\Re^{n_2}) \times U \to \mathcal{Y}$ (with $\mathcal{Y}$ being a normed linear space) are locally bounded mappings with $f_i(t,d,0,0,0) = 0$, $i = 1,2$, $H(t,0,0,0) = 0$ for all $(t,d) \in \Re^+ \times D$. The coupled system is depicted in Fig. 1.1.

Specifically, we consider systems of the form (1.18), (1.19), (1.20) with initial conditions $x_1(t_0+\theta) = x_{10}(\theta), \theta \in [-r_1,0]$, and $x_2(t_0+\theta) = x_{20}(\theta), \theta \in [-r_2,0]$,

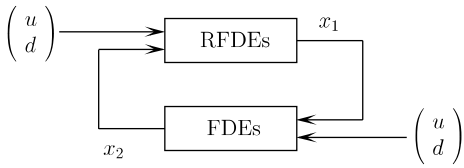

Fig. 1.1 Control systems described by coupled RFDEs and FDEs

with $x_{10} \in C^0([-r_1, 0]; \Re^{n_1})$ and $x_{20} \in \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2})$, under the following hypotheses:

(P1) The function $\tau : \Re^+ \to (0, +\infty)$ is continuous with $\sup_{t \geq 0} \tau(t) \leq r_2$.

(P2) There exist functions $a \in K_\infty$, $\beta \in K^+$ such that

$$\begin{aligned}&\left|f_i\big(t, d, x_1, T_{r_2-\tau(t)}\big(-\tau(t)\big)x_2, u\big)\right|\\&\quad\leq a\big(\beta(t)\|x_1\|_{r_1}\big) + a\big(\beta(t)\big\|T_{r_2-\tau(t)}\big(-\tau(t)\big)x_2\big\|_{r_2-\tau(t)}\big) + a\big(\beta(t)|u|\big)\\&\quad\text{for all } (t, d, x_1, x_2, u) \in \Re^+ \times D \times C^0\big([-r_1, 0]; \Re^{n_1}\big)\\&\qquad\qquad\qquad \times \mathcal{L}^\infty\big([-r_2, 0]; \Re^{n_2}\big) \times U.\end{aligned}$$

(P3) For all $x_1 \in C^0([-r_1, +\infty); \Re^{n_1})$, $d \in \mathcal{L}^\infty_{\text{loc}}(\Re^+; D)$, $u \in \mathcal{L}^\infty_{\text{loc}}(\Re^+; U)$, and $x_2 \in \mathcal{L}^\infty_{\text{loc}}([-r_2, +\infty); \Re^{n_2})$, the mappings $t \to f_i(t, d(t), T_{r_1}(t)x_1, T_{r_2-\tau(t)}(t - \tau(t))x_2, u(t))$, $i = 1, 2$, are measurable. Moreover, for each fixed $(t, d, x_2, u) \in \Re^+ \times D \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times U$, the mapping $f_1(t, d, x_1, T_{r_2-\tau(t)}(-\tau(t))x_2, u)$ is continuous with respect to $x_1 \in C^0([-r_1, 0]; \Re^{n_1})$.

(P4) There exists a symmetric positive definite matrix $P \in \Re^{n_1 \times n_1}$ such that for every pair of bounded sets $I \subset \Re^+$ and $\Omega \subset C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times U$, there exists $L := L(I, \Omega) \geq 0$ such that

$$\begin{aligned}&\big(x_1(0) - y_1(0)\big)' P\big(f_1\big(t, d, x_1, T_{r_2-\tau(t)}\big(-\tau(t)\big)x_2, u\big)\\&\quad - f_1\big(t, d, y_1, T_{r_2-\tau(t)}\big(-\tau(t)\big)x_2, u\big)\big) \leq L\|x_1 - y_1\|^2_{r_1}\\&\quad\text{for all } (t, d) \in I \times D,\ (x_1, x_2, u) \in \Omega, \text{ and } (y_1, x_2, u) \in \Omega.\end{aligned} \tag{1.21}$$

(P5) The mapping $H : \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times U \to \mathcal{Y}$ is continuous with $H(t, 0, 0, 0) = 0$ for all $t \geq 0$. Moreover, the image set $H(\Omega)$ is bounded for each bounded set $\Omega \subset \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times U$.

For example, Hypotheses (P1), (P2), (P3) are satisfied if $D \subset \Re^l$ is compact and there exist continuous functions $\tau_i : \Re^+ \to (0, +\infty)$ $(i = 1, \ldots, p)$, $\tau : \Re^+ \to (0, +\infty)$ with $0 < \tau_1(t) < \tau_2(t) < \cdots < \tau_p(t) \leq \tau(t)$ for all $t \geq 0$ and $\sup_{t \geq 0} \tau(t) \leq r_2$, and continuous mappings $g_i : \Re^+ \times D \times C^0([-r_1, 0]; \Re^{n_1}) \times \Re^{pn_2} \times \Re^k \times U \to \Re^{n_i}$, $i = 1, 2$, $h : \Re^+ \times [-r_2, 0] \times \Re^{n_2} \to \Re^k$ with $g_i(t, d, 0, 0, 0, 0) = 0$, $h(t, \theta, 0) = 0$ for all $(t, \theta, d) \in \Re^+ \times [-r - T, 0] \times D$, such that, for each $i = 1, 2$,

$$\begin{aligned}&f_i\big(t, d, x_1, T_{r_2-\tau(t)}\big(-\tau(t)\big)x_2, u\big)\\&\quad = g_i\left(t, d, x_1, x_2(-\tau_1), x_2(-\tau_2), \ldots, x_2(-\tau_p), \int_{-r_2}^{-\tau(t)} h\big(t, \theta, x_2(\theta)\big)\, d\theta, u\right)\end{aligned}$$

$$\text{for all } (t,d,x_1,x_2,u) \in \Re^+ \times D \times C^0\big([-r_1,0];\Re^{n_1}\big) \times \mathcal{L}^\infty\big([-r_2,0];\Re^{n_2}\big) \times U$$

Systems of the form (1.18), (1.19), (1.20) arise in many problems in Mathematical Control Theory and Mathematical Systems Theory (see, for instance, [31, 35, 38, 39] and the references therein). For example, consider the stabilization problem for the scalar system:

$$\begin{aligned} &\dot{x}(t) = f\big(x(t)\big) + u(t) + au(t-r) \\ &x(t) \in \Re, u(t) \in \Re \end{aligned} \tag{1.22}$$

where $f : \Re \to \Re$ is a continuous function with $f(0) = 0$, and $r > 0$ and $a \in \Re$ are constants. If the designer selects to apply the feedback linearization approach for system (1.22), then we have

$$u(t) = -Kx(t) - f\big(x(t)\big) - au(t-r) \tag{1.23}$$

where $K > 0$. Consequently, if $a \neq 0$, the closed-loop system (1.22) with (1.23) is described by the following system of coupled RFDEs and FDEs:

$$\begin{aligned} &\dot{x}(t) = -Kx(t) \\ &u(t) = -Kx(t) - f\big(x(t)\big) - au(t-r) \\ &x(t) \in \Re, u(t) \in \Re \end{aligned} \tag{1.24}$$

Notice that system (1.24) has the form of system (1.18), (1.19), (1.20) with $u(t)$ in place of $x_2(t)$ and $x(t)$ in place of $x_1(t)$. Moreover, Hypotheses (P1–4) are satisfied for system (1.24).

However, the strongest motive for the study of systems of the form (1.18), (1.19), (1.20) is that systems of the form (1.18), (1.19), (1.20) allow the consideration of *discontinuous solutions* to systems described by Neutral Functional Differential Equations. For example, consider the scalar system described by a Neutral Functional Differential Equation

$$\frac{d}{dt}\big(x(t) - x(t-2r)\big) = x(t-r) \quad x(t) \in \Re \tag{1.25}$$

with initial condition $T_{2r}(t_0) = x_0 \in C^0([-2r,0];\Re)$. The solution of (1.25) for $t \in [t_0, t_0 + r]$ is given by

$$x(t) = x(t-2r) + x(t_0) - x(t_0 - 2r) + \int_{t_0-r}^{t-r} x(s)\,ds \tag{1.26}$$

It is clear from (1.26) that the solution of (1.25) can be defined even if the initial condition is discontinuous, i.e., $T_{2r}(t_0) = x_0 \in \mathcal{L}^\infty([-2r,0];\Re)$. How to obtain such a (weak) solution?

The following idea was proposed in [5] for linear systems (though it was expressed in a different way). First, define

$$\begin{aligned} &x_1(t) = x(t) - x(t-2r) \\ &x_2(t) = x(t) \end{aligned}$$

Then, (1.25) is transformed into the following system of coupled RFDEs and FDEs of the form (1.18), (1.19):

$$\begin{aligned}\dot{x}_1(t) &= x_2(t-r)\\ x_2(t) &= x_1(t) + x_2(t-2r)\\ x_1(t) &\in \Re,\ x_2(t) \in \Re\end{aligned} \tag{1.27}$$

Notice that Hypotheses (P1–4) are satisfied for system (1.27) (Hypothesis (P5) is irrelevant since there is no output). The solution of (1.27) for $t \in (t_0, t_0 + r]$ is given by:

$$\begin{aligned}x_1(t) &= x_1(t_0) + \int_{t_0-r}^{t-r} x_2(s)\,ds\\ x_2(t) &= x_2(t-2r) + x_1(t_0) + \int_{t_0-r}^{t-r} x_2(s)\,ds\end{aligned} \tag{1.28}$$

Notice that $x_2(t) = x(t)$ does not coincide with the solution of (1.25) given by (1.26) unless $x_1(t_0) = x_2(t_0) - x_2(t_0 - 2r)$, the so-called "matching condition." It should be emphasized that, if the "matching condition" does not hold, then the solution of (1.27), given by (1.28), is discontinuous, even if the initial condition is smooth. Consequently, system (1.27) provides a generalized framework for the study of the Neutral Functional Differential Equation (1.25). The idea described for the simple example (1.25) can be generalized for nonlinear control systems described by Neutral Functional Differential Equations of the following form (special case of the so-called Hale's form, see [14]):

$$\frac{d}{dt}\Big(x(t) - g\big(t, T_{r-\tau(t)}\big(t-\tau(t)\big)x\big)\Big) = f\big(t, d(t), T_r(t)x, u(t)\big) \quad x(t) \in \Re^n \tag{1.29}$$

Without loss of generality, we may assume that the continuous function $\tau : \Re^+ \to (0, +\infty)$ with $\sup_{t\geq 0} \tau(t) \leq r$ is nonincreasing. If we define $x_1(t) = x(t) - g(t, T_{r-\tau(t)}(t-\tau(t))x)$, $x_2(t) = x(t)$, and the operator

$$\begin{aligned}&\Re^+ \times C^0\big([-r,0];\Re^n\big) \times \mathcal{L}^\infty\big([-2r,0];\Re^n\big) \ni (t, x_1, x_2)\\ &\quad \to G\big(t, x_1, T_{2r-\tau(t)}\big(-\tau(t)\big)x_2\big)\\ &G\big(t, x_1, T_{2r-\tau(t)}\big(-\tau(t)\big)x_2\big)\\ &\quad := \begin{cases} x_2(\theta) & \theta \in [-r, -\tau(t)]\\ x_1(\theta) + g(t+\theta, T_{r-\tau(t+\theta)}(\theta - \tau(t+\theta))x_2) & \theta \in (-\tau(t), 0)\end{cases}\end{aligned}$$

then system (1.29) is associated with the following system described by coupled RFDEs and FDEs:

$$\begin{aligned}\dot{x}_1(t) &= f\big(t, d(t), G(t, T_r(t)x_1, T_{2r-\tau(t)}\big(t-\tau(t)\big)x_2), u(t)\big)\\ x_2(t) &= x_1(t) + g\big(t, T_{r-\tau(t)}(t-\tau(t))x_2\big)\end{aligned} \tag{1.30}$$

Notice that system (1.30) is in the form of (1.18), (1.19). The component x_2 of the solution of (1.30) coincides with the solution x of (1.29) if and only if the initial data are continuous functions which satisfy the "matching condition" $G(t_0, T_r(t_0)x_1, T_{2r-\tau(t_0)}(t_0 - \tau(t_0))x_2) = T_r(t_0)x_2$. However, notice that even if the matching condition holds, the solution of (1.30) can be defined for discontinuous initial data. Consequently, if the matching condition holds and the initial data are discontinuous, then the component x_2 of the solution of (1.30) is a discontinuous mapping which satisfies the differential equation $\frac{d}{dt}(x_2(t) - g(t, T_{r-\tau(t)}(t-\tau(t))x_2)) = f(t, d(t), T_r(t)x_2, u(t))$ almost everywhere for $t \geq t_0$. Thus, if the matching condition holds, then system (1.30) provides "weak" solutions to the Neutral Functional Differential Equation (1.29).

The approach described above is not restricted to Neutral Functional Differential Equations of the form (1.29). We can also consider Neutral Functional Differential Equations of the form (Bellman's form, see [3])

$$\dot{x}(t) = f\big(t, d(t), T_r(t)x, T_{r-\tau(t)}\big(t - \tau(t)\big)\dot{x}, u(t)\big) \quad x(t) \in \Re^n \tag{1.31}$$

In this case, the corresponding system of coupled RFDEs and FDEs is

$$\begin{aligned} \dot{x}_1(t) &= f\big(t, d(t), T_r(t)x_1, T_{r-\tau(t)}\big(t-\tau(t)\big)x_2, u(t)\big) \\ x_2(t) &= f\big(t, d(t), T_r(t)x_1, T_{r-\tau(t)}\big(t-\tau(t)\big)x_2, u(t)\big) \\ &x_1(t) \in \Re^n, x_2(t) \in \Re^n, d(t) \in D, u(t) \in U, t \geq 0 \end{aligned} \tag{1.32}$$

Notice that, if $T_r(t_0)x_2 = T_r(t_0)\dot{x}$ and $T_r(t_0)x_1 = T_r(t_0)x$, then the component x_1 of the solution of (1.32) coincides with the solution x of (1.31) for all $t \geq t_0$.

Consequently, it should be emphasized that the study of coupled RFDEs and FDEs offers a great advantage: Neutral Functional Differential Equations of the form (1.29) and Neutral Functional Differential Equations of the form (1.31) can be studied *in the same way and in the same framework*.

Another field which motivates the study of systems of the form (1.18), (1.19), (1.20) is the field of control (or dynamical) systems described by hyperbolic partial differential equations of the form

$$\begin{aligned} &\frac{\partial v_i}{\partial t}(t,z) + a_i \frac{\partial v_i}{\partial z}(t,z) = f_i\big(t, v_i(t,z), \xi(t)\big) \quad i = 1, \ldots, p \\ &v_i(t,z) \in \Re^{n_i}, \xi(t) \in \Re^k, z \in [0,1] \end{aligned} \tag{1.33}$$

where $a_i > 0$ $(i = 1, \ldots, p)$ are constants, along with boundary conditions of the form

$$v_i(t,0) = F_i\big(t, d(t), \xi(t), u(t), v(t,z); z \in \big[\tau(t), 1\big]\big) \quad i = 1, \ldots, p \tag{1.34}$$

where $v(t,z) = (v_1(t,z), \ldots, v_p(t,z))'$ and

$$\dot{\xi}(t) = g\big(t, d(t), \xi(t), u(t), v(t,z); z \in \big[\tau(t), 1\big]\big) \tag{1.35}$$

The system of equations (1.33), (1.34), (1.35) is accompanied with initial conditions $v_i(t_0, z) = v_{0i}(z)$ and $\xi(t_0) = \xi_0 \in \Re^k$.

If we define $x_1(t) = \xi(t)$ and $x_2(t) = (v_1(t,0), \ldots, v_p(t,0))'$, then it can be shown that the state variables $x_1(t)$, $x_2(t)$ satisfy a system of coupled RFDEs and FDEs for any $t \geq t_0 + \max_{i=1,\ldots,p} a_i^{-1}$. Consequently, the asymptotic behavior of system (1.33), (1.34), (1.35) is determined by the associated system of coupled RFDEs and FDEs. The discontinuous solutions generated by the associated system of coupled RFDEs and FDEs are important, since such solutions correspond to "weak" solutions of the problem (1.33), (1.34), (1.35).

We next provide an example of a physical system, which is described by partial differential equations and can be represented by equations (1.18), (1.19).

Example 1.2.3 (The linearized St. Venant equations) We consider the following system of partial differential equations:

$$\begin{aligned} &\frac{\partial h}{\partial t}(t,z)+\frac{\partial v}{\partial z}(t,z)=0 \\ &\frac{\partial v}{\partial t}(t,z)+\frac{\partial h}{\partial z}(t,z)=-u(t) \end{aligned} \tag{1.36}$$

where $z \in [0, 1]$ and $u(t) \in \Re$. The above system of partial differential equations is the linearization of the nonlinear system of partial differential equations describing the height $h(t,z)$ and the horizontal velocity $v(t,z)$ of an inviscid incompressible fluid contained in a tank at time $t \geq 0$ and position within the tank $z \in [0, 1]$. The tank is constrained to move only in the horizontal direction, and the position and the velocity of the tank are denoted by $D(t)$ and $s(t)$, respectively. Since $u(t) \in \Re$ is the horizontal acceleration of the tank (i.e., the force acting on the tank), we have in addition the following ordinary differential equations:

$$\dot{D}(t)=s(t) \qquad \dot{s}(t)=u(t) \tag{1.37}$$

Finally, the system is accompanied by the "no-flow" boundary conditions

$$v(t,0)=v(t,1)=0 \tag{1.38}$$

The reader should notice that the controllability and stabilizability problem of system (1.36), (1.37) has attracted attention (see [8], p. 212, and [41]), since it is known that system (1.36), (1.37) is uncontrollable (see [40]). By defining $A(t,z) := h(t,z)+v(t,z)$, $B(t,z) := h(t,z)-v(t,z)$, $x_{2,1}(t) := A(t,0)$, and $x_{2,2}(t) := B(t,1)$, we obtain by direct integration on the characteristic lines:

$$A(t,z)=\begin{cases} h_0(z-t)+v_0(z-t)-\int_0^t u(\tau)\,d\tau & \text{for } t \leq z \\ x_{2,1}(t-z)-\int_{t-z}^t u(\tau)\,d\tau & \text{for } t > z \end{cases}$$

$$B(t,z)=\begin{cases} h_0(z+t)-v_0(z+t)+\int_0^t u(\tau)\,d\tau & \text{for } t \leq 1-z \\ x_{2,2}(t+z-1)+\int_{t+z-1}^t u(\tau)\,d\tau & \text{for } t > 1-z \end{cases}$$

where $h_0(z) := h(0,z)$ and $v_0(z) := v(0,z)$. By defining $x_{2,1}(-w) := h_0(w) + v_0(w)$ and $x_{2,2}(-w) := h_0(1-w) - v_0(1-w)$ for $w \in [0, 1]$ and using the relations $h(t,z)=\frac{1}{2}(A(t,z)+B(t,z))$ and $v(t,z)=\frac{1}{2}(A(t,z)-B(t,z))$, we obtain for all $t \geq 0$:

$$h(t,z)=\frac{1}{2}x_{2,1}(t-z)+\frac{1}{2}x_{2,2}(t+z-1)+\frac{1}{2}\int_{\max(0,t+z-1)}^{\max(0,t-z)}u(\tau)\,d\tau \quad (1.39)$$

$$v(t,z)=\frac{1}{2}x_{2,1}(t-z)-\frac{1}{2}x_{2,2}(t+z-1)-\frac{1}{2}\int_{\max(0,t+z-1)}^{t}u(\tau)\,d\tau$$
$$-\frac{1}{2}\int_{\max(0,t-z)}^{t}u(\tau)\,d\tau \quad (1.40)$$

It should be noticed that the above equations give discontinuous solutions for system (1.36), (1.37) if the initial conditions $h_0(z):=h(0,z)$ and $v_0(z):=v(0,z)$ are discontinuous functions. Finally, we exploit the boundary conditions (1.38). Using (1.38) and (1.40), we obtain for all $t\ge 0$:

$$x_{2,1}(t)=x_{2,2}(t-1)+\int_{\max(0,t-1)}^{t}u(\tau)\,d\tau$$
$$x_{2,2}(t)=x_{2,1}(t-1)-\int_{\max(0,t-1)}^{t}u(\tau)\,d\tau$$

Consequently, we are led to the study of the following linear autonomous control system of coupled RFDEs and FDEs:

$$\dot{x}_{1,1}(t)=x_{1,2}(t)$$
$$\dot{x}_{1,2}(t)=u(t)$$
$$x_{2,1}(t)=x_{2,2}(t-1)+x_{1,2}(t)-x_{1,2}(t-1)$$
$$x_{2,2}(t)=x_{2,1}(t-1)-x_{1,2}(t)+x_{1,2}(t-1)$$
$$x_1(t)=\big(x_{1,1}(t),x_{1,2}(t)\big)\in\Re^2,\ x_2(t)=\big(x_{2,1}(t),x_{2,2}(t)\big)\in\Re^2,\ u(t)\in\Re \quad (1.41)$$

where $x_{1,1}(t)$ is the position of the tank $D(t)$, and $x_{1,2}(t)$ is the horizontal velocity of the tank $s(t)$. Notice that system (1.41) gives weak solutions for the original system (1.36) if $x_{2,1}(-w):=h(0,w)+v(0,w)$, $x_{2,2}(-w):=h(0,1-w)-v(0,1-w)$, and $x_{1,2}(-w)=x_{1,2}(0)$ for $w\in[0,1]$ by means of the formulae

$$h(t,z)=\frac{1}{2}x_{2,1}(t-z)+\frac{1}{2}x_{2,2}(t+z-1)+\frac{1}{2}x_{1,2}(t-z)-\frac{1}{2}x_{1,2}(t+z-1)$$
$$v(t,z)=\frac{1}{2}x_{2,1}(t-z)-\frac{1}{2}x_{2,2}(t+z-1)-x_{1,2}(t)+\frac{1}{2}x_{1,2}(t+z-1)$$
$$+\frac{1}{2}x_{1,2}(t-z)$$

Questions of controllability and stabilizability for system (1.36), (1.37) can be addressed by studying system (1.41).

The following theorem is the major tool for the existence of solutions of the system of coupled RFDEs and FDEs of the form (1.18), (1.19), (1.20).

Theorem 1.1 *Consider system* (1.18), (1.19) *under Hypotheses* (P1–4). *Then, for every* $t_0\ge 0$, $(x_{10},x_{20})\in C^0([-r_1,0];\Re^{n_1})\times\mathcal{L}^\infty([-r_2,0];\Re^{n_2})$, $d\in\mathcal{L}^\infty_{loc}(\Re^+;D)$,

$u \in \mathcal{L}^\infty_{loc}(\Re^+; U)$, there exist $t_{max} \in (t_0, +\infty]$ and a unique pair of mappings $x_1 \in C^0([t_0 - r_1, t_{max}); \Re^{n_1})$, $x_2 \in \mathcal{L}^\infty_{loc}([t_0 - r_2, t_{max}); \Re^{n_2})$ with $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$, $x_1 \in C^0([t_0 - r_1, t_{max}); \Re^{n_1})$ being absolutely continuous on $[t_0, t_{max})$ such that (1.18) holds a.e. for $t \in [t_0, t_{max})$ and (1.19) holds for all $t \in (t_0, t_{max})$. In addition, if $t_{max} < +\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{max})$ with $\|T_{r_1}(t)x_1\|_{r_1} > M$.

Theorem 1.1 guarantees that $t_{max} \in (t_0, +\infty]$ is the maximal existence time for the solution of (1.18), (1.19). The idea behind the proof of Theorem 1.1 is the method of steps, used already in [35].

Proof of Theorem 1.1 Let $t_0 \geq 0$, $(x_{10}, x_{20}) \in C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2})$, $d \in \mathcal{L}^\infty_{loc}(\Re^+; D)$, and $u \in \mathcal{L}^\infty_{loc}(\Re^+; U)$ (arbitrary).

Define $h := \min(1; \min\{\tau(t_0 + s) : s \in [0, 1]\})$. Notice that by virtue of the definition of $h > 0$, it holds that $t - \tau(t) \leq t_0$ for all $t \in [t_0, t_0 + h]$. By virtue of Theorem 2.1 in [14] (and its extension for Caratheodory conditions in p. 58 of the same book), there exists $\delta \in (0, h]$ and $x_1 \in C^0([t_0 - r_1, t_0 + \delta); \Re^{n_1})$ with $T_{r_1}(t_0)x_1 = x_{10}$ being absolutely continuous on $[t_0, t_0 + \delta)$ such that the differential equation $\dot{x}_1(t) = f_1(t, d(t), T_{r_1}(t)x_1, T_{r_2 - \tau(t)}(t - \tau(t))x_2, u(t))$ is satisfied a.e. for $t \in [t_0, t_0 + \delta)$. Moreover, Hypothesis (P4) guarantees that the mapping $x_1 \in C^0([t_0 - r_1, t_0 + \delta); \Re^{n_1})$ is unique. Without loss of generality, we may assume that δ is the maximal time of existence of the solution of differential equation (1.18) in the interval $(0, h]$. There exist two cases for the mapping $x_1 \in C^0([t_0 - r_1, t_0 + \delta); \Re^{n_1})$:

(a) For every $M > 0$, there exists $t \in [t_0, t_0 + \delta)$ with $\|T_{r_1}(t)x_1\|_{r_1} > M$.
(b) $\|T_{r_1}(t)x_1\|_{r_1}$ is bounded for all $t \in [t_0, t_0 + h)$. In this case, the mapping $x_1 \in C^0([t_0 - r_1, t_0 + \delta); \Re^{n_1})$ can be extended continuously in a unique way on $[t_0 - r_1, t_0 + h]$.

Next, we consider the FDE $x_2(t) = f_2(t, d(t), T_{r_1}(t)x_1, T_{r_2 - \tau(t)}(t - \tau(t))x_2, u(t))$. By (P2) and (P3), there exists a unique mapping $x_2 \in \mathcal{L}^\infty_{loc}([t_0 - r_2, t_0 + \delta); \Re^{n_2})$ with $T_{r_2}(t_0)x_2 = x_{20}$ such that $x_2(t) = f_2(t, d(t), T_{r_1}(t)x_1, T_{r_2 - \tau(t)}(t - \tau(t))x_2, u(t))$ holds for all $t \in (t_0, t_0 + \delta)$. Moreover, if the mapping $x_1 \in C^0([t_0 - r_1, t_0 + \delta); \Re^{n_1})$ can be extended continuously in a unique way on $[t_0 - r_1, t_0 + h]$, then, similarly, $x_2 \in \mathcal{L}^\infty([t_0 - r_2, t_0 + \delta); \Re^{n_2})$ can be extended on $[t_0 - r_2, t_0 + h]$ (notice that Hypothesis (P2) implies that x_2 is bounded as long as x_1 is bounded).

If case (a) holds, the proof is completed by defining $t_{max} = t_0 + \delta$. If case (b) holds, all arguments can be repeated with $t_0 + h$ in place of t_0 (next step). We continue the same procedure of construction of the solution step-by-step. The procedure may be stopped after some steps (if case (a) is encountered) or may be continued indefinitely (if case (a) is never encountered). In the latter case, for each step i, we obtain a pair of mappings $x_1 \in C^0([t_0 - r_1, t_{i+1}]; \Re^{n_1})$, $x_2 \in \mathcal{L}^\infty([t_0 - r_2, t_{i+1}]; \Re^{n_2})$ with $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$, $x_1 \in C^0([t_0 - r_1, t_{i+1}]; \Re^{n_1})$ being absolutely continuous on $[t_0, t_{i+1}]$ such that (1.18) holds a.e. for $t \in [t_0, t_{i+1}]$ and (1.19) holds for all $t \in (t_0, t_{i+1}]$, where the sequence $\{t_i\}_{i=0}^{\infty}$ satisfies

$$t_{i+1} = t_i + \min\bigl(1; \min\bigl\{\tau(t_i + s) : s \in [0, 1]\bigr\}\bigr) \quad \text{for all } i = 0, 1, 2, \ldots.$$

Notice that the sequence $\{t_i\}_{i=0}^{\infty}$ is increasing and consequently $\lim t_i = \sup t_i$. The assumption that $L = \lim t_i = \sup t_i < +\infty$ implies that $t_{i+1} \geq t_i + \mu$ for all $i = 0, 1, 2, \ldots$, where $\mu = \min(1; \min\{\tau(s) : s \in [0, L + 1]\})$, which gives the contradiction $t_i \geq t_0 + (i - 1)\mu$ for all $i = 1, 2, \ldots$. It follows that $\lim t_i = \sup t_i = +\infty$.

The proof is thus completed. □

Let M_U and M_D denote the sets of Lebesgue-measurable and locally essentially bounded functions $u : \Re^+ \to U$ and $d : \Re^+ \to D$, respectively. Let $\phi : A_\phi \to C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2})$ be the map $\phi(t, t_0, x_0, u, d) = (T_{r_1}(t)x_1, T_{r_2}(t)x_2)$, which for each $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times M_D \times M_U$ and $t \in [t_0, t_{\max})$ gives the "history" of the unique solution of (1.18), (1.19) with $T_r(t_0)x = x_0$. The mapping $\phi : A_\phi \to C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2})$ is defined on the set

$$A_\phi = \bigcup_{(t_0, x_0, u, d) \in \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times M_U \times M_D} [t_0, t_{\max}) \bigl\{(t_0, x_0, u, d)\bigr\}$$

It should be clear from all the above that the mapping $\phi : A_\phi \to C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2})$ is well defined and satisfies the following properties:

(1) *Existence*: For each $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times M_D \times M_U$, there exists $t > t_0$ such that $[t_0, t] \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$.
(2) *Identity Property*: For each $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times M_D \times M_U$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.
(3) *Causality*: For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ for all $\tau \in [t_0, t]$, it holds that $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ and $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.
(4) *The "Boundedness-Implies-Continuation" (BIC) Property*: For each $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times M_D \times M_U$, there exists a maximal existence time, i.e., there exists $t_{\max} \in (t_0, +\infty]$ such that $[t_0, t_{\max}) \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$ and for all $t \geq t_{\max}$, it holds that $(t, t_0, x_0, u, d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|T_{r_1}(t)x_1\|_{r_1} > M$, where $\phi(t, t_0, x_0, u, d) := (T_{r_1}(t)x_1, T_{r_2}(t)x_2)$.
(5) *The Classical Semigroup Property*: For each $t \in [t_0, t_{\max})$, it holds that $\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$ for all $\tau \in [t_0, t]$.

Again, the reason for allowing the output to take values in an abstract normed linear space is that the case (1.18), (1.19), (1.20) allows the study of:

- outputs with no delays, e.g., $Y(t) = h(t, x_1(t), x_2(t))$ with $\mathcal{Y} = \Re^k$,
- outputs with discrete or distributed delay, e.g., $Y(t) = h(x(t), x(t - r))$ or $Y(t) = \int_{t-r_1}^{t} h(t, \theta, x_1(\theta))\, d\theta$ with $\mathcal{Y} = \Re^k$,

- functional outputs with memory, e.g., $(Y(t))(\theta) = h(t, \theta, x_1(t+\theta))$ for $\theta \in [-r_1, 0]$ or the identity output

$$Y(t) = \begin{cases} x_1(t+\theta) & \theta \in [-r_1, 0] \\ x_2(t+\theta) & \theta \in [-r_2, 0] \end{cases}$$

  with

$$\mathcal{Y} = C^0\big([-r_1, 0]; \Re^{n_1}\big) \times \mathcal{L}^\infty\big([-r_2, 0]; \Re^{n_2}\big).$$

1.2.4 Control Systems Described by Functional Difference Equations (FDEs)

Consider the class of systems described by the following FDEs:

$$\begin{aligned} x(t) &= f\big(t, d(t), T_{r-\tau(t)}\big(t-\tau(t)\big)x, u(t)\big) \\ Y(t) &= H\big(t, T_r(t)x, u(t)\big) \\ &\quad x(t) \in \Re^n, d(t) \in D, u(t) \in U, t \geq 0 \end{aligned} \tag{1.42}$$

where $D \subseteq \Re^l$ is a nonempty set, $U \subseteq \Re^m$ is a nonempty set with $0 \in U$, $r > 0$, $f : \bigcup_{t\geq 0}\{t\} \times D \times B(t) \times U \to \Re^n$, and $B(t)$ denotes the set of all bounded mappings $x : [-r+\tau(t), 0] \to \Re^n$ under the following hypotheses:

(Q1) The function $\tau : \Re^+ \to (0, +\infty)$ is continuous with $\sup_{t\geq 0} \tau(t) \leq r$.

(Q2) There exist functions $a \in K_\infty$, $\beta \in K^+$ such that $|f(t, d, T_{r-\tau(t)}(-\tau(t))x, u)| \leq a(\beta(t)\|T_{r-\tau(t)}(-\tau(t))x\|_{r-\tau(t)}) + a(\beta(t)|u|)$ for all $(t, d, x, u) \in \Re^+ \times D \times \mathcal{X} \times U$, where $\mathcal{X}$ is the normed linear space of bounded functions $x : [-r, 0] \to \Re^n$ with norm $\|x\|_\mathcal{X} = \sup_{-r\leq\tau\leq 0} |x(\tau)|$.

(Q3) The mapping $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$, where $\mathcal{Y}$ is a normed linear space, is continuous with $H(t, 0, 0) = 0$ for all $t \geq 0$. Moreover, the image set $H(\Omega)$ is bounded for each bounded set $\Omega \subset \Re^+ \times \mathcal{X} \times U$.

Using the method of steps, exactly as in the proof of Theorem 1.1, we can obtain the following result.

Proposition 1.1 *Let M_U and M_D denote the sets of locally bounded functions $u : \Re^+ \to U$ and $d : \Re^+ \to D$, respectively, and consider system* (1.42) *under Hypotheses* (Q1), (Q2). *Then, for all $t_0 \geq 0$, $x_0 \in \mathcal{X}$, and $(u, d) \in M_U \times M_D$, there exists a unique locally bounded mapping $x : [t_0 - r, +\infty) \to \Re^n$ with $T_r(t_0)x = x_0$ such that* (1.42) *holds for all $t > t_0$.*

Systems of the form (1.42) are important and arise in many situations. We next provide an example, which shows that systems of the form (1.42) can be used for the study of time behavior of dynamic games.

Example 1.2.4 (The dynamic Cournot oligopoly) We consider the case of Cournot oligopoly where n players produce quantities of a single homogeneous product. The payoff function for each player is expressed by

$$\pi_i = pq_i - c_i q_i - \frac{1}{2} K_i q_i^2 \quad i = 1, \dots, n \tag{1.43}$$

where $K_i, c_i, i = 1, \dots, n$, are constants, $q_i \in [0, Q_i]$, $i = 1, \dots, n$, is the quantity of the commodity produced by the ith player, $Q_i > 0$ is the maximum level of production of the product for the ith player, and $p \geq 0$ is the price of the commodity.

Assuming a linear demand function

$$p = b\left(a - \sum_{i=1}^{n} q_i\right) \tag{1.44}$$

where $a, b > 0$ are constants satisfying $a \geq \sum_{i=1}^{n} Q_i$ and $b > -\frac{1}{2} \min_{i=1,\dots,n} K_i$, we obtain the best reply mapping for each one of the players:

$$q_i = f_i(q_{-i}) := \min\left\{Q_i, \max\left\{0, \frac{ab - c_i}{2b + K_i} - \frac{b}{2b + K_i} \sum_{j \neq i} q_j\right\}\right\}$$
$$i = 1, \dots, n \tag{1.45}$$

We define

$$S := [0, Q_1] \times [0, Q_2] \times \cdots \times [0, Q_n] \subset \Re^n \tag{1.46}$$

$$q = (q_1, \dots, q_n) \in S \tag{1.47}$$

$$F(q) := \begin{bmatrix} f_1(q_{-1}) \\ \vdots \\ f_n(q_{-n}) \end{bmatrix} = \begin{bmatrix} \min\{Q_1, \max\{0, \frac{ab - c_1}{2b + K_1} - \frac{b}{2b + K_1} \sum_{j \neq 1} q_j\}\} \\ \vdots \\ \min\{Q_n, \max\{0, \frac{ab - c_n}{2b + K_n} - \frac{b}{2b + K_n} \sum_{j \neq n} q_j\}\} \end{bmatrix} \tag{1.48}$$

and we notice that the set $S \subset \Re^n$ as defined by (1.46) is compact and convex and that the map $F : S \to S$ as defined by (1.48) is continuous. Consequently, Brouwer's fixed point theorem guarantees the existence of at least one Nash equilibrium $q^* \in S$ with $q_i^* = f_i(q_{-i}^*) = \min\{Q_i, \max\{0, \frac{ab - c_i}{2b + K_i} - \frac{b}{2b + K_i} \sum_{j \neq i} q_j^*\}\}$ for $i = 1, \dots, n$.

Next, we assume that the dynamics of the game are described in continuous time as follows:

- every player forms an expectation for the behavior of all other players at each time $t \geq 0$: the expectation of the ith player for the production level of the jth player at time $t \geq 0$ will be denoted by $q_{i,j}^{\exp}(t) \in [0, Q_j]$ $(j \neq i, i, j = 1, \dots, n)$,
- every player determines her production level as a convex combination of a past production level and the best reply response based on the expectations for the behavior of all other players at each time $t \geq 0$, i.e., for each $i = 1, 2, \dots, n$,

$$q_i(t) = \theta_i(t) \min\{Q_i, \max\{0, q_i(t - \tau_i(t))\}\} + (1 - \theta_i(t))$$
$$\times \min\left\{Q_i, \max\left\{0, \frac{ab - c_i}{2b + K_i} - \frac{b}{2b + K_i} \sum_{j \neq i} q_{i,j}^{\exp}(t)\right\}\right\} \tag{1.49}$$

where $\theta_i : \Re^+ \to [0, \Theta]$ and $\tau_i : \Re^+ \to [T, r]$ are in general unknown functions, and $0 \le \Theta < 1$ and $0 < T \le r$ are constants (in general unknown).

The reader should notice that (1.49) is a model that evolves in continuous time, i.e., $t \in \Re^+$. If the expectation rules $q_{i,j}^{\exp}(t)$ ($j \ne i$, $i, j = 1, \ldots, n$) and the functions $\theta_i : \Re^+ \to [0, \Theta]$, $\tau_i : \Re^+ \to [T, r]$ ($i = 1, \ldots, n$) were known, we would have an accurate description of the dynamics of the Cournot oligopoly game. However, we will not assume exact knowledge of the expectation rules but a specific consistency condition. More specifically, we will assume thatYYYYYYYYY:

(H) All expectation rules $q_{i,j}^{\exp}(t)$ ($j \ne i$, $i, j = 1, \ldots, n$) are Consistent Backward-looking expectations with respect to the Nash equilibrium point $q^* \in S$, i.e., there exist constants $0 < T \le r$ such that, the following inequalities hold for all $j \ne i$, $i, j = 1, \ldots, n$ and for all $t \ge 0$:

$$\left|q_{i,j}^{\exp}(t) - q_j^*\right| \le \sup_{t-r \le \tau \le t-T} \left|q_j(\tau) - q_j^*\right| = \left\|q_j - q_j^*\right\|_{[t-r,t-T]} \quad (1.50)$$

In other words, the consistency condition (1.50) recognizes that it is not logical for the ith player to expect that the production level of the jth manufacturer will deviate from its equilibrium level more than the highest deviation she has experienced in the past. Next, we present some examples of Consistent Backward-looking expectation rules:

(1) $q_{i,j}^{\exp}(t) = a_{i,j}(t) \sum_{l=1}^{m} w_{i,j,l}(t) q_j(t - \tau_{i,j,l}(t)) + (1 - a_{i,j}(t)) q_j^*$, where $a_{i,j}(t) \in [0, 1]$, $r \ge \tau_{i,j,l}(t) \ge T > 0$, $w_{i,j,l}(t) \ge 0$ with $1 = \sum_{l=1}^{m} w_{i,j,l}(t)$ for all $t \ge 0$ and $l = 1, \ldots, m$. In discrete-time models the case $\tau_{i,j,l}(t) = t + l - [t]$, $a_{i,j}(t) \equiv 1$, $w_{i,j,l}(t) \equiv w_{i,j,l} \ge 0$ with $1 = \sum_{l=1}^{m} w_{i,j,l}$ is the usual backward-looking expectation, which gives $q_{i,j}^{\exp}(t) = \sum_{l=1}^{m} w_{i,j,l} q_j(k - l)$ for $t \in [k, k+1)$.

(2) $q_{i,j}^{\exp}(t) = a_{i,j}(t) \int_{-r}^{-T} h_{i,j}(s) q_j(t+s)\, ds + (1 - a_{i,j}(t)) q_j^*$, where $0 < T < r$, $a_{i,j}(t) \in [0, 1]$ for all $t \ge 0$, $h_{i,j} : [-r, -T] \to \Re$ is a Lebesgue-integrable function with $h_{i,j}(s) \ge 0$ for almost all $s \in [-r, -T]$ and $1 = \int_{-r}^{-T} h_{i,j}(s)\, ds$. Of course, in this case it is additionally required that $q_j(t)$ must be Lebesgue integrable and essentially bounded.

We notice the following important fact for consistent backward-looking expectations:

Fact $q_{i,j}^{\exp}(t)$ *(where* $j \ne i$, $i, j = 1, \ldots, n$*) is a Consistent Backward-looking expectation with respect to the Nash equilibrium point* $q^* \in S$ *if and only if there exist constants* $0 < T \le r$ *and a function* $d_{i,j} : \Re^+ \to [-1, 1]$ *such that*

$$q_{i,j}^{\exp}(t) = \min\left\{Q_j, \max\left\{0, q_j^* + d_{i,j}(t) \left\|q_j - q_j^*\right\|_{[t-r,t-T]}\right\}\right\}$$
$$\textit{for all } t \ge 0 \quad (1.51)$$

Proof of Fact Assume first that $q_{i,j}^{\exp}(t)$ (where $j \ne i$, $i, j = 1, \ldots, n$) is a Consistent Backward-looking expectation with respect to the Nash equilibrium point $q^* \in S$, i.e., that (1.50) holds. We distinguish the following cases.

Case 1: If $q_{i,j}^{\exp}(t) = 0$, then (1.50) implies that $q_j^* \le \|q_j - q_j^*\|_{[t-r,t-T]}$. In this case we define $d_{i,j}(t) = -1$, and equality (1.51) holds.
Case 2: If $q_{i,j}^{\exp}(t) = Q_j$, then (1.50) implies that $Q_j - q_j^* \le \|q_j - q_j^*\|_{[t-r,t-T]}$. In this case we define $d_{i,j}(t) = 1$, and equality (1.51) holds.
Case 3: If $q_{i,j}^{\exp}(t) \in (0, Q_j)$ and $\|q_j - q_j^*\|_{[t-r,t-T]} > 0$, then equality (1.51) holds with $d_{i,j}(t) = \operatorname{sgn}(q_{i,j}^{\exp}(t) - q_j^*)\frac{|q_{i,j}^{\exp}(t) - q_j^*|}{\|q_j - q_j^*\|_{[t-r,t-T]}}$. Inequality (1.50) implies that $|d_{i,j}(t)| \le 1$.
Case 4: If $q_{i,j}^{\exp}(t) \in (0, Q_j)$ and $\|q_j - q_j^*\|_{[t-r,t-T]} = 0$, then inequality (1.50) implies that $q_{i,j}^{\exp}(t) = q_j^*$. In this case equality (1.51) holds for arbitrary $d_{i,j}(t) \in [-1, 1]$.

On the other hand, if (1.51) holds, then $q_{i,j}^{\exp}(t) \in [0, Q_j]$ for all $t \ge 0$. Moreover, the reader can verify that inequality (1.50) holds. The proof is complete. □

The previous fact shows that Hypothesis (H) is equivalent to the existence of constants $0 < T \le r$ and functions $d_{i,j} : \Re^+ \to [-1, 1]$ ($j \ne i, i, j = 1, \ldots, n$) such that the following equalities hold for all $i = 1, \ldots, n$:

$$\begin{aligned} q_i(t) = {} & \theta_i(t) \min\bigl\{Q_i, \max\bigl\{0, q_i\bigl(t - \tau_i(t)\bigr)\bigr\}\bigr\} \\ & + \bigl(1 - \theta_i(t)\bigr) \min\Biggl\{Q_i, \max\Biggl\{0, \frac{ab - c_i}{2b + K_i} \\ & - \frac{b}{2b + K_i} \sum_{j \ne i} \min\bigl\{Q_j, \max\bigl\{0, q_j^* + d_{i,j}(t)\bigl\|q_j - q_j^*\bigr\|_{[t-r,t-T]}\bigr\}\bigr\}\Biggr\}\Biggr\} \end{aligned} \tag{1.52}$$

In general, the functions $\theta_i : \Re^+ \to [0, \Theta]$, $\tau_i : \Re^+ \to [T, r]$, $d_{i,j} : \Re^+ \to [-1, 1]$ ($j \ne i$, $i, j = 1, \ldots, n$) and the constants $0 \le \Theta < 1$, $0 < T \le r$ are unknown. Therefore, the dynamical system (1.52) is an uncertain dynamical system described by Functional Difference Equations (FDEs). In order to recast (1.52) to the form (1.42) under Hypotheses (Q1), (Q2), we define the dimensionless deviation variables $x_i(t) = \frac{q_i(t) - q_i^*}{Q_i}$ ($i = 1, \ldots, n$), and we obtain from (1.45) and (1.52) for $i = 1, \ldots, n$:

$$\begin{aligned} x_i(t) = {} & \theta_i(t) \min\bigl\{1 - L_i, \max\bigl\{-L_i, x_i\bigl(t - \tau_i(t)\bigr)\bigr\}\bigr\} \\ & + \bigl(1 - \theta_i(t)\bigr) \min\Biggl\{1 - L_i, \max\Biggl\{-L_i, M_i - L_i \\ & - R_i \sum_{j \ne i} g_{i,j} \min\bigl\{1, \max\bigl\{0, L_j + d_{i,j}(t)\|x_j\|_{[t-r,t-T]}\bigr\}\bigr\}\Biggr\}\Biggr\} \end{aligned} \tag{1.53}$$

where $L_i = \frac{q_i^*}{Q_i} \in [0, 1]$, $M_i = \frac{ab - c_i}{(2b + K_i)Q_i}$, $R_i = \frac{b}{2b + K_i} > 0$, $g_{i,j} = \frac{Q_j}{Q_i} > 0$ for $j \ne i$, $i = 1, \ldots, n$, are constants which satisfy $L_i = \min\{1, \max\{0, M_i - R_i \sum_{j \ne i} g_{i,j} L_j\}\}$ for all $i = 1, \ldots, n$. The reader should notice that Hypotheses (Q1), (Q2) hold for system (1.53) with $\tau(t) \equiv T$.

Finally, notice that

- All discrete-time models of the form ($i = 1, \dots, n$)

$$q_i(k+1) = \theta_i(k) q_i(k) + \big(1 - \theta_i(k)\big) \times \min\left\{Q_i, \max\left\{0, \frac{ab - c_i}{2b + K_i} - \frac{b}{2b + K_i} \sum_{j \neq i} q_{i,j}^{\exp}(k+1)\right\}\right\} \quad (1.54)$$

with

$$q_{i,j}^{\exp}(k+1) = a_{i,j}(k) \sum_{l=0}^{m} w_{i,j,l}(k) q_j(k-l) + \big(1 - a_{i,j}(k)\big) q_j^* \quad (1.55)$$

where k, m are nonnegative integers, $a_{i,j}(k) \in [0,1]$ ($i, j = 1, \dots, n$), $\theta_i(k) \in [0, \Theta]$ ($i = 1, \dots, n$) with $\Theta \in [0, 1)$, $w_{i,j,l}(k) \geq 0$ with $1 = \sum_{l=0}^{m} w_{i,j,l}(k)$ for all $k \geq 0$ and $l = 0, \dots, m$ ($i, j = 1, \dots, n$), are immersed in the uncertain model (1.52) and its equivalent expression (1.53) in the sense that for every model of the form (1.54), (1.55) one can give functions $\theta_i : \Re^+ \to [0, \Theta]$, $\tau_i : \Re^+ \to [T, r]$, $d_{i,j} : \Re^+ \to [-1, 1]$ ($j \neq i$, $i, j = 1, \dots, n$) such that the solution of (1.52) coincides with the solution obtained by the discrete-time model (1.54), (1.55).
- All continuous-time models of the form

$$\dot{q}_i(t) = \mu_i \min\left\{Q_i, \max\left\{0, \frac{ab - c_i}{2b + K_i} - \frac{b}{2b + K_i} \sum_{j \neq i} q_{i,j}^{\exp}(t)\right\}\right\} - \mu_i q_i(t) \quad (1.56)$$

where, for each $i = 1, \dots, n$, $\mu_i > 0$ are constants, and $q_{i,j}^{\exp}(t)$ ($j \neq i$, $j = 1, \dots, n$) are Consistent Backward-looking expectations with respect to the Nash equilibrium point $q^* \in S$, are immersed in the uncertain model (1.52). Indeed, for all $i = 1, 2, \dots, n$ and $t \geq r > 0$, the solution of (1.56) implies the following integral equations:

$$q_i(t) = \exp(-\mu_i r) q_i(t-r) + \int_{t-r}^{t} \exp\big(-\mu_i(t-\tau)\big) \times \left\{Q_i, \max\left\{0, \frac{ab - c_i}{2b + K_i} - \frac{b}{2b + K_i} \sum_{j \neq i} q_{i,j}^{\exp}(\tau)\right\}\right\} d\tau.$$

From the above expression under the assumption that the mappings $t \to q_{i,j}^{\exp}(t)$ ($j \neq i$, $i, j = 1, \dots, n$) are continuous, we can conclude that for all $t \geq r$ and $i = 1, \dots, n$, there exist $g_i(t) \in [t - r, t]$, $i = 1, \dots, n$, such that

$$q_i(t) = \exp(-\mu_i r) q_i(t-r) + \big(1 - \exp(-\mu_i r)\big) \times \min\left\{Q_i, \max\left\{0, \frac{ab - c_i}{2b + K_i} - \frac{b}{2b + K_i} \sum_{j \neq i} q_{i,j}^{\exp}\big(g_i(t)\big)\right\}\right\}.$$

The reader may verify that for Consistent Backward-looking expectations with respect to the Nash equilibrium point $q^* \in S$, the above model can be described by the uncertain model (1.52) with $\theta_i(t) \equiv \exp(-\mu_i r)$, $\tau_i(t) \equiv r$, $i = 1, \dots, n$, and $\Theta := \max_{i=1,\dots,n} \exp(-\mu_i r) < 1$.

We define the output mapping to be the identity mapping $H(t, x, u) := x$ for all $x \in \mathcal{X}$. This example will be studied further in Chap. 5.

Let $\mathcal{X}$ be the normed linear space of bounded functions $x : [-r, 0] \to \Re^n$ with norm $\|x\|_{\mathcal{X}} = \sup_{-r \le \tau \le 0} |x(\tau)|$, $\phi : A_\phi \to \mathcal{X}$ be the mapping $\phi(t, t_0, x_0, u, d) := T_r(t)x$, which for each $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$ and $t \in [t_0, +\infty)$ gives the "history" of the unique solution of (1.42) with $T_r(t_0)x = x_0$. The mapping $\phi : A_\phi \to \mathcal{X}$ is defined on the set $A_\phi = \bigcup_{(t_0,x_0,u,d)\in\Re^+\times\mathcal{X}\times M_U\times M_D} [t_0, +\infty) \times \{(t_0, x_0, u, d)\}$. It should be clear from all the above that the mapping $\phi : A_\phi \to \mathcal{X}$ is well defined and satisfies the following properties:

(1) *Existence*: For each $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$, there exists $t > t_0$ such that $[t_0, t] \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$.
(2) *Identity Property*: For each $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.
(3) *Causality*: For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ for all $\tau \in [t_0, t]$, it holds that $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ and $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.
(4) *The "Boundedness-Implies-Continuation" (BIC) Property*: For each $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$, there exists a maximal existence time, i.e., there exists $t_{\max} \in (t_0, +\infty]$, such that $[t_0, t_{\max}) \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$ and for all $t \ge t_{\max}$, it holds that $(t, t_0, x_0, u, d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|T_r(t)x\|_r = \|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$.
(5) *The Classical Semigroup Property*: For each $t \in [t_0, t_{\max})$, it holds that $\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$ for all $\tau \in [t_0, t]$.

The reader should notice that Proposition 1.1 guarantees that the "Boundedness-Implies-Continuation" (BIC) Property and the Classical Semigroup Property hold with $t_{\max} = +\infty$.

1.2.5 Control Systems with Variable Sampling Partition

Given a pair of sets $D \subseteq \Re^l$ and $U \subseteq \Re^m$ with $0 \in U$ a closed set, a positive function $h : \Re^+ \times \Re^n \times U \times D \to (0, r]$ which is bounded by certain constant $r > 0$, and a triplet of vector fields $f : \Re^+ \times \Re^+ \times \Re^n \times \Re^n \times U \times U \times D \times D \to \Re^n$, $H : \Re^+ \times \Re^n \times U \to \Re^p$, $R : \Re^+ \times \Re^n \times \Re^n \times U \times U \times D \times D \to \Re^n$, we consider the hybrid system that produces, for each $(t_0, x_0) \in \Re^+ \times \Re^n$ and for each pair of measurable and locally bounded inputs $u : \Re^+ \to U$ and $d : \Re^+ \to D$, the piecewise absolutely continuous function $t \to x(t) \in \Re^n$, according to the following algorithm:

Step i:

(1) Given τ_i and $x(\tau_i)$, calculate τ_{i+1} using the recursive equation

$$\tau_{i+1} = \tau_i + h\big(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i)\big);$$

(2) Compute the state trajectory $x(t), t \in [\tau_i, \tau_{i+1})$ as the solution of the differential equation

$$\dot{x}(t) = f\big(t, \tau_i, x(t), x(\tau_i), u(t), u(\tau_i), d(t), d(\tau_i)\big);$$

(3) Calculate $x(\tau_{i+1})$ using the equation

$$x(\tau_{i+1}) = R\Big(\tau_i, \lim_{t \to \tau_{i+1}^-} x(t), x(\tau_i), u(\tau_{i+1}), u(\tau_i), d(\tau_{i+1}), d(\tau_i)\Big);$$

(4) Compute the output trajectory $Y(t), t \in [\tau_i, \tau_{i+1}]$ using the equation

$$Y(t) = H\big(t, x(t), u(t)\big).$$

For $i = 0$, we take $\tau_0 = t_0$ and $x(\tau_0) = x_0$ (initial condition). Schematically, we write

$$\begin{aligned}
&\dot{x}(t) = f\big(t, \tau_i, x(t), x(\tau_i), u(t), u(\tau_i), d(t), d(\tau_i)\big) \quad t \in [\tau_i, \tau_{i+1}) \\
&\quad \tau_0 = t_0, \tau_{i+1} = \tau_i + h\big(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i)\big),\ i = 0, 1, \dots \\
&x(\tau_{i+1}) = R\Big(\tau_i, \lim_{t \to \tau_{i+1}^-} x(t), x(\tau_i), u(\tau_{i+1}), u(\tau_i), d(\tau_{i+1}), d(\tau_i)\Big) \\
&Y(t) = H\big(t, x(t), u(t)\big)
\end{aligned} \tag{1.57}$$

with initial condition $x(t_0) = x_0 \in \Re^n$.

We consider hybrid systems of the form (1.57) under the following hypotheses:

(A1) $f(t, \tau, x, x_0, u, u_0, d, d_0)$ is measurable with respect to $t \geq 0$ and continuous with respect to $(x, d, u) \in \Re^n \times D \times U$. Moreover, there exists a symmetric positive definite matrix $P \in \Re^{n\times n}$ such that for every bounded $S \subset \Re^+ \times \Re^+ \times \Re^n \times \Re^n \times U \times U$, there exists a constant $L \geq 0$ such that

$$\begin{aligned}
&(x-y)' P\big(f(t, \tau, x, x_0, u, u_0, d, d_0) - f(t, \tau, y, x_0, u, u_0, d, d_0)\big) \\
&\quad \leq L|x-y|^2 \\
&\quad \text{for all } (t, \tau, x, x_0, u, u_0, d, d_0) \in S \times D \times D \text{ and } y \in \Re^n.
\end{aligned} \tag{1.58}$$

(A2) There exist functions $\gamma \in K^+$, $a \in K_\infty$ such that

$$\begin{aligned}
&\big|f(t, \tau, x, x_0, u, u_0, d, d_0)\big| \leq \gamma(t) a\big(|x| + |x_0| + |u| + |u_0|\big) \\
&\quad \text{for all } t \geq \tau,\ (\tau, u, u_0, d, d_0, x, x_0) \in \Re^+ \times U \times U \times D \times D \times \Re^n \times \Re^n
\end{aligned} \tag{1.59}$$

$$\begin{aligned}
&\big|R(t, x, x_0, u, u_0, d, d_0)\big| \leq \gamma(t) a\big(|x| + |x_0| + |u| + |u_0|\big) \\
&\quad \text{for all } (t, u, u_0, d, d_0, x, x_0) \in \Re^+ \times U \times U \times D \times D \times \Re^n \times \Re^n.
\end{aligned} \tag{1.60}$$

(A3) $H : \Re^+ \times \Re^n \times U \to \Re^p$ is a continuous map with $H(t, 0, 0) = 0$ for all $t \geq 0$,

(A4) There exists a partition $b = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ (i.e., an increasing sequence of time instants with $T_0 = 0$ and $T_i \to +\infty$) such that for every bounded $S \subset \Re^+ \times \Re^n \times U \times D$, there exists $s > 0$ with $h(t, x, u, d) \geq \min\{q_b(t) - t, s\}$ for all $(t, x, u, d) \in S$, where $q_b(t) := \min\{T \in b; t < T\}$.

Occasionally, we will use the following stronger hypothesis:

(A5) There exist a positive, continuous, and bounded function $h_l : \Re^+ \times \Re^n \times U \to (0, r]$ and a partition $b = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ such that

$$h(t, x, u, d) \geq \min\{q_b(t) - t, h_l(t, x, u)\}$$
$$\forall (t, x, u, d) \in \Re^+ \times \Re^n \times U \times D \tag{1.61}$$

where $q_b(t) := \min\{T \in b; t < T\}$.

Systems of the form (1.57) under Hypotheses (A1–4) arise frequently in certain applications in mathematical control theory and numerical analysis. We mention here two important applications:

(1) *Application of "sampled-data" feedback*: For example, consider the finite-dimensional continuous-time control system $\dot{x}(t) = f(t, x(t), v(t))$, where $x(t) \in \Re^n$, $v(t) \in \Re^m$, and the vector field $f : \Re^+ \times \Re^n \times \Re^m \to \Re^n$ is continuous and locally Lipschitz in $x \in \Re^n$. Suppose that there exists a family of measurable and locally bounded controls $t \to v(t, t_0, x_0)$ parameterized by $(t_0, x_0) \in \Re^+ \times \Re^n$ with the following property: for every $(t_0, x_0) \in \Re^+ \times \Re^n$, the unique solution of $\dot{x}(t) = f(t, x(t), v(t, t_0, x_0))$ with initial condition $x(t_0) = x_0$ exists for all $t \geq t_0$ and satisfies $\lim_{t \to +\infty} x(t) = 0 \in \Re^n$. Then, by application of the measurable and locally bounded controls $t \to v(t, t_0, x_0)$ on the interval $[t_0, t_0 + h(t_0, x_0))$, where $h : \Re^+ \times \Re^n \to (0, r]$ is a positive function bounded by certain constant $r > 0$, we obtain a control system that produces for each $(t_0, x_0) \in \Re^+ \times \Re^n$ and for each pair of measurable and locally bounded inputs $u : \Re^+ \to \Re^m$, $e : \Re^+ \to \Re^n$ the absolutely continuous function $[t_0, +\infty) \ni t \to x(t) \in \Re^n$ that satisfies a.e. the following differential equation:

$$\dot{x}(t) = f\big(t, x(t), v\big(t, \tau_i, x(\tau_i) + e(\tau_i)\big) + u(t)\big) \quad t \in [\tau_i, \tau_{i+1})$$
$$\tau_0 = t_0, \tau_{i+1} = \tau_i + h\big(\tau_i, x(\tau_i) + e(\tau_i)\big), i = 0, 1, \ldots \tag{1.62}$$
$$Y(t) = H\big(t, x(t)\big)$$

with initial condition $x(t_0) = x_0$. In this case, the measurable and locally bounded inputs $u : \Re^+ \to \Re^m$ and $e : \Re^+ \to \Re^n$ represent the control actuator error and the measurement error, respectively.

(2) *Numerical solutions of ordinary differential equations*: For example, consider the finite-dimensional continuous-time dynamical system $\dot{x}(t) = f(t, x(t))$, where $x(t) \in \Re^n$. Let $b = \{T_i\}_{i=0}^{\infty}$ be a partition of $\Re^+$, i.e., an increasing sequence of time instants with $T_0 = 0$ and $T_i \to +\infty$, and define $q_b(t) := \min\{T \in b; t < T\}$. Consider the explicit Euler discretization scheme with state-dependent (adaptive) time step $\Re^+ \times \Re^n \ni (t, x) \to h(t, x) > 0$ that produces for each $(t_0, x_0) \in \Re^+ \times \Re^n$ the absolutely continuous function $[t_0, +\infty) \ni t \to x(t) \in \Re^n$ (Euler arc) satisfying the evolution equation with initial condition $x(t_0) = x_0$,

$$\dot{x}(t) = f\big(\tau_i, x(\tau_i)\big) \quad t \in [\tau_i, \tau_{i+1})$$
$$\tau_0 = t_0, \tau_{i+1} = \min\big\{q_b(\tau_i), \tau_i + h\big(\tau_i, x(\tau_i)\big)\big\}, i = 0, 1, \ldots \tag{1.63}$$

Clearly system (1.63) is a system of the form (1.57).

Consider system (1.57) under Hypotheses (A1–4). Clearly, under Hypothesis (A1), for each pair of measurable and locally bounded inputs $u : \Re^+ \to U$ and $d : \Re^+ \to D$ and for each $(t_0, x_0) \in \Re^+ \times \Re^n$, the piecewise absolutely continuous function $t \to x(t) \in \Re^n$ that satisfies (1.57) with initial condition $x(t_0) = x_0$ is unique.

Standard arguments from the theory of existence of solutions of ordinary differential equations (see, for instance, [14]) show that if $x(t)$ is defined for some $t > t_0$, then there exists $\varepsilon > 0$ such that the solution $x(\tau)$ is also defined for $\tau \in [t, t + \varepsilon)$. Thus, for each $(t_0, x_0, u, d) \in \Re^+ \times \Re^n \times M_U \times M_D$, there exists a maximal existence time, i.e., there exists $t_{\max} \in (t_0, +\infty]$ for which the solution $x(t)$ of (1.57) is defined on $[t_0, t_{\max})$ and cannot be continued further.

We next show that if $t_{\max} < +\infty$, then the solution $x(t)$ of (1.57) cannot be bounded on $[t_0, t_{\max})$. The proof of this implication depends on the following claim.

Claim *Let $s > 0$ be a constant. Every infinite sequence $\{\tau_i\}_{i=0}^{\infty}$ with $\tau_{i+1} \geq \min\{q_b(\tau_i), \tau_i + s\}$ for each i and $\tau_0 \geq 0$ satisfies $\tau_i \to +\infty$, where $q_b(t)$ equals $\min\{T \in b; t < T\}$, and $b = \{T_i\}_{i=0}^{\infty}$ is the partition involved in Hypothesis* (A4).

Proof of Claim Since $q_b(\tau_i) > \tau_i$, it follows that $\{\tau_i, i = 0, 1, \ldots\}$ is an increasing sequence. Consequently, we have $\tau_i \to \sup\{\tau_i, i = 0, 1, \ldots\}$. Suppose that $\sup\{\tau_i, i = 0, 1, \ldots\} < +\infty$. In this case, there exist $N, \tilde{N} > 0$ such that $T_N, T_{N-1} \in \pi$ and $T_{N-1} < \tau_i < T_N, i = \tilde{N}, \tilde{N} + 1, \ldots$. Thus, we obtain $q_b(\tau_i) = T_N, i = \tilde{N}, \tilde{N} + 1, \ldots$, and consequently $\tau_{i+1} \geq \min\{T_N, \tau_i + s\}, i = \tilde{N}, \tilde{N} + 1, \ldots$. Clearly, since $\tau_{i+1} < T_N$, $i = \tilde{N}, \tilde{N} + 1, \ldots$, it follows that we must have $\tau_i + s = \min\{T_N, \tau_i + s\}, i = \tilde{N}, \tilde{N} + 1, \ldots$, and this implies $\tau_{i+1} \geq \tau_i + s, i = \tilde{N}, \tilde{N} + 1, \ldots$. Thus, we obtain $\tau_i \geq T_{N-1} + (i - \tilde{N})s$, $i = \tilde{N}, \tilde{N} + 1, \ldots$, which shows that $\sup\{\tau_i, i = 0, 1, \ldots\} = +\infty$, a contradiction. The proof of the claim is complete. □

We are now ready to show the required implication. Suppose that $t_{\max} < +\infty$. Let $\{t_0 = \tau_0, \tau_1, \ldots\}$ be the sequence of time instants satisfying

$$\tau_{i+1} = \tau_i + h\big(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i)\big) \quad i = 0, 1, 2, \ldots$$

and $R := \max\{\sup\{|u(t)|; t \in [t_0, t_{\max}]\}, \sup\{|d(t)|; t \in [t_0, t_{\max}]\}\}$. We consider the following cases:

(1) The cardinal number of the set $\{\tau_0, \tau_1, \ldots\}$ is finite. Standard arguments from the theory of existence of solutions of ordinary differential equations show that, in this case, we have $\limsup_{t \to t_{\max}^-} |x(t)| = +\infty$.
(2) The cardinal number of the set $\{\tau_0, \tau_1, \ldots\}$ is infinite. In this case, we have $\sup\{\tau_0, \tau_1, \ldots\} \leq t_{\max} + r$. However, if $x(t)$ is bounded (say $x(t) \in B[0, \rho]$ for some $\rho > 0$), then we may define the bounded set $S = [t_0, t_{\max} + r] \times B[0, \rho] \times B_U[0, R] \times B_D[0, R]$. By virtue of Hypothesis (A4), there exist a partition $b = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ and $s > 0$ with $h(t, x, u, d) \geq \min\{q_b(t) - t, s\}$ for all $(t, x, u, d) \in S$. Consequently, it holds that $\tau_{i+1} \geq \min\{q_b(\tau_i), \tau_i + s\}$,

$i = 0, 1, \ldots,$ with $\tau_0 \geq 0$. It follows from the above claim that $\tau_i \to +\infty$, which contradicts the fact that $\sup\{\tau_0, \tau_1, \ldots\} \leq t_{\max} + r < +\infty$. Thus, we conclude that the solution $x(t)$ of (1.57) is not bounded.

In any case the hypothesis $t_{\max} < +\infty$ leads to the conclusion that the solution $x(t)$ of (1.57) is not bounded.

Let M_U and M_D be the sets of measurable and locally bounded inputs $u : \Re^+ \to U$ and $d : \Re^+ \to D$, respectively. Let $\phi : A_\phi \to \Re^n$ be the mapping $\phi(t, t_0, x_0, u, d) := x(t)$ which for each $(t_0, x_0, d, u) \in \Re^+ \times \Re^n \times M_D \times M_U$ and $t \in [t_0, t_{\max})$ gives the value $x(t) \in \Re^n$ of the unique solution of (1.57) with $x(t_0) = x_0$ and $\tau_0 = t_0 \geq 0$. The mapping $\phi : A_\phi \to \Re^n$ is defined on the set

$$A_\phi = \bigcup_{(t_0, x_0, u, d) \in \Re^+ \times \Re^n \times M_U \times M_D} \big[t_0, t_{\max}\big) \times \big\{(t_0, x_0, u, d)\big\}$$

It should be clear from all the above that the mapping $\phi : A_\phi \to \Re^n$ is well defined and satisfies the following properties:

(1) *Existence*: For each $(t_0, x_0, d, u) \in \Re^+ \times \Re^n \times M_D \times M_U$, there exists $t > t_0$ such that $[t_0, t] \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$.
(2) *Identity Property*: For each $(t_0, x_0, d, u) \in \Re^+ \times \Re^n \times M_D \times M_U$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.
(3) *Causality*: For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ for all $\tau \in [t_0, t]$, it holds that $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ and $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.
(4) *The "Boundedness-Implies-Continuation" (BIC) Property*: For each $(t_0, x_0, u, d) \in \Re^+ \times \Re^n \times M_U \times M_D$, there exists a maximal existence time, i.e., there exists $t_{\max} \in (t_0, +\infty]$ such that $[t_0, t_{\max}) \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$ and for all $t \geq t_{\max}$, it holds that $(t, t_0, x_0, u, d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $|\phi(t, t_0, x_0, u, d)| > M$.

An important feature of systems of the form (1.57) under Hypotheses (A1–4) is that they do not satisfy the classical "semigroup property": for example, the solution $x(t)$ of (1.57) with initial condition $x(t_0) = x_0$ does not coincide (in general) for $t \geq t_1 > t_0$ with the solution $\tilde{x}(t)$ of (1.57) with initial condition $\tilde{x}(t_1) = x(t_1)$ corresponding to the same measurable and locally bounded inputs $u : \Re^+ \to U$ and $d : \Re^+ \to D$. The following example illustrates this point.

Example 1.2.5 Consider the following system:

$$\begin{aligned} \dot{x}(t) &= -x(\tau_i) \quad t \in [\tau_i, \tau_{i+1}) \\ \tau_{i+1} &= \tau_i + 1 \\ x(t) &\in \Re \end{aligned} \tag{1.64}$$

with initial condition $x(t_0) = x_0 \in \Re$ and $\tau_0 = t_0 \geq 0$. In this case we can determine analytically the transition map for all $t \geq t_0$ (u, d in this example are irrelevant):

$$\phi(t, t_0, x_0) = \begin{cases} (1 - t + t_0)x_0 & \text{for } t \in [t_0, t_0 + 1) \\ 0 & \text{for } t \geq t_0 + 1 \end{cases}$$

It is clear that the state space is $\Re$ and that the classical semigroup property does not hold for this system, i.e., the property

$$\text{“for each } t \in [t_0, t_{\max}),\ \text{it holds that}$$
$$\phi(t, \tau, \phi(\tau, t_0, x_0)) = \phi(t, t_0, x_0) \text{ for all } \tau \in [t_0, t]\text{”}$$
$$\text{(Classical Semigroup Property)}$$

does not hold. However, notice that the equality $\phi(t, \tau, \phi(\tau, t_0, x_0)) = \phi(t, t_0, x_0)$ holds if $\tau \in \pi(t_0, x_0) = \{t_0, t_0 + 1, t_0 + 2, \ldots\}$.

Let $\pi(t_0, x_0, u, d) \subseteq [t_0, +\infty)$ be the set of the sampling instants, i.e., $\pi(t_0, x_0, u, d) = \{\tau_i\}_{i=0}^{\infty}$, where the sampling instants are generated by the recursive relation $\tau_{i+1} = \tau_i + h(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i))$, $i = 0, 1, \ldots$, with $\tau_0 = t_0$. Then the following property holds:

(5) *Weak Semigroup Property*: There exists a constant $r > 0$ such that for each $t \geq t_0$ with $(t, t_0, x_0, u, d) \in A_\phi$:
   (a) $(\tau, t_0, x_0, u, d) \in A_\phi$ for all $\tau \in [t_0, t]$,
   (b) $\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$ for all $\tau \in [t_0, t] \cap \pi(t_0, x_0, u, d)$,
   (c) if $(t + r, t_0, x_0, u, d) \in A_\phi$, then it holds that $\pi(t_0, x_0, u, d) \cap [t, t + r] \neq \emptyset$,
   (d) for all $\tau \in \pi(t_0, x_0, u, d)$ with $(\tau, t_0, x_0, u, d) \in A_\phi$, we have $\pi(\tau, \phi(\tau, t_0, x_0, u, d), u, d) = \pi(t_0, x_0, u, d) \cap [\tau, +\infty)$.

1.3 Deterministic Control Systems

Based on the examples of the previous section, we want to introduce the notion of a deterministic control system. Notice that a more relaxed notion of a deterministic control system will be adopted here than what has been widely used in past literature. This extension allows us to consider the important class of control systems with variable sampling partition.

Definition 1.1 A deterministic control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with outputs consists of

(i) a set U (control set) which is a subset of a normed linear space $\mathcal{U}$ with $0 \in U$ and a set $M_U \subseteq \mathcal{M}(U)$ (allowable control inputs) which contains at least the identically zero input $u_0 \in M_U$, (i.e., the input that satisfies $u_0(t) = 0 \in U$ for all $t \geq 0$);
(ii) a set D (disturbance set) and a set $M_D \subseteq \mathcal{M}(D)$, which is called the “set of allowable disturbances”;
(iii) a pair of normed linear spaces $\mathcal{X}$ and $\mathcal{Y}$ called the “state space” and the “output space,” respectively;
(iv) a map $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ that maps bounded sets of $\Re^+ \times \mathcal{X} \times U$ into bounded sets of $\mathcal{Y}$, called the “output map”;

(v) a set-valued map $\Re^+ \times \mathcal{X} \times M_U \times M_D \ni (t_0, x_0, u, d) \to \pi(t_0, x_0, u, d) \subseteq [t_0, +\infty)$, with $t_0 \in \pi(t_0, x_0, u, d)$ for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, called the set of "sampling times"; and

(vi) the map $\phi : A_\phi \to \mathcal{X}$, where $A_\phi \subseteq \Re^+ \times \Re^+ \times \mathcal{X} \times M_U \times M_D$, called the "transition map," which has the following properties:

(1) *Existence*: For each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, there exists $t > t_0$ such that $[t_0, t] \times (t_0, x_0, u, d) \subseteq A_\phi$.

(2) *Identity Property*: For each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.

(3) *Causality*: For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ for all $\tau \in [t_0, t]$, it holds that $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ with $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.

(4) *Weak Semigroup Property*: There exists a constant $r > 0$ such that for each $t \geq t_0$ with $(t, t_0, x_0, u, d) \in A_\phi$:

(a) $(\tau, t_0, x_0, u, d) \in A_\phi$ for all $\tau \in [t_0, t]$,

(b) $\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$ for all $\tau \in [t_0, t] \cap \pi(t_0, x_0, u, d)$,

(c) if $(t + r, t_0, x_0, u, d) \in A_\phi$, then it holds that $\pi(t_0, x_0, u, d) \cap [t, t + r] \neq \emptyset$,

(d) for all $\tau \in \pi(t_0, x_0, u, d)$ with $(\tau, t_0, x_0, u, d) \in A_\phi$, we have $\pi(\tau, \phi(\tau, t_0, x_0, u, d), u, d) = \pi(t_0, x_0, u, d) \cap [\tau, +\infty)$.

A deterministic control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with no external inputs, i.e., $U = \{0\}$, is called a deterministic dynamical system.

A deterministic control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with $\phi(t, t_0, x, u, d) = x$ for all $(t_0, x, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, $t \geq t_0$, and $H : \Re^+ \times U \to \mathcal{Y}$ (i.e., the output is independent of the state) is called a static map.

Definition 1.2 A deterministic control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with outputs is called *T-periodic* for some constant $T > 0$ if

(a) $H(t + T, x, u) = H(t, x, u)$ for all $(t, x, u) \in \Re^+ \times \mathcal{X} \times U$;

(b) for every $(u, d) \in M_U \times M_D$ and integer k, there exist inputs $P_{kT}u \in M_U$ and $P_{kT}d \in M_D$ with $(P_{kT}u)(t) = u(t + kT)$ and $(P_{kT}d)(t) = d(t + kT)$ for all $t + kT \geq 0$;

(c) for each $(t, t_0, x_0, u, d) \in A_\phi$ with $t \geq t_0$ and for each integer k with $t_0 - kT \geq 0$, it follows that $(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d) \in A_\phi$ and $\pi(t_0 - kT, x_0, P_{kT}u, P_{kT}d) = \bigcup_{\tau \in \pi(t_0, x_0, u, d)} \{\tau - kT\}$ with $\phi(t, t_0, x_0, u, d) = \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d)$.

Definition 1.3 A deterministic control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with outputs is called *time-invariant*, or *autonomous*, if it is T-periodic for all $T > 0$.

Remark 1.3 (a) Section 1.2.1 shows that systems (1.3) described by ODEs under (H1)–(H4) define deterministic control systems $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with $\mathcal{X} = \Re^n$ and $\mathcal{Y} = \Re^k$. If the mappings $f : \Re^+ \times \Re^n \times U \times D \to \Re^n$ and

$H : \Re^+ \times \Re^n \times U \to \Re^k$ are T-periodic with respect to t, then system (1.3) is T-periodic. Moreover, if the mappings $f : \Re^+ \times \Re^n \times U \times D \to \Re^n$ and $H : \Re^+ \times \Re^n \times U \to \Re^k$ are independent of t, then system (1.3) is autonomous.

(b) Section 1.2.2 shows that systems (1.10) described by RFDEs under Hypotheses (S1–4) define deterministic control systems $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with $\mathcal{X} = C^0([-r, 0]; \Re^n)$. If the mappings $f : \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times D \to \Re^n$ and $H : \Re^+ \times C^0([-r, 0]; \Re^n) \times U \to \mathcal{Y}$ are T-periodic with respect to t, then system (1.10) is T-periodic. Moreover, if the mappings $f : \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times D \to \Re^n$ and $H : \Re^+ \times C^0([-r, 0]; \Re^n) \times U \to \mathcal{Y}$ are independent of t, then system (1.10) becomes autonomous.

(c) Section 1.2.3 shows that systems (1.18), (1.19), (1.20) described by coupled RFDEs and FDEs under Hypotheses (P1–5) define deterministic control systems $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with $\mathcal{X} = C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2})$. If the mappings $f_i : \bigcup_{t\geq 0}\{t\} \times D \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2 + \tau(t), 0]; \Re^{n_2}) \times U \to \Re^{n_i}$, $i = 1, 2$, $H : \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times U \to \mathcal{Y}$, and $\tau : \Re^+ \to (0, +\infty)$ are T-periodic with respect to t, then system (1.18), (1.19), (1.20) is T-periodic. Furthermore, if the mappings $f_i : \bigcup_{t\geq 0}\{t\} \times D \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2 + \tau(t), 0]; \Re^{n_2}) \times U \to \Re^{n_i}$, $i = 1, 2$, $H : \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) \times U \to \mathcal{Y}$, and $\tau : \Re^+ \to (0, +\infty)$ are independent of t, then system (1.18), (1.19), (1.20) is autonomous.

(d) Section 1.2.4 shows that systems of the form (1.42) described by FDEs under (Q1)–(Q3) define deterministic control systems $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with $\mathcal{X}$ being the normed linear space of bounded functions $x : [-r, 0] \to \Re^n$ with norm $\|x\|_{\mathcal{X}} = \sup_{-r\leq\tau\leq 0} |x(\tau)|$. If the mappings $f : \bigcup_{t\geq 0}\{t\} \times D \times B(t) \times U \to \Re^n$, $\tau : \Re^+ \to (0, +\infty)$, and $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$, where $B(t)$ denotes the set of all bounded mappings $x : [-r + \tau(t), 0] \to \Re^n$, are T-periodic with respect to t, then system (1.42) is T-periodic. Moreover, if the mappings $f : \bigcup_{t\geq 0}\{t\} \times D \times B(t) \times U \to \Re^n$, $\tau : \Re^+ \to (0, +\infty)$, and $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ are independent of t, then system (1.42) is autonomous.

(e) Section 1.2.5 shows that systems of the form (1.57) described by ODEs under (A1)–(A4) define deterministic control systems $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with $\mathcal{X} = \Re^n$ and $\mathcal{Y} = \Re^p$. Notice that if $h(\tau + T, x, u, d) = h(\tau, x, u, d)$, $f(t + T, \tau + T, x, x_0, u, u_0, d, d_0) = f(t, \tau, x, x_0, u, u_0, d, d_0)$, $R(\tau + T, x, x_0, u, u_0, d, d_0) = R(\tau, x, x_0, u, u_0, d, d_0)$ and $H(t + T, x, u) = H(t, x, u)$ for certain $T > 0$ and for $(t, \tau, u, u_0, d, d_0, x, x_0) \in \Re^+ \times \Re^+ \times U \times U \times D \times D \times \Re^n \times \Re^n$ with $t \geq \tau$, then system (1.57) is *T-periodic*. Furthermore, if $h(\tau, x, u, d) = h(x, u, d)$ and if $f(t, \tau, x, x_0, u, u_0, d, d_0) = f(t - \tau, x, x_0, u, u_0, d, d_0)$, $R(\tau, x, x_0, u, u_0, d, d_0) = R(x, x_0, u, u_0, d, d_0)$, and $H(t, x, u) = H(x, u)$ for $(t, \tau, u, u_0, d, d_0, x, x_0) \in \Re^+ \times \Re^+ \times U \times U \times D \times D \times \Re^n \times \Re^n$ with $t \geq \tau$, then system (1.57) is autonomous.

We next give definitions of some important classes of control systems.

Definition 1.4 Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with outputs. We say that system Σ:

(1) satisfies the "*Boundedness-Implies-Continuation*" (*BIC*) property if for each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, there exists a maximal existence time, i.e., there exists $t_{\max} \in (t_0, +\infty]$ such that $[t_0, t_{\max}) \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$ and for all $t \geq t_{\max}$, it holds that $(t, t_0, x_0, u, d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$;
(2) is robustly forward complete (RFC) from the input $u \in M_U$ if it has the BIC property and for every $R \geq 0$, $T \geq 0$, it holds that

$$\sup\bigl\{\bigl\|\phi(t_0 + s, t_0, x_0, u, d)\bigr\|_{\mathcal{X}}; u \in \mathcal{M}\bigl(B_U[0, R]\bigr) \cap M_U, s \in [0, T], \|x_0\|_{\mathcal{X}} \leq R, t_0 \in [0, T], d \in M_D\bigr\} < +\infty.$$

Remark 1.4 Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property. It follows that Σ satisfies the (classical) semigroup property (see [21, 22, 46]) if the weak semigroup property holds with $\pi(t_0, x_0, u, d) = [t_0, t_{\max})$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the transition map for Σ that corresponds to $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, i.e.,

"for each $t \in [t_0, t_{\max})$, it holds that
$\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$ for all $\tau \in [t_0, t]$"
(Classical Semigroup Property)

Notice that the RFC property is more demanding than the BIC property. All classes of systems presented in the previous section satisfy the BIC property. However, only for systems described by FDEs of the form (1.42) under Hypotheses (Q1), (Q2), we can prove that the RFC property holds. Indeed, the following result guarantees the RFC property for systems described by FDEs.

Theorem 1.2 *Consider the control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *described by* (1.42) *under Hypotheses* (Q1), (Q2), *where* $\mathcal{X}$ *is the normed linear space of bounded functions* $x : [-r, 0] \to \Re^n$ *with norm* $\|x\|_{\mathcal{X}} = \sup_{-r \leq \tau \leq 0} |x(\tau)|$. *For every* $T \geq 0$, *there exists* $G_T \in K_\infty$ *such that the following estimate holds for all* $R \geq 0$:

$$\sup\bigl\{\bigl\|\phi(t_0 + s, t_0, x_0, u, d)\bigr\|_{\mathcal{X}}; u \in \mathcal{M}\bigl(B_U[0, R]\bigr) \cap M_U, s \in [0, T], \|x_0\|_{\mathcal{X}} \leq R, t_0 \in [0, T], d \in M_D\bigr\} \leq G_T(R) \tag{1.65}$$

Proof Let $T \geq 0$, $R \geq 0$ be given and define

$$s = \min\bigl\{\tau(t) : t \in [0, 2T + r]\bigr\} \tag{1.66}$$

By virtue of (Q1), it follows that $s > 0$. Consider the solution of (1.42) with initial condition $T_r(t_0)x = x_0 \in \mathcal{X}$ corresponding to inputs $u \in \mathcal{M}(B_U[0, R]) \cap M_U$, $d \in M_D$ and such that $\|x_0\|_{\mathcal{X}} \leq R$, $t_0 \in [0, T]$. Hypothesis (Q2), in conjunction with definition (1.66), implies that, for all $t \in [t_0, t_0 + T]$,

$$\sup_{t_0 - r \leq \tau \leq t + s} \bigl|x(\tau)\bigr| \leq \tilde{a}\Bigl(\sup_{t_0 - r \leq \tau \leq t} \bigl|x(\tau)\bigr|\Bigr) + \tilde{a}(R) \tag{1.67}$$

where $\tilde{a}(y) := a(y \max_{0 \le t \le 2T+r} \beta(t))$, $\theta(y) := y + 2\tilde{a}(y)$, and

$$\max\left\{R, \sup_{t_0 - r \le \tau \le t+s} \left|x(\tau)\right|\right\} \le \theta\left(\max\left\{R, \sup_{t_0 - r \le \tau \le t} \left|x(\tau)\right|\right\}\right) \tag{1.68}$$

Inequality (1.68) and induction arguments show that the following estimate holds for all integers $j \ge 0$ with $js \le T$:

$$\max\left\{R, \sup_{t_0 - r \le \tau \le t_0 + (j+1)s} \left|x(\tau)\right|\right\} \le \theta^{(j+1)}(R) \tag{1.69}$$

where $\theta^{(j)} = \underbrace{\theta \circ \cdots \circ \theta}_{j \text{ times}}$. It follows from (1.69) that inequality (1.65) holds with $G_T = \theta^{(i)}$, where $i = [\frac{T}{s}] + 1$. The proof is complete. □

1.4 Equilibrium Points

The following definition clarifies the notion of an equilibrium point for control systems with outputs in the sense of Definition 1.1.

Definition 1.5 Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ and let $u_0 \in M_U$ be the identically zero input, i.e., $u_0(t) = 0$ for all $t \ge 0$. Suppose that $H(t, 0, 0) = 0$ for all $t \ge 0$. We say that $0 \in \mathcal{X}$ is a *robust equilibrium point from the input $u \in M_U$ for Σ* if

(1) for every $(t, t_0, d) \in \Re^+ \times \Re^+ \times M_D$ with $t \ge t_0$, it holds that $\phi(t, t_0, 0, u_0, d) = 0$;
(2) for all $\varepsilon > 0$ and $T, h \in \Re^+$, there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that for all $(t_0, x, u) \in [0, T] \times \mathcal{X} \times M_U$ and $\tau \in [t_0, t_0 + h]$ with $\|x\|_{\mathcal{X}} + \sup_{t \ge 0} \|u(t)\|_{\mathcal{U}} \le \delta$, it holds that $(\tau, t_0, x, u, d) \in A_\phi$ for all $d \in M_D$ and

$$\sup\left\{\left\|\phi(\tau, t_0, x, u, d)\right\|_{\mathcal{X}}; d \in M_D, \tau \in [t_0, t_0 + h], t_0 \in [0, T]\right\} \le \varepsilon.$$

The reader should not be surprised by the previous definition of a robust equilibrium point. The usual definition of equilibrium point does not require property (2) of Definition 1.5 to hold. However, in most cases the control systems studied satisfy the property of *continuous dependence on the input and initial conditions* of the transition map, i.e.,

for each $(t, T, x_0, u) \in \Re^+ \times \Re^+ \times \mathcal{X} \times M_U$ with $t \ge T$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $(x, v) \in \mathcal{X} \times M_U$ with $\|x - x_0\|_{\mathcal{X}} + \sup_{t \ge 0} \|v(\tau) - u(\tau)\|_{\mathcal{U}} < \delta$ and for every $(t_0, d) \in [0, T] \times M_D$ with $(t, t_0, x_0, u, d) \in A_\phi$, it follows that $(t, t_0, x, v, d) \in A_\phi$ and

$$\sup\left\{\left\|\phi(\tau, t_0, x, v, d) - \phi(\tau, t_0, x_0, u, d)\right\|_{\mathcal{X}}; (t_0, d) \in [0, T] \times M_D, \right.$$
$$\left. \tau \in [t_0, t] \text{ with } (t, t_0, x_0, u, d) \in A_\phi\right\} < \varepsilon$$

It can be immediately verified that if the transition map depends continuously on the input and initial conditions, then the usual definition of an equilibrium point is equivalent to Definition 1.5 (since property (2) of Definition 1.5 is automatically satisfied). Since our effort is to provide results for systems that do not necessarily satisfy the property of continuous dependence on the initial conditions, we do not assume this property.

Definition 1.5 clarifies the main reason for which there is a distinction of the inputs acting on the system in Definition 1.1 of control systems ("inputs" and "disturbances"). The inputs that belong to the "set of allowed disturbances" ($M_D \subseteq \mathcal{M}(D)$) do not alter the position of equilibrium points. On the other hand, inputs that belong to the set of "allowable control inputs" ($M_U \subseteq \mathcal{M}(U)$) are allowed to alter the position of equilibrium points. This reminds the difference between "additive" and "multiplicative" uncertainties in linear system theory.

Example 1.4.1 (Robust equilibrium points for control systems described by ODEs) Consider system (1.3) under Hypotheses (H1–4). Since $f(t,0,0,d)=0$ for all $(t,d)\in\Re^+\times D$, it follows that $\phi(t,t_0,0,u_0,d)=0\in\Re^n$ for all $(t_0,d)\in\Re^+\times M_D$ and $t\geq t_0$. Let $\varepsilon>0$, $T,h\in\Re^+$ (arbitrary), and let $\delta:=\delta(\varepsilon,T,h)>0$ be the unique solution of the equation

$$\frac{K_2}{K_1}\exp\left(h\left(\frac{L}{K_1}+1\right)\right)\delta^2+\frac{K_2}{K_1}\exp\left(h\left(\frac{L}{K_1}+1\right)\right)\left(\max_{0\leq t\leq T+h}\gamma(t)\right)^2\big(a(\delta)\big)^2=\frac{\varepsilon^2}{4} \tag{1.70}$$

where $L:=L(T+h,\varepsilon)$, $L(\cdot)$ is the function involved in (1.1), $K_1,K_2>0$ are constants that satisfy $K_1|x|^2\leq x'Px\leq K_2|x|^2$ for all $x\in\Re^n$ for the symmetric positive definite matrix $P\in\Re^{n\times n}$ involved in Hypothesis (H1), and $\gamma\in K^+$ and $a\in K_\infty$ are the functions involved in Hypothesis (H3). We next show that property (2) of Definition 1.5 holds with $\delta:=\delta(\varepsilon,T,h)>0$ defined by (1.70). Notice that since system (1.3) under Hypotheses (H1–4) satisfies the BIC property, it suffices to show that, for all $(x_0,u)\in\Re^n\times M_U$ with $|x_0|+\sup_{t\geq 0}|u(t)|\leq\delta$, it holds that $\sup\{|\phi(\tau,t_0,x_0,u,d)|; d\in M_D, \tau\in[t_0,t_0+h], t_0\in[0,T]\}\leq\varepsilon$.

The proof will be made by contradiction. Suppose that there exists $(x_0,u)\in\Re^n\times M_U$ with $|x_0|+\sup_{t\geq 0}|u(t)|\leq\delta$ and

$$\sup\big\{\big|\phi(\tau,t_0,x_0,u,d)\big|; d\in M_D, \tau\in[t_0,t_0+h], t_0\in[0,T]\big\}>\varepsilon$$

Consequently, there exist $d\in M_D$, $t_0\in[0,T]$, and $t\in[t_0,t_0+h]$ with $|x(t)|\geq\frac{3\varepsilon}{4}$ (where $x(t)=\phi(t,t_0,x_0,u,d)$). Consider the nonempty set $A=\{t\geq t_0:|x(t)|>\frac{5\varepsilon}{8}\}$. Let $t_1=\inf A$. Notice that since $|x_0|\leq\delta\leq\frac{\varepsilon}{2}$ and $|x(t)|\geq\frac{3\varepsilon}{4}$, it follows that $t_1>t_0$ and $t_1\leq t_0+h$. Furthermore, by virtue of definition $t_1=\inf A$, we have $|x(t_1)|=\frac{5\varepsilon}{8}$ and $|x(t)|\leq\varepsilon$ for all $t\in[t_0,t_1]$. In this case, we consider the absolutely continuous function $V(t)=x'(t)Px(t)$, which by (1.1) satisfies the following inequality a.e. for $t\in[t_0,t_1]$:

$$\dot{V}(t)\leq L\big|x(t)\big|^2+2x'(t)Pf\big(t,0,u(t),d(t)\big) \tag{1.71}$$

With (1.71), using the inequalities $2x'Pf(t,0,u,d) \le x'Px + f'(t,0,u,d)Pf(t,0,u,d)$, $K_1|x|^2 \le x'Px \le K_2|x|^2$, and $|f(t,0,u(t),d(t))| \le \gamma(t)a(|u|) \le \gamma(t)a(\delta)$, we conclude that the absolutely continuous function $V(t) = x'(t)Px(t)$ satisfies the following inequality a.e. for $t \in [t_0, t_1]$:

$$\dot{V}(t) \le \left(\frac{L}{K_1} + 1\right)V(t) + K_2\gamma^2(t)\big(a(\delta)\big)^2 \tag{1.72}$$

The above differential inequality, in conjunction with the inequality $K_1|x|^2 \le x'Px \le K_2|x|^2$, implies that, for all $t \in [t_0, t_1]$,

$$\begin{aligned} \big|x(t)\big|^2 \le{}& \frac{K_2}{K_1}\exp\left(h\left(\frac{L}{K_1}+1\right)\right)\delta^2 \\ &+ \frac{K_2}{K_1}\exp\left(h\left(\frac{L}{K_1}+1\right)\right)\Big(\max_{0\le t\le T+h}\gamma(t)\Big)^2\big(a(\delta)\big)^2 \end{aligned}$$

The previous inequality in conjunction with (1.70) implies $|x(t_1)| \le \frac{\varepsilon}{2}$, which contradicts the fact that $|x(t_1)| = \frac{5\varepsilon}{8}$.

Thus we conclude that $0 \in \Re^n$ is a robust equilibrium point for system (1.3) under Hypotheses (H1–4).

Example 1.4.2 (Robust equilibrium points for control systems described by RFDEs) Consider system (1.10) under Hypotheses (S1–4). Since $f(t,0,0,d) = 0$ for all $(t,d) \in \Re^+ \times D$, it follows that $\phi(t,t_0,0,u_0,d) = 0 \in C^0([-r,0];\Re^n)$ for all $(t_0,d) \in \Re^+ \times M_D$ and $t \ge t_0$. Let $\varepsilon > 0$, $T, h \in \Re^+$ (arbitrary), and let $\delta := \delta(\varepsilon,T,h) > 0$ be the unique solution of the equation

$$\frac{K_2}{K_1}\exp\left(\frac{L+K_2}{K_1}h\right)\left(\delta^2 + h\Big(\max_{0\le t\le T+h}\gamma(t)\Big)^2\big(a(\delta)\big)^2\right) = \frac{\varepsilon^2}{4} \tag{1.73}$$

where $L := L(T+h,\varepsilon)$, $L(\cdot)$ is the function involved in (1.8), $K_1, K_2 > 0$ are constants that satisfy $K_1|x|^2 \le x'Px \le K_2|x|^2$ for all $x \in \Re^n$ for the symmetric positive definite matrix $P \in \Re^{n\times n}$ involved in Hypothesis (S1), and $\gamma \in K^+$ and $a \in K_\infty$ are the functions involved in Hypothesis (S2). We next show that property (2) of Definition 1.5 holds with $\delta := \delta(\varepsilon,T,h) > 0$ defined by (1.73). Notice that since system (1.10) under Hypotheses (S1–4) satisfies the BIC property, it suffices to show that for all $(x_0,u) \in C^0([-r,0];\Re^n) \times M_U$ with $\|x_0\|_r + \sup_{t\ge0}|u(t)| \le \delta$, it holds that $\sup\{\|\phi(\tau,t_0,x_0,u,d)\|_r; d \in M_D, \tau \in [t_0,t_0+h], t_0 \in [0,T]\} \le \varepsilon$.

The proof will be made by contradiction. Suppose that there exists $(x_0,u) \in C^0([-r,0];\Re^n) \times M_U$ with $\|x_0\|_r + \sup_{t\ge0}|u(t)| \le \delta$ and

$$\sup\left\{\big\|\phi(\tau,t_0,x_0,u,d)\big\|_r; d \in M_D, \tau \in [t_0,t_0+h], t_0 \in [0,T]\right\} > \varepsilon$$

Consequently, there exists $d \in M_D$, $t_0 \in [0,T]$ and $t \in [t_0,t_0+h]$ with $\|T_r(t)x\|_r \ge \frac{3\varepsilon}{4}$ (where $T_r(t)x = \phi(t,t_0,x_0,u,d)$). By virtue of Lemma 2.1, p. 40 in [14], we know that the mapping $\tau \to \|T_r(\tau)x\|_r$ is continuous on $[t_0,t]$. Consider the nonempty set $A = \{\tau \ge t_0 : \|T_r(\tau)x\| > \frac{5\varepsilon}{8}\}$. Let $t_1 = \inf A$. Notice that since $\|x_0\|_r \le \delta \le \frac{\varepsilon}{2}$ and $\|T_r(t)x\|_r \ge \frac{3\varepsilon}{4}$, it follows that $t_1 > t_0$ and $t_1 \le t_0 + h$. Furthermore, by the definition of $t_1 = \inf A$, we have $\|T_r(t_1)x\|_r = \frac{5\varepsilon}{8}$ and $\|T_r(\tau)x\|_r \le \varepsilon$

for all $\tau \in [t_0, t_1]$. In this case, we consider the absolutely continuous function $V(t) = x'(t)Px(t)$, which by (1.8) satisfies the following inequality a.e. for $t \in [t_0, t_1]$:

$$\dot{V}(t) \le L\left\|T_r(t)x\right\|_r^2 + 2x'(t)Pf\big(t, 0, u(t), d(t)\big) \tag{1.74}$$

With (1.74), using the inequalities $2x'Pf(t,0,u,d) \le x'Px + f'(t,0,u,d)Pf(t,0,u,d)$, $K_1|x|^2 \le x'Px \le K_2|x|^2$, and $|f(t,0,u(t),d(t))| \le \gamma(t)a(|u|) \le \gamma(t)a(\delta)$, we conclude that the following inequality holds for all $t \in [t_0, t_1]$:

$$\left|x(t)\right|^2 \le \frac{L+K_2}{K_1}\int_{t_0}^{t}\left\|T_r(\tau)x\right\|_r^2 d\tau + \frac{K_2}{K_1}\|x_0\|_r^2 + h\frac{K_2}{K_1}\Big(\max_{0\le t\le T+h}\gamma(t)\Big)^2\big(a(\delta)\big)^2$$

The above integral inequality gives for all $t \in [t_0, t_1]$:

$$\begin{aligned}\left\|T_r(t)x\right\|_r^2 \le{}& \frac{L+K_2}{K_1}\int_{t_0}^{t}\left\|T_r(\tau)x\right\|_r^2 d\tau + \frac{K_2}{K_1}\|x_0\|_r^2\\ &+ h\frac{K_2}{K_1}\Big(\max_{0\le t\le T+h}\gamma(t)\Big)^2\big(a(\delta)\big)^2\end{aligned} \tag{1.75}$$

Inequality (1.75) in conjunction with the Gronwall–Bellman lemma implies that

$$\left\|T_r(t)x\right\|_r^2 \le \frac{K_2}{K_1}\exp\left(\frac{L+K_2}{K_1}h\right)\left(\|x_0\|_r^2 + h\Big(\max_{0\le t\le T+h}\gamma(t)\Big)^2\big(a(\delta)\big)^2\right)$$
$$\text{for all } t \in [t_0, t_1].$$

The previous inequality in conjunction with $\|x_0\|_r \le \delta$ and (1.73) implies $\|T_r(t_1)x\|_r \le \frac{\varepsilon}{2}$, which contradicts the fact that $\|T_r(t_1)x\|_r = \frac{5\varepsilon}{8}$.

Thus, $0 \in C^0([-r,0];\Re^n)$ is a robust equilibrium point from the input $u \in M_U$ for system (1.10) under Hypotheses (S1–4).

Example 1.4.3 (Robust equilibrium points for control systems described by coupled RFDEs and FDEs) For systems described by coupled RFDEs and FDEs of the form (1.18), (1.19) under Hypotheses (P1–4), the fact that $(0,0) \in C^0([-r_1,0];\Re^{n_1}) \times \mathcal{L}^\infty([-r_2,0];\Re^{n_2})$ is a robust equilibrium point from the input $u \in M_U$ follows from the following result.

Theorem 1.3 *Consider system* (1.18), (1.19) *under Hypotheses* (P1–4). *Then for all $\varepsilon > 0$ and $T, h \in \Re^+$, there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that for all $(t_0, x_{10}, x_{20}) \in [0,T] \times C^0([-r_1,0];\Re^{n_1}) \times \mathcal{L}^\infty([-r_2,0];\Re^{n_2})$ and $(u,d) \in \mathcal{L}^\infty_{\text{loc}}(\Re^+;U) \times \mathcal{L}^\infty_{\text{loc}}(\Re^+;D)$ with $\|x_{10}\|_{r_1} + \|x_{20}\|_{r_2} + \sup_{t\ge 0}|u(t)| \le \delta$, there exist $t_{\max} \in (t_0 + h, +\infty]$ and a unique pair of mappings $x_1 \in C^0([t_0 - r_1, t_{\max});\Re^{n_1})$, $x_2 \in \mathcal{L}^\infty_{\text{loc}}([t_0 - r_2, t_{\max});\Re^{n_2})$ with $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$, $x_1 \in C^0([t_0 - r_1, t_{\max});\Re^{n_1})$ being absolutely continuous on $[t_0, t_{\max})$, such that* (1.18) *holds a.e. for $t \in [t_0, t_{\max})$,* (1.19) *holds for all $t \in (t_0, t_{\max})$, and*

$$\sup\Big\{\left\|T_{r_1}(t)x_1\right\|_{r_1} + \left\|T_{r_2}(t)x_2\right\|_{r_2}; t \in [t_0, t_0 + h]\Big\} \le \varepsilon. \tag{1.76}$$

For the proof of Theorem 1.3, we need the following technical lemma.

Lemma 1.1 *For all* $\varepsilon > 0$ *and* $T \in \Re^+$, *there exists* $\tilde{\delta} := \tilde{\delta}(\varepsilon, T) > 0$ *such that for all* $(t_0, x_{10}, x_{20}) \in [0, T] \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2})$ *and* $(u, d) \in \mathcal{L}^\infty_{\text{loc}}(\Re^+; U) \times \mathcal{L}^\infty_{\text{loc}}(\Re^+; D)$ *with* $\|x_{10}\|_{r_1} + \|x_{20}\|_{r_2} + \sup_{t \geq 0} |u(t)| \leq \tilde{\delta}$, *there exist* $t_{\max} \in (t_0 + h, +\infty]$, *where* $h := h(T) = \min(1; \min\{\tau(s) : s \in [0, T + 1]\})$, *and a unique pair of mappings* $x_1 \in C^0([t_0 - r_1, t_{\max}); \Re^{n_1})$, $x_2 \in \mathcal{L}^\infty_{\text{loc}}([t_0 - r_2, t_{\max}); \Re^{n_2})$ *with* $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$ *such that* (1.18) *holds a.e. for* $t \in [t_0, t_{\max})$, (1.19) *holds for all* $t \in (t_0, t_{\max})$, *and* (1.76) *holds.*

Proof Without loss of generality, we may assume that the function $a \in K_\infty$ involved in Hypothesis (P2) satisfies $a(s) \geq s$ for all $s \geq 0$ and that the function $\beta \in K^+$ involved in Hypothesis (P2) is nondecreasing with $\beta(t) \geq 1$ for all $t \geq 0$. Let $\varepsilon > 0$, $T \in \Re^+$, and define

$$\tilde{\delta}(\varepsilon, T) := \frac{1}{\beta(T+1)} a^{-1}\left(\sqrt{\frac{K_1}{K_2}} \frac{\exp\left(-\frac{\tilde{L}(\varepsilon,T)+K_2}{K_1}\right)}{4\beta(T+1)} a^{-1}\left(\frac{\varepsilon}{9}\right)\right) \tag{1.77}$$

where $\tilde{L}(\varepsilon, T)$ is the constant that corresponds to the bounded sets $I := [0, T + 1] \subset \Re^+$, $\Omega \subset \{x_1 \in C^0([-r_1, 0]; \Re^{n_1}) : \|x_1\|_{r_1} \leq \varepsilon\} \times \{x_2 \in \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) : \|x_2\|_{r_2} \leq \varepsilon\} \times \{u \in U : |u| \leq \varepsilon\}$ and satisfies (1.21), and $K_1, K_2 > 0$ are constants that satisfy $K_1|x|^2 \leq x'Px \leq K_2|x|^2$ for all $x \in \Re^{n_1}$ for the symmetric positive definite matrix $P \in \Re^{n_1 \times n_1}$ involved in Hypothesis (P4).

Let $(t_0, x_{10}, x_{20}) \in [0, T] \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2})$ and $(u, d) \in \mathcal{L}^\infty_{\text{loc}}(\Re^+; U) \times \mathcal{L}^\infty_{\text{loc}}(\Re^+; D)$ with $\|x_{10}\|_{r_1} + \|x_{20}\|_{r_2} + \sup_{t \geq 0} |u(t)| < \tilde{\delta}$ (but otherwise arbitrary). By Theorem 1.1, there exist $t_{\max} \in (t_0, +\infty]$ and a unique pair of mappings $x_1 \in C^0([t_0 - r_1, t_{\max}); \Re^{n_1})$, $x_2 \in \mathcal{L}^\infty_{\text{loc}}([t_0 - r_2, t_{\max}); \Re^{n_2})$ with $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$, x_1 being absolutely continuous on $[t_0, t_{\max})$, such that (1.18) holds a.e. for $t \in [t_0, t_{\max})$ and (1.19) holds for all $t \in (t_0, t_{\max})$. In addition, if $t_{\max} < +\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|T_{r_1}(t)x_1\|_{r_1} > M$.

Define the set

$$A = \left\{t \in [t_0, t_{\max}) : \left\|T_{r_1}(t)x_1\right\|_{r_1} > \frac{1}{\beta(T+1)} a^{-1}\left(\frac{\varepsilon}{9}\right)\right\} \tag{1.78}$$

We distinguish two cases:

(1) $A \cap [t_0, t_0 + h] = \emptyset$;
(2) $A \cap [t_0, t_0 + h] \neq \emptyset$.

Case 1: We will show that (1.76) holds in this case with $\tilde{\delta} := \tilde{\delta}(\varepsilon, T) > 0$ as defined by (1.77). If $A \cap [t_0, t_0 + h] = \emptyset$, where $h := h(T) = \min(1; \min\{\tau(s) : s \in [0, T + 1]\})$, then x_1 is bounded on $[t_0, t_0 + h]$, and consequently we have $t_{\max} > t_0 + h$. Moreover, by virtue of Hypothesis (P2), we have, for all $t \in [t_0, t_0 + h]$,

$$\begin{aligned}\left|x_2(t)\right| \leq{} & a\left(\beta(t)\left\|T_{r_1}(t)x_1\right\|_{r_1}\right) + a\left(\beta(t)\left\|T_{r_2-\tau(t)}\left(t - \tau(t)\right)x_2\right\|_{r_2-\tau(t)}\right) \\ & + a\left(\beta(t)\left|u(t)\right|\right)\end{aligned} \tag{1.79}$$

Since $A \cap [t_0, t_0 + h] = \emptyset$ (which implies $a(\beta(t)\|T_{r_1}(t)x_1\|_{r_1}) \leq \frac{\varepsilon}{9}$ for all $t \in [t_0, t_0 + h]$; see (1.78)), $\|x_{20}\|_{r_2} \leq \tilde{\delta}$ (which implies $a(\beta(t)\|T_{r_2-\tau(t)}(t - \tau(t))x_2\|_{r_2-\tau(t)}) \leq$

$\frac{\varepsilon}{9}$ for all $t \in [t_0, t_0 + h]$; see (1.77)), and $\sup_{t\geq 0}|u(t)| \leq \tilde{\delta}$ (which implies $a(\beta(t)|u(t)|) \leq \frac{\varepsilon}{9}$ for all $t \in [t_0, t_0 + h]$; see (1.77)), from (1.79) we obtain

$$\left|x_2(t)\right| \leq \frac{\varepsilon}{3} \quad \text{for all } t \in [t_0, t_0 + h] \tag{1.80}$$

Inequality (1.80), in conjunction with the fact that $\|x_{20}\|_{r_2} < \tilde{\delta} \leq \frac{\varepsilon}{3}$ and the assumption $A \cap [t_0, t_0 + h] = \emptyset$ (which implies $\|T_{r_1}(t)x_1\|_{r_1} \leq a(\beta(t)\|T_{r_1}(t)x_1\|_{r_1}) \leq \frac{\varepsilon}{9}$ for all $t \in [t_0, t_0 + h]$), shows that (1.76) holds in this case.

Case 2: We will show that this case cannot happen by contradiction. Assume that $A \cap [t_0, t_0 + h] \neq \emptyset$ and define $t_1 = \inf A$. By continuity of the mapping $t \to \|T_{r_1}(t)x_1\|_{r_1}$ and since $\|x_{10}\|_{r_1} \leq \tilde{\delta} < \frac{1}{\beta(T+1)} a^{-1}(\frac{\varepsilon}{9})$, it follows that $t_1 > t_0$. Hence, by continuity of the map $t \to \|T_{r_1}(t)x_1\|_{r_1}$ and definition (1.78), we have $\|T_{r_1}(t_1)x_1\|_{r_1} = \frac{1}{\beta(T+1)} a^{-1}(\frac{\varepsilon}{9})$. Evaluating the derivative of the absolutely continuous map $V(t) = x_1'(t)Px_1(t)$ on $[t_0, t_1]$, in conjunction with Hypothesis (P4), gives

$$\begin{aligned}\frac{d}{dt}V(t) &= 2x_1'(t)Pf_1\big(t, d(t), T_{r_1}(t)x_1, T_{r_2-\tau(t)}\big(t-\tau(t)\big)x_2, u(t)\big)\\ &\leq 2\tilde{L}\big\|T_{r_1}(t)x_1\big\|_{r_1}^2\\ &\quad + 2x_1'(t)Pf_1\big(t, d(t), 0, T_{r_2-\tau(t)}\big(t-\tau(t)\big)x_2, u(t)\big)\end{aligned} \tag{1.81}$$

where $\tilde{L}$ is the constant that corresponds to the bounded sets $I := [0, T+1] \subset \Re^+$, $\Omega \subset \{x_1 \in C^0([-r_1, 0]; \Re^{n_1}) : \|x_1\|_{r_1} \leq \varepsilon\} \times \{x_2 \in \mathcal{L}^\infty([-r_2, 0]; \Re^{n_2}) : \|x_2\|_{r_2} \leq \varepsilon\}$ $\times\{u \in U : |u| \leq \varepsilon\}$ and satisfies (1.21). Inequality (1.81), in conjunction with Hypothesis (P2), the inequality $2x_1'Pf_1 \leq K_2|x_1|^2 + K_2|f_1|^2$ and the facts that $\|x_{20}\|_{r_2} < \tilde{\delta}$ (which implies

$$a\big(\beta(t)\big\|T_{r_2-\tau(t)}\big(t-\tau(t)\big)x_2\big\|_{r_2-\tau(t)}\big) \leq \sqrt{\frac{K_1}{K_2}\frac{\exp\big(-\frac{\tilde{L}(\varepsilon,T)+K_2}{K_1}\big)}{4\beta(T+1)}} a^{-1}\left(\frac{\varepsilon}{9}\right)$$
$$\text{for all } t \in [t_0, t_0 + h];$$

see (1.77)) and $\sup_{t\geq 0}|u(t)| \leq \tilde{\delta}$ (which implies

$$a\big(\beta(t)\big|u(t)\big|\big) \leq \sqrt{\frac{K_1}{K_2}\frac{\exp\big(-\frac{\tilde{L}(\varepsilon,T)+K_2}{K_1}\big)}{4\beta(T+1)}} a^{-1}\left(\frac{\varepsilon}{9}\right)$$

for all $t \in [t_0, t_0 + h]$ see (1.77)), gives

$$\dot{V}(t) \leq 2\big(\tilde{L} + K_2\big)\big\|T_{r_1}(t)x_1\big\|_{r_1}^2 + K_1\frac{\exp\big(-2\frac{\tilde{L}+K_2}{K_1}\big)}{4\beta^2(T+1)}\left(a^{-1}\left(\frac{\varepsilon}{9}\right)\right)^2 \quad \text{a.e.}$$
$$\text{on } [t_0, t_1] \tag{1.82}$$

Integrating both sides of (1.82) and using $K_1|x|^2 \leq x'Px \leq K_2|x|^2$, we get, for all $t \in [t_0, t_1]$,

$$\left|x_1(t)\right|^2 \le \frac{K_2}{K_1}\left|x_1(t_0)\right|^2 + 2\frac{\tilde{L}+K_2}{K_1}\int_{t_0}^{t}\left\|T_{r_1}(s)x_1\right\|_{r_1}^2 ds$$
$$+ \frac{\exp\left(-2\frac{\tilde{L}+K_2}{K_1}\right)}{4\beta^2(T+1)}\left(a^{-1}\left(\frac{\varepsilon}{9}\right)\right)^2 \tag{1.83}$$

The following inequality is a direct consequence of (1.83) and holds for all $t \in [t_0, t_1]$:

$$\left\|T_{r_1}(t)x_1\right\|_{r_1}^2 \le \frac{K_2}{K_1}\left\|T_{r_1}(t_0)x_1\right\|_{r_1}^2 + 2\frac{\tilde{L}+K_2}{K_1}\int_{t_0}^{t}\left\|T_{r_1}(s)x_1\right\|_{r_1}^2 ds$$
$$\times \frac{\exp\left(-2\frac{\tilde{L}+K_2}{K_1}\right)}{4\beta^2(T+1)}\left(a^{-1}\left(\frac{\varepsilon}{9}\right)\right)^2 \tag{1.84}$$

Since the map $t \to \|T_{r_1}(t)x_1\|_{r_1}$ is continuous and (1.84) holds on $[t_0, t_1]$, we may apply the Gronwall–Bellman lemma. We obtain, for all $t \in [t_0, t_1]$,

$$\left\|T_{r_1}(t)x_1\right\|_{r_1} \le \sqrt{\frac{K_2}{K_1}}\exp\left(\frac{\tilde{L}+K_2}{K_1}\right)\left\|T_{r_1}(t_0)x_1\right\|_{r_1} + \frac{1}{2\beta(T+1)}a^{-1}\left(\frac{\varepsilon}{9}\right) \tag{1.85}$$

Since $\|x_{10}\|_{r_1} \le \tilde{\delta}$ (which implies $\|T_{r_1}(t_0)x_1\|_{r_1} \le \sqrt{\frac{K_1}{K_2}}\frac{\exp(-\frac{\tilde{L}+K_2}{K_1})}{4\beta(T+1)}a^{-1}(\frac{\varepsilon}{9})$), from (1.85) we get

$$\left\|T_{r_1}(t_1)x_1\right\|_{r_1} \le \frac{3}{4\beta(T+1)}a^{-1}\left(\frac{\varepsilon}{9}\right) < \frac{1}{\beta(T+1)}a^{-1}\left(\frac{\varepsilon}{9}\right) \tag{1.86}$$

Inequality (1.86) contradicts the equality $\|T_{r_1}(t_1)x_1\|_{r_1} = \frac{1}{\beta(T+1)}a^{-1}(\frac{\varepsilon}{9})$. Thus, the case $A \cap [t_0, t_0+h] \neq \emptyset$ cannot happen.

The proof is complete. □

We continue with the proof of Theorem 1.3.

Proof of Theorem 1.3 Consider the sequence $\{T_i\}_{i=0}^{\infty}$ generated by the recursive relation

$$\begin{aligned} T_0 &= T \ge 0 \\ T_{i+1} &= T_i + \min\left(1; \min\left\{\tau(s) : s \in [0, T_i+1]\right\}\right) \quad i = 1, 2, \ldots \end{aligned} \tag{1.87}$$

A standard contradiction argument shows that $\lim T_i = \sup T_i = +\infty$ for all $T_0 \ge 0$. Consequently, given arbitrary $T, h \in \Re^+$, there exists some nonnegative integer $l(T,h)$ such that the sequence $\{T_i\}_{i=0}^{\infty}$ defined by (1.87) with the initial condition $T_0 = T$ satisfies $T_i \ge T + h$ for all $i \ge l(T,h)$. The following fact exploits the properties of the sequence $\{T_i\}_{i=0}^{\infty}$ defined by (1.87).

Fact *For all $\varepsilon > 0$, $T \in \Re^+$, and nonnegative integers i, there exists $\delta_i := \delta_i(\varepsilon, T) > 0$ such that for all $(t_0, x_{10}, x_{20}) \in [0, T] \times C^0([-r_1, 0]; \Re^{n_1}) \times \mathcal{L}^{\infty}([-r_2, 0]; \Re^{n_2})$ and $(u, d) \in \mathcal{L}^{\infty}_{\text{loc}}(\Re^+; U) \times \mathcal{L}^{\infty}_{\text{loc}}(\Re^+; D)$ with $\|x_{10}\|_{r_1} +$*

$\|x_{20}\|_{r_2} + \sup_{t\geq 0}|u(t)| \leq \delta_i$, *there exists* $t_{\max} \in (t_0 + h, +\infty]$, *where* $h := h(T) = T_{i+1} - T$, $\{T_i\}_{i=0}^{\infty}$ *is the sequence that satisfies* (1.87), *and a unique pair of mappings* $x_1 \in C^0([t_0 - r_1, t_{\max}); \Re^{n_1})$, $x_2 \in \mathcal{L}^{\infty}_{\text{loc}}([t_0 - r_2, t_{\max}); \Re^{n_2})$ *with* $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$ *such that* (1.18) *holds a.e. for* $t \in [t_0, t_{\max})$, (1.19) *holds for all* $t \in (t_0, t_{\max})$, *and* (1.76) *holds.*

The proof of the fact will be made by induction. By virtue of Lemma 1.1 it is clear that the fact holds for $i = 0$. Suppose that the fact holds for a certain nonnegative integer i. Let $\varepsilon > 0$, $T \in \Re^+$, and define

$$\delta_{i+1}(\varepsilon, T) := \min\left\{\delta_i(\varepsilon, T);\, \delta_i\left(\frac{1}{2}\tilde{\delta}(\varepsilon, T_{i+1}), T\right);\, \frac{1}{2}\tilde{\delta}(\varepsilon, T_{i+1})\right\} > 0 \quad (1.88)$$

Next consider the solution $x_1 \in C^0([t_0 - r_1, t_{\max}); \Re^{n_1})$, $x_2 \in \mathcal{L}^{\infty}_{\text{loc}}([t_0 - r_2, t_{\max}); \Re^{n_2})$ of (1.18), (1.19) with $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$ corresponding to inputs $(u, d) \in \mathcal{L}^{\infty}_{\text{loc}}(\Re^+; U) \times \mathcal{L}^{\infty}_{\text{loc}}(\Re^+; D)$ with $\|x_{10}\|_{r_1} + \|x_{20}\|_{r_2} + \sup_{t\geq 0}|u(t)| \leq \delta_{i+1}$. Since $\delta_{i+1} \leq \delta_i$, from the assumption it follows that the fact holds for the nonnegative integer i:

$$\sup\left\{\left\|T_{r_1}(t)x_1\right\|_{r_1} + \left\|T_{r_2}(t)x_2\right\|_{r_2};\, t \in [t_0, t_0 + T_{i+1} - T]\right\} \leq \varepsilon \qquad (1.89)$$

Moreover, since $\delta_{i+1}(\varepsilon, T) \leq \delta_i(\frac{1}{2}\tilde{\delta}(\varepsilon, T_{i+1}), T)$, it follows from the assumption that the fact holds for the nonnegative integer i:

$$\left\|T_{r_1}(t_0 + T_{i+1} - T)x_1\right\|_{r_1} + \left\|T_{r_2}(t_0 + T_{i+1} - T)x_2\right\|_{r_2} \leq \frac{1}{2}\tilde{\delta}(\varepsilon, T_{i+1}) \qquad (1.90)$$

Furthermore, since $\|x_{10}\|_{r_1} + \|x_{20}\|_{r_2} + \sup_{t\geq 0}|u(t)| \leq \delta_{i+1}$ and $\delta_{i+1}(\varepsilon, T) \leq \frac{1}{2}\tilde{\delta}(\varepsilon, T_{i+1})$, we obtain that $\sup_{t\geq 0}|u(t)| \leq \frac{1}{2}\tilde{\delta}(\varepsilon, T_{i+1})$. Combining (1.90) and the previous inequality, we get

$$\left\|T_{r_1}(t_0 + T_{i+1} - T)x_1\right\|_{r_1} + \left\|T_{r_2}(t_0 + T_{i+1} - T)x_2\right\|_{r_2} + \sup_{t\geq 0}\left|u(t)\right| \leq \tilde{\delta}(\varepsilon, T_{i+1}) \qquad (1.91)$$

Notice that since $t_0 \in [0, T]$, we obtain that $t_0 + T_{i+1} - T \in [0, T_{i+1}]$. The solution $x_1 \in C^0([t_0 - r_1, t_{\max}); \Re^{n_1})$, $x_2 \in \mathcal{L}^{\infty}_{\text{loc}}([t_0 - r_2, t_{\max}); \Re^{n_2})$ of (1.18), (1.19) with $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$ corresponding to inputs $(u, d) \in \mathcal{L}^{\infty}_{\text{loc}}(\Re^+; U) \times \mathcal{L}^{\infty}_{\text{loc}}(\Re^+; D)$ with $\|x_{10}\|_{r_1} + \|x_{20}\|_{r_2} + \sup_{t\geq 0}|u(t)| \leq \delta_{i+1}$ coincides, over the time interval $[t_0 + T_{i+1} - T, \infty)$, with the solution $x_1 \in C^0([t_0 + T_{i+1} - T - r_1, t_{\max}); \Re^{n_1})$, $x_2 \in \mathcal{L}^{\infty}_{\text{loc}}([t_0 + T_{i+1} - T - r_2, t_{\max}); \Re^{n_2})$ of (1.18), (1.19) with initial condition $(T_{r_1}(t_0 + T_{i+1} - T)x_1, T_{r_2}(t_0 + T_{i+1} - T)x_2)$ corresponding to same inputs $(u, d) \in \mathcal{L}^{\infty}_{\text{loc}}(\Re^+; U) \times \mathcal{L}^{\infty}_{\text{loc}}(\Re^+; D)$ satisfying (1.91). Using Lemma 1.1, in conjunction with (1.91), definition (1.87), and the fact that $t_0 + T_{i+1} - T \in [0, T_{i+1}]$, we obtain

$$\sup\left\{\left\|T_{r_1}(t)x_1\right\|_{r_1} + \left\|T_{r_2}(t)x_2\right\|_{r_2};\, t \in [t_0 + T_{i+1} - T, t_0 + T_{i+2} - T]\right\} \leq \varepsilon \quad (1.92)$$

Combining (1.89) with (1.92), we may conclude that the fact holds for $i + 1$.

By virtue of the fact, it follows that Theorem 1.3 holds with $\delta(\varepsilon, T, h) := \delta_{l(T,h)}(\varepsilon, T) > 0$, where $l(T, h)$ is the nonnegative integer with the property that the sequence $\{T_i\}_{i=0}^{\infty}$ defined by (1.87) with initial condition $T_0 = T$ satisfies $T_i \geq T + h$ for all $i \geq l(T, h)$. The proof is complete. □

Example 1.4.4 (Robust equilibrium points for control systems described by FDEs) Theorem 1.2 shows that $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ for system (1.42) under Hypotheses (Q1–3).

Example 1.4.5 (Robust equilibrium points for control systems with variable sampling partition) The following proposition guarantees that system (1.57) under Hypotheses (A1–5) has a robust equilibrium point from the input $u \in M_U$.

Proposition 1.2 $0 \in \Re^n$ *is a robust equilibrium point from the input* $u \in M_U$ *for system* (1.57) *under Hypotheses* (A1–5).

Proof Since $f(t, \tau, 0, 0, 0, 0, d, d_0) = 0$ and $R(\tau, 0, 0, 0, 0, d, d_0) = 0$ for all $(\tau, d, d_0) \in \Re^+ \times D \times D$ and $t \geq \tau$, and $H(t, 0, 0) = 0$ for all $t \geq 0$, it follows that property (1) of Definition 1.5 is automatically satisfied. It suffices to show that for all $\varepsilon > 0$ and $T, T' \in \Re^+$, there exists $\delta := \delta(\varepsilon, T, T') > 0$ such that for all $(t_0, x_0, u, d) \in [0, T] \times \Re^n \times M_U \times M_D$ and $t \in [t_0, t_0 + T']$ with $|x_0| + \sup_{t\geq 0} |u(t)| \leq \delta$, it holds that the solution $x(t)$ of (1.57) with initial condition $x(t_0) = x_0$ corresponding to inputs $(u, d) \in M_U \times M_D$ exists and satisfies $\sup\{|x(t)|; d \in M_D, t \in [t_0, t_0 + T'], t_0 \in [0, T]\} \leq \varepsilon$.

Claim 1 *For all* $\varepsilon > 0$ *and* $T > 0$, *there exists* $\delta := \delta(\varepsilon, T) > 0$ *such that if* $|x(t_0)| + \sup_{t\geq 0} |u(t)| \leq \delta$, *then the unique solution of* (1.57) *starting from* $x(t_0)$ *at time* $t_0 \in [0, T]$ *and corresponding to inputs* $(u, d) \in M_U \times M_D$ *exists for all* $t \in [t_0, \tau_1]$ *and satisfies* $|x(t)| \leq \varepsilon$ *for all* $t \in [t_0, \tau_1]$, *where* $\tau_1 = t_0 + h(t_0, x(t_0), u(t_0), d(t_0))$.

Proof of Claim 1 Let $L > 0$ be the constant that satisfies (1.58) for the bounded set $S := [0, T + r] \times [0, T + r] \times B[0, \varepsilon] \times B[0, \varepsilon] \times B_U[0, \varepsilon] \times B_U[0, \varepsilon]$. It follows from (1.58), (1.59) that the following inequality holds for all $x, x_0 \in B[0, \varepsilon]$, $u, u_0 \in B_U[0, \varepsilon]$, $\tau \in [0, T]$, $t \in [\tau, \tau + r]$, and $d, d_0 \in D$:

$$\begin{aligned} &2x' P f(t, \tau, x, x_0, u, u_0, d, d_0) \\ &\quad \leq (2L + K_2)|x|^2 + M K_2 a^2\big(|x_0| + |u| + |u_0|\big) \end{aligned} \tag{1.93}$$

where $M := \max\{\gamma^2(t); t \in [0, T + r]\}$, $K_1, K_2 > 0$ are constants that satisfy $K_1|x|^2 \leq x' P x \leq K_2|x|^2$ for all $x \in \Re^n$ for the symmetric positive definite matrix $P \in \Re^{n\times n}$ involved in Hypothesis (A1), and $\gamma \in K^+$ and $a \in K_\infty$ are the functions involved in (1.59), (1.60). Let $\rho > 0$ be the unique solution of the equation

$$\varepsilon_1^2 = \frac{K_2}{K_1} \exp\left(\frac{(2L + K_2)r}{K_1}\right)\big(\rho^2 + r M a^2(3\rho)\big) \tag{1.94}$$

where $r > 0$ is the upper bound for h, and

$$\varepsilon_1 := \min\left\{\frac{\varepsilon}{2}, \frac{1}{4} a^{-1}\left(\frac{\varepsilon}{\max\{\gamma(t); t \in [0, T + r]\}}\right)\right\} > 0 \tag{1.95}$$

Define

$$\delta = \min\left\{\frac{\varepsilon_1}{2}, \rho\right\} \tag{1.96}$$

For arbitrary $(x(t_0), u, d) \in \Re^n \times M_U \times M_D$ with $|x(t_0)| + \sup_{t\geq 0}|u(t)| < \delta$, consider the unique solution $x(t) \in \Re^n$ of (1.57) starting from $x(t_0)$ and corresponding to input $(u, d) \in M_U \times M_D$. Since (1.96) implies $\delta < \varepsilon_1$, it follows that $|x(t_0)| < \varepsilon_1$. Next, we show that $|x(t)| < \varepsilon_1$ for all $t \in [t_0, \tau_1)$. The proof will be made by contradiction. Suppose that there exists $t_1 \in (t_0, \tau_1)$ with $|x(t_1)| \geq \varepsilon_1$. Let t_ε the maximal time in the interval $[t_0, t_1]$ such that $|x(t)| < \varepsilon_1$ for all $t \in [t_0, t_\varepsilon)$. By virtue of continuity of the solution with respect to time on the interval $[t_0, \tau_1)$, the maximal time t_ε is well defined. By continuity of the solution with respect to time we must have $|x(t_\varepsilon)| = \varepsilon_1$. On the other hand, inequality (1.93), in conjunction with the fact $|x(t_0)| + \sup_{t\geq 0}|u(t)| \leq \delta$ and definition (1.96), implies that the absolutely continuous function $V(t) = x'(t)Px(t)$ satisfies $\dot{V}(t) \leq (2L + K_2)|x(t)|^2 + K_2 M a^2(3\rho)$ for almost all $t \in [t_0, t_\varepsilon]$. Using the previous differential inequality and inequality $K_1|x|^2 \leq x'Px \leq K_2|x|^2$, we obtain $|x(t)|^2 \leq \frac{K_2}{K_1}|x(t_0)|^2 + \frac{2L+K_2}{K_1}\int_{t_0}^{t}|x(s)|^2\,ds + r\frac{K_2}{K_1}Ma^2(3\rho)$ for all $t \in [t_0, t_\varepsilon]$. The previous inequality, in conjunction with the Gronwall–Bellman lemma, implies $|x(t)|^2 \leq \exp(\frac{(2L+K_2)r}{K_1})\frac{K_2}{K_1}(|x(t_0)|^2 + rMa^2(3\rho))$ for all $t \in [t_0, t_\varepsilon]$. Again, the previous inequality, in conjunction with (1.94), the fact that $t_\varepsilon < \tau_1 \leq t_0 + r$, and inequality $|x(t_0)| \leq \rho$, directly implies that $|x(t_\varepsilon)| \leq \frac{\varepsilon_1}{2} < \varepsilon_1$, which contradicts $|x(t_\varepsilon)| = \varepsilon_1$. We conclude that $|x(t)| < \varepsilon_1$ for all $t \in [t_0, \tau_1)$.

By virtue of uniform continuity of the solution on the interval $[t_0, \tau_1)$ (notice that by (1.59) $\dot{x}(t)$ is bounded on $[t_0, \tau_1)$), it follows that the limit $\lim_{t\to\tau_1^-} x(t)$ exists and satisfies $|\lim_{t\to\tau_1^-} x(t)| \leq \varepsilon_1$. Using (1.60) in conjunction with (1.95), (1.96), the facts that $t_0 < \tau_1 \leq t_0 + r$, $|x(t_0)| + \sup_{t\geq 0}|u(t)| < \delta$, and $|\lim_{t\to\tau_1^-} x(t)| \leq \varepsilon_1$, we conclude that $|x(\tau_1)| \leq \varepsilon$. The previous inequality, combined with the facts that $\varepsilon_1 \leq \varepsilon$ and $|x(t)| \leq \varepsilon_1$ for all $t \in [t_0, \tau_1)$, implies that $|x(t)| \leq \varepsilon$ for all $t \in [t_0, \tau_1]$. Consequently, Claim 1 is proved. □

Using induction, the fact that $\tau_i \leq t_0 + ir$ for all nonnegative integers i (where $r > 0$ is the upper bound for h) and Claim 1, we may conclude that the following claim holds.

Claim 2 *For all $\varepsilon > 0$, $T > 0$, and integers $N > 0$, there exists $\delta := \delta(\varepsilon, T, N) > 0$ such that if $|x(t_0)| + \sup_{t\geq 0}|u(t)| \leq \delta$, then the unique solution of (1.57) starting from $x(t_0)$ at time $t_0 \in [0, T]$ and corresponding to inputs $(u, d) \in M_U \times M_D$ exists for all $t \in [t_0, \tau_N]$ and satisfies $|x(t)| \leq \varepsilon$ for all $t \in [t_0, \tau_N]$, where $\tau_{i+1} = \tau_i + h(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i))$, $i = 1, \ldots, N-1$, with $\tau_0 = t_0$.*

Next, consider the continuous function h_l involved in Hypothesis (A5). For this function, we have the following claim.

Claim 3 *For all $\varepsilon > 0$ and $T > 0$, there exists an integer $N := N(\varepsilon, T) > 0$ such that $\tau_N > T$, where $s := \min\{h_l(t, x, u); (t, x, u) \in [0, T] \times B[0, \varepsilon] \times B_U[0, \varepsilon]\}$,* *and the sequence* $\{\tau_i\}_{i=0}^{N}$ *satisfies* $\tau_{i+1} \geq \min\{q_b(\tau_i), \tau_i + s\}$, $i = 1, \ldots, N-1$, *with arbitrary initial condition* $\tau_0 \geq 0$, *where* $q_b(t) := \min\{T \in b; t < T\}$, *and* $b = \{T_i\}_{i=0}^{\infty}$ *is the partition involved in Hypothesis* (A5).

Proof of Claim 3 Let arbitrary $\varepsilon > 0$, $T > 0$. Since the set $[0, T] \times B[0, \varepsilon] \times B_U[0, \varepsilon]$ is compact ($U \subseteq \Re^m$ is closed) and h_l is continuous, we have $s > 0$. Consider the infinite sequence $\{y_i\}_{i=0}^{\infty}$ which satisfies $y_{i+1} = \min\{q_b(y_i), y_i + s\}$, $i = 1, 2, \ldots,$ with $y_0 = 0$. Using a contradiction argument, we can show that $y_i \to +\infty$, and consequently for every $T > 0$, there exists an integer $N > 0$ such that $y_N > T$. Consider arbitrary $\tau_0 \geq 0$ and an arbitrary sequence $\{\tau_i\}_{i=0}^{N}$ that satisfies $\tau_{i+1} \geq \min\{q_b(\tau_i), \tau_i + s\}$, $i = 1, \ldots, N-1$. By virtue of Theorem 1.6.1 in [33] (Comparison principle), we have $\tau_i \geq y_i$, $i = 1, \ldots, N$, which implies $\tau_N > T$. The proof of Claim 3 is complete. □

We are now ready to show the required property. Consider arbitrary $\varepsilon > 0$, $T, T' \in \Re^+$. Claim 3 implies that there exists an integer $N := N(\varepsilon, T + T' + r) > 0$ such that $\tau_N > T + T' + r$, where $s := \min\{h_l(t, x, u); (t, x, u) \in [0, T + T' + r] \times B[0, \varepsilon] \times B_U[0, \varepsilon]\}$, and the sequence $\{\tau_i\}_{i=0}^{N}$ satisfies $\tau_{i+1} \geq \min\{q_b(\tau_i), \tau_i + s\}$, $i = 1, \ldots, N-1$, with arbitrary initial condition $\tau_0 \geq 0$. On the other hand, by virtue of Claim 2, there exists $\delta := \delta(\varepsilon, T, N) > 0$ such that if $|x(t_0)| + \sup_{t \geq 0} |u(t)| \leq \delta$, then the unique solution of (1.57) starting from $x(t_0)$ at time $t_0 \in [0, T]$ and corresponding to inputs $(u, d) \in M_U \times M_D$ exists for all $t \in [t_0, \tau_N]$ and satisfies $|x(t)| \leq \varepsilon$ for all $t \in [t_0, \tau_N]$, where $\tau_{i+1} = \tau_i + h(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i))$, $i = 1, \ldots, N-1$, with $\tau_0 = t_0$. Hypothesis (A5) implies that the sequence $\{\tau_i\}_{i=0}^{N}$ satisfies the inequality $\tau_{i+1} \geq \min\{q_b(\tau_i), \tau_i + s\}$ for all integers i for which $\tau_i \leq T + T' + r$. If we assume that $\tau_N \leq T + T'$, then we obtain a contradiction, and thus we conclude that $\tau_N > T + T'$. It follows that if $|x(t_0)| + \sup_{t \geq 0} |u(t)| < \delta$, then the unique solution of (1.57) starting from $x(t_0)$ at time $t_0 \in [0, T]$ and corresponding to inputs $(u, d) \in M_U \times M_D$ exists for all $t \in [t_0, t_0 + T']$ and satisfies $|x(t)| \leq \varepsilon$ for all $t \in [t_0, t_0 + T']$. □

Since our interest is focused on sampled-data systems of the form (1.62) or (1.63) (with no impulses), we can use the following result.

Proposition 1.3 *Consider system* (1.57) *under Hypotheses* (A1–4) *and suppose that* $R(t, x, x_0, u, u_0, d, d_0) = x$ *for all* $(t, u, u_0, d, d_0, x, x_0) \in \Re^+ \times U \times U \times D \times D \times \Re^n \times \Re^n$. *Then* $0 \in \Re^n$ *is a robust equilibrium point from the input* $u \in M_U$ *for system* (1.57).

Proof Let $\varepsilon > 0$, $T, T' \in \Re^+$, and let $L > 0$ be the constant that satisfies (1.58) for the bounded set $S := [0, T + T'] \times [0, T + T'] \times B[0, \varepsilon] \times B[0, \varepsilon] \times B_U[0, \varepsilon] \times B_U[0, \varepsilon]$. Let $\rho > 0$ be the unique solution of the equation

$$\varepsilon^2 = \frac{4K_2}{K_1} \exp\left(\frac{(2L + K_2)(T + T')}{K_1}\right)\left(\rho^2 + \left(T + T'\right) M a^2(3\rho)\right) \tag{1.97}$$

where $r > 0$ is the upper bound for h, and define

$$\delta = \min\left\{\frac{\varepsilon}{2}, \rho\right\} \tag{1.98}$$

The proof will be made by contradiction. Suppose that there exists $(x_0, u) \in \Re^n \times M_U$ with $|x_0| + \sup_{t\geq 0}|u(t)| \leq \delta$ and

$$\sup\left\{\left|\phi(\tau, t_0, x_0, u, d)\right|; d \in M_D, \tau \in \left[t_0, t_0 + T'\right], t_0 \in [0, T]\right\} > \varepsilon$$

Consequently, there exist $d \in M_D$, $t_0 \in [0, T]$, and $t \in [t_0, t_0 + T']$ with $|x(t)| \geq \frac{3\varepsilon}{4}$ (where $x(t) = \phi(t, t_0, x_0, u, d)$). Consider the nonempty set $A = \{t \geq t_0 : |x(t)| > \frac{5\varepsilon}{8}\}$ and let $t_1 = \inf A$. Notice that since $|x_0| \leq \delta \leq \frac{\varepsilon}{2}$ and $|x(t)| \geq \frac{3\varepsilon}{4}$, it follows that $t_1 > t_0$ and $t_1 \leq t_0 + T'$. Furthermore, by the definition $t_1 = \inf A$ and the continuity of $x(t)$ (this holds only for the impulse-free case), we have $|x(t_1)| = \frac{5\varepsilon}{8}$ and $|x(t)| \leq \varepsilon$ for all $t \in [t_0, t_1]$. In this case, we consider the absolutely continuous function $V(t) = x'(t)Px(t)$, which by (1.93) satisfies the following inequality a.e. for $t \in [t_0, t_1]$:

$$\dot{V}(t) \leq (2L + K_2)\left|x(t)\right|^2 + K_2 M a^2(3\rho) \tag{1.99}$$

Using (1.99) and the inequality $K_1|x|^2 \leq x'Px \leq K_2|x|^2$, we obtain $|x(t)|^2 \leq \frac{K_2}{K_1}|x(t_0)|^2 + \frac{2L+K_2}{K_1}\int_{t_0}^{t}|x(s)|^2\,ds + (T + T')\frac{K_2}{K_1}Ma^2(3\rho)$ for all $t \in [t_0, t_1]$. The previous inequality, in conjunction with the Gronwall–Bellman lemma, implies

$$\left|x(t)\right|^2 \leq \exp\left(\frac{(2L + K_2)(T + T')}{K_1}\right)\frac{K_2}{K_1}\left(\left|x(t_0)\right|^2 + \left(T + T'\right)Ma^2(3\rho)\right)$$
$$\forall t \in [t_0, t_1]$$

The previous inequality, together with (1.97) and the inequality $|x(t_0)| \leq \rho$, directly implies that $|x(t_1)| \leq \frac{\varepsilon}{2}$, which contradicts $|x(t_1)| = \frac{5\varepsilon}{8}$. The proof is complete. □

1.5 Feedback Interconnection of Systems

We next give the notion of the interconnection or feedback connection of control systems. See Fig. 1.2 for an illustration.

Definition 1.6 Consider a pair of control systems

$$\Sigma_1 = \left(\mathcal{X}_1, Y_1, M_{S_2\times U}, M_D, \tilde{\phi}_1, \pi_1, H_1\right)$$
$$\Sigma_2 = \left(\mathcal{X}_2\mathcal{Y}_2, M_{S_1\times U}, M_D, \tilde{\phi}_2, \pi_2, H_2\right)$$

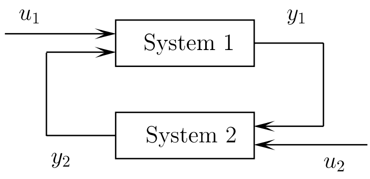

Fig. 1.2 Configuration of interconnected systems

with outputs $H_1 : \Re^+ \times \mathcal{X}_1 \times \mathcal{Y}_2 \times U \to S_1 \subseteq \mathcal{Y}_1$, $H_2 : \Re^+ \times \mathcal{X}_2 \times \mathcal{Y}_1 \times U \to S_2 \subseteq \mathcal{Y}_2$, and the BIC property and for which $0 \in \mathcal{X}_i$, $i = 1, 2$, are robust equilibrium points from the inputs $(v_2, u) \in M_{S_2 \times U}$ and $(v_1, u) \in M_{S_1 \times U}$, respectively. Suppose that there exists a *unique* mapping $\phi = (\phi_1, \phi_2) : A_\phi \to \mathcal{X}$ and a set-valued map $\Re^+ \times \mathcal{X} \times M_U \times M_D \ni (t_0, x_0, u, d) \to \pi(t_0, x_0, u, d) \subseteq [t_0, +\infty)$, where $A_\phi \subseteq \Re^+ \times \Re^+ \times \mathcal{X} \times M_U \times M_D$ and $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, such that, for every $(t, t_0, x_0, u, d) \in A_\phi$ with $t \geq t_0$ and $x_0 = (x_{1,0}, x_{2,0}) \in \mathcal{X}_1 \times \mathcal{X}_2$, it holds that there exists a pair of external inputs $v_i \subseteq \mathcal{M}(S_i)$, $i = 1, 2$, with $v_1(\tau) = H_1(\tau, \phi_1(\tau, t_0, x_0, u, d), v_2(\tau), u(\tau))$, $v_2(\tau) = H_2(\tau, \phi_2(\tau, t_0, x_0, u, d), v_1(\tau), u(\tau))$ for all $\tau \in [t_0, t]$, $(v_i, u) \in M_{S_i \times U}$, $i = 1, 2$, so that $\pi(t_0, x_0, u, d) = \pi_1(t_0, x_{1,0}, (v_2, u), d) \cap \pi_2(t_0, x_{2,0}, (v_1, u), d)$ and $\phi_1(\tau, t_0, x_0, u, d) = \tilde{\phi}_1(\tau, t_0, x_{1,0}, (v_2, u), d)$, $\phi_2(\tau, t_0, x_0, u, d) = \tilde{\phi}_2(\tau, t_0, x_{2,0}, (v_1, u), d)$ for all $\tau \in [t_0, t]$.

Moreover, let $\mathcal{Y}$ be a normed linear space, and $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ a continuous map that maps bounded sets of $\Re^+ \times \mathcal{X} \times U$ into bounded sets of $\mathcal{Y}$ with $H(t, 0, 0) = 0$ for all $t \geq 0$ and suppose that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is a control system with outputs and the BIC property, for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Then, system Σ is said to be the *feedback connection* or the *interconnection* of systems Σ_1 and Σ_2.

It should be emphasized that the feedback interconnection of two systems may create a system which has different qualitative properties from each one of the systems connected. For example, if we connect a control system described by RFDEs with a control system with impulses at fixed times, then the overall system will be a system with both "memory" and impulses (discontinuous systems described by RFDEs; see [49]).

The feedback interconnection of two systems is not always well defined. For example, the uniqueness of the map $\phi = (\phi_1, \phi_2) : A_\phi \to \mathcal{X}_1 \times \mathcal{X}_2$ requires regularity properties which must be satisfied. The nature of the additional properties that guarantee that the feedback interconnection of two systems is well defined depends heavily on the nature of the overall system. The following example illustrates this point.

Example 1.5.1 Consider the following feedback interconnection of two systems described by ODEs:

$$\begin{aligned} \dot{x} &= f(t, d, x, u) \\ v &= p(t, x, u) \\ &x \in \Re^n, u \in U \subseteq \Re^m, d \in D \subseteq \Re^l \end{aligned} \tag{1.100}$$

and

$$\begin{aligned} \dot{w} &= g(t, w, v) \\ u &= k(t, w, v) \\ &w \in \Re^k, v \in V \subseteq \Re^q \end{aligned} \tag{1.101}$$

where systems (1.100) and (1.101) satisfy Hypotheses (H1–4). In order the feedback interconnection of systems (1.100), (1.101) to be well defined, we must require that for any given (t,x,w), there exists a unique pair of solutions $u=k^*(t,x,w)$, $v=p^*(t,x,w)$ to the algebraic equations $v=p(t,x,u)$ and $u=k(t,w,v)$. For example, this requirement is not fulfilled if $k(t,w,v):=v\in\Re$ and $p(t,x,u)=2u-u^3$ with $u\in\Re$. If for any given (t,x,w), there exists a unique pair of solutions $u=k^*(t,x,w)$, $v=p^*(t,x,w)$ to the algebraic equations $v=p(t,x,u)$, $u=k(t,w,v)$ and if all maps (f,g,p^*,k^*) are assumed to be locally Lipschitz, then the total interconnected system takes the following form:

$$\begin{aligned}\dot{x}&=f\big(t,d,x,k^*(t,x,w)\big)\\ \dot{w}&=g\big(t,w,p^*(t,x,w)\big)\\ &x\in\Re^n,\ w\in\Re^k,\ d\in D\subseteq\Re^l\end{aligned}\tag{1.102}$$

Indeed, if all maps (f,g,p^*,k^*) are assumed to be locally Lipschitz, then Hypotheses (H1–4) automatically hold for system (1.102), and the feedback interconnection of systems (1.100), (1.101) is well defined.

1.6 Transformation of Systems

Definition 1.7 Let $\mathcal{X}$ be a normed linear space. We say that a map $\Phi:\Re^+\times\mathcal{X}\to\mathcal{X}$ with $\Phi(t,0)=0$ for all $t\geq 0$ is *a change of coordinates* if:

(i) $\Phi(t,\mathcal{X})=\mathcal{X}$ for all $t\geq 0$;
(ii) For every $t\geq 0$, the following implication holds:

$$\Phi(t,x_1)=\Phi(t,x_2)\quad\Rightarrow\quad x_1=x_2;\tag{1.103}$$

(iii) there exist a continuous map (called the inverse map) $\Phi^{-1}:\Re^+\times\mathcal{X}\to\mathcal{X}$ and functions $a\in K_\infty$, $\beta\in K^+$ such that

$$x=\Phi\big(t,\Phi^{-1}(t,x)\big)=\Phi^{-1}\big(t,\Phi(t,x)\big)\quad\text{for all }(t,x)\in\Re^+\times\mathcal{X}\tag{1.104}$$

$$\begin{aligned}&\big\|\Phi(t,x)\big\|_{\mathcal{X}}\leq a\big(\beta(t)\|x\|_{\mathcal{X}}\big)\quad\text{and}\quad\big\|\Phi^{-1}(t,x)\big\|_{\mathcal{X}}\leq a\big(\beta(t)\|x\|_{\mathcal{X}}\big)\\ &\quad\text{for all }(t,x)\in\Re^+\times\mathcal{X}.\end{aligned}\tag{1.105}$$

Definition 1.8 Let U, V be subsets of the normed linear spaces $\mathcal{U}_1,\mathcal{U}_2$, respectively, with $0\in U$ and $0\in V$. A continuous mapping $q:\Re^+\times V\to U$ with $q(t,V)=U$ and $q(t,0)=0$ for all $t\geq 0$ for which there exist functions $a\in K_\infty$ and $\mu\in K^+$ such that $\|q(t,v)\|_{\mathcal{U}_1}\leq a(\mu(t)\|v\|_{\mathcal{U}_2})$ is called a transformation of V onto U.

The following lemma can be verified easily by the reader.

Lemma 1.2 *Consider a deterministic control system* $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,\pi,H)$ *with outputs satisfying the BIC property. Let* $\Phi:\Re^+\times\mathcal{X}\to\mathcal{X}$ *be a change of coordinates, and* $q:\Re^+\times V\to U$ *a transformation of* V *onto* U. *Then*

$\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ is a deterministic control system satisfying the BIC property, where M_V is the set of all mappings $v : \Re^+ \to V$ with the property that the mapping $Qv : \Re^+ \to U$ defined by $(Qv)(t) = q(t, v(t))$ for all $t \geq 0$ satisfies $Qv \in M_U$ and ϕ', π', H' are defined by the following equations:

$$\phi'(t, t_0, x_0, v, d) = \Phi\big(t, \phi\big(t, t_0, \Phi^{-1}(t_0, x_0), Qv, d\big)\big) \tag{1.106}$$

$$\pi'(t_0, x_0, v, d) = \pi\big(t_0, \Phi^{-1}(t_0, x_0), Qv, d\big) \tag{1.107}$$

$$H'(t, x, v) := H\big(t, \Phi^{-1}(t, x), q(t, v)\big) \tag{1.108}$$

Moreover, if $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ for Σ, then $0 \in \mathcal{X}$ is a robust equilibrium point from the input $v \in M_V$ for Σ'.

The system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', H')$ provided by Lemma 1.2 is called the *transformed system Σ under the change of coordinates $\Phi : \mathcal{X} \to \mathcal{X}$ and the input transformation $q : \Re^+ \times V \to U$*. Notice that if the weak semigroup property holds for Σ, then the weak semigroup property holds for Σ'.

Lemma 1.2 will help in the study of a given system, since very often it is possible to find a change of coordinates $\Phi : \mathcal{X} \to \mathcal{X}$ and an input transformation $q : \Re^+ \times V \to U$ such that the study of the transformed system Σ' is easier than the study of the original system Σ. Moreover, transformations of systems help us to study systems which do not necessarily fall into one of the categories of systems studied in Sect. 1.2. The following example illustrates the effect of the transformation for systems described by ODEs.

Example 1.6.1 Consider system (1.3) under Hypotheses (H1–4). Let $\Phi : \Re^+ \times \Re^n \to \Re^n$ be a change of coordinates, and $q : \Re^+ \times V \to U$ a transformation of $V \subseteq \Re^l$ onto $U \subseteq \Re^m$. Moreover, suppose that $\Phi : \Re^+ \times \Re^n \to \Re^n$ is continuously differentiable. Then, the transformed system $\Sigma' := (\Re^n, \Re^k, M_V, M_D, \phi', H')$ under the change of coordinates $\Phi : \Re^+ \times \Re^n \to \Re^n$ and the input transformation $q : \Re^+ \times V \to U$ is a control system described by the following equations:

$$\begin{aligned}
\dot{x}(t) &= \frac{\partial \Phi}{\partial t}\big(t, \Phi^{-1}\big(t, x(t)\big)\big) \\
&\quad + \frac{\partial \Phi}{\partial x}\big(t, \Phi^{-1}\big(t, x(t)\big)\big) f\big(t, \Phi^{-1}\big(t, x(t)\big), q\big(t, v(t)\big), d(t)\big) \\
Y(t) &= H\big(t, \Phi^{-1}\big(t, x(t)\big), q\big(t, v(t)\big)\big) \\
&\quad x(t) \in \Re^n, Y(t) \in \Re^k, v(t) \in V, d(t) \in D
\end{aligned} \tag{1.109}$$

Notice that Hypotheses (H1–4) are not necessarily satisfied for system (1.109). However, Lemma 1.2 guarantees that the system $\Sigma' := (\Re^n, \Re^k, M_V, M_D, \phi', H')$ is a deterministic control system which satisfies both the classical semigroup property and the BIC property. Moreover, $0 \in \Re^n$ is a robust equilibrium point from the input $v \in M_V$ for Σ'. Finally, notice that although Hypothesis (H1) is not necessarily satisfied for (1.109), if $\Phi^{-1} : \Re^+ \times \Re^n \to \Re^n$ is continuously differentiable, we can guarantee that the initial value problem for (1.109) has a unique solution for all $(v, d) \in M_V \times M_D$.

1.7 Discrete-Time Systems

Let $\mathcal{X}, \mathcal{Y}, \mathcal{U}$ be a triplet of normed linear spaces, D, $U \subseteq \mathcal{U}$ nonempty sets with $0 \in U$, $\pi = \{\tau_i\}_{i=0}^{\infty}$ a partition of $\Re^+$ with diameter $r > 0$, i.e., an increasing sequence of times with $\tau_0 = 0$, $\sup\{\tau_{i+1} - \tau_i; i = 0, 1, 2, \ldots\} = r$ and $\tau_i \to +\infty$, and $f : \pi \times D \times \mathcal{X} \times U \to \mathcal{X}$, $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ mappings with $f(t, d, 0, 0) = 0$ and $H(t, 0, 0) = 0$ for all $(t, d) \in \pi \times D$. We define $p_\pi(t) = \max\{\tau \in \pi : \tau \leq t\}$ and $q_\pi(t) = \min\{\tau \in \pi : \tau > t\}$. We consider the system that produces for each $(t_0, x_0) \in \Re^+ \times \mathcal{X}$ and for each pair of piecewise constant, right-continuous mappings $u : \Re^+ \to U$ and $d : \Re^+ \to D$ which are continuous on $\Re^+ \backslash \pi$, the piecewise constant function $t \to x(t) \in \mathcal{X}$, based on the following algorithm:

Step 0: Set $x(t) = x_0$, for all $t \in [t_0, q_\pi(t_0))$ and $x(q_\pi(t_0)) = f(p_\pi(t_0), d(p_\pi(t_0)), x_0, u(p_\pi(t_0)))$.
Step i: Given $\tau_i \in \pi$ and $x(\tau_i)$, set $x(t) = x(\tau_i)$ for all $t \in [\tau_i, \tau_{i+1})$ and $x(\tau_{i+1}) = f(\tau_i, d(\tau_i), x(\tau_i), u(\tau_i))$.

Schematically, we write

$$\begin{aligned} x(t) &= x(\tau_i) \quad t \in [\tau_i, \tau_{i+1}) \\ x(\tau_{i+1}) &= f\big(\tau_i, d(\tau_i), x(\tau_i), u(\tau_i)\big) \\ Y(t) &= H\big(t, x(t), u(t)\big) \end{aligned} \tag{1.110}$$

with initial condition $x(t_0) = x_0 \in \mathcal{X}$. We consider systems of the form (1.110) under the following hypotheses:

(L1) There exist functions $a \in K_\infty$ and $\beta \in K^+$ such that $\|f(t, d, x, u)\|_{\mathcal{X}} \leq a(\beta(t)\|x\|_{\mathcal{X}}) + a(\beta(t)\|u\|_{\mathcal{U}})$ for all $(t, x, d, u) \in \pi \times \mathcal{X} \times D \times U$.
(L2) For all $(\tau_i, x, u) \in \pi \times \mathcal{X} \times U$ and $t \in [\tau_i, \tau_{i+1})$, it holds that $H(t, x, u) = H(\tau_i, x, u)$. Moreover, for each fixed $\tau_i \in \pi$, the mapping $\mathcal{X} \times U \ni (x, u) \to H(\tau_i, x, u)$ is uniformly continuous on each bounded set in $\mathcal{X} \times U$.
(L3) For every pair of bounded sets $I \subset \pi$, $S \subset \mathcal{X} \times U$, the set $H(I \times S)$ is bounded.

Let M_D be the set of piecewise constant, right-continuous mappings $d : \Re^+ \to D$ that are continuous on $\Re^+ \backslash \pi$ (i.e., mappings $d : \Re^+ \to D$ for which there exists a sequence $\{d_i \in D\}_{i=0}^{\infty}$ such that for every $\tau_i \in \pi$, it holds that $d(t) = d_i$ for all $t \in [\tau_i, \tau_{i+1})$). Similarly, let M_U be the set of piecewise constant, right-continuous mappings $u : \Re^+ \to U$ that are continuous on $\Re^+ \backslash \pi$.

Systems of the form (1.110) are called discrete-time systems and have been studied extensively in the literature for the case $\pi = Z^+$. We next provide an example of an economic system that can be modeled by means of a discrete-time system.

Example 1.7.1 (The Cobweb model with stocks) We consider the evolution of the price of a certain commodity in a completely competitive market at discrete periods: each period $t + 1$ is characterized by the release in the market of $S(t + 1)$ units of the commodity, which are supplied by certain agents and by the release $G(t)$ units of the commodity by the government. Let us denote by $P(t)$ the price of the commodity at period t, and by $Q(t)$ the quantity of the commodity stored by the government at period t. When $G(t) < 0$, then the government actually buys and stores $|G(t)|$ commodity units at period $t+1$. Let $Q_{\max} > 0$ denote the storage capability for the particular commodity. Consequently, the following intrinsic constraints must be satisfied for all periods:

$$G(t) \le Q(t) \quad \text{and} \quad 0 \le Q(t) \le Q_{\max} \quad \text{for all } t \tag{1.111}$$

Furthermore, the quantity of the commodity stored by the government must obey the following difference equation for all periods:

$$Q(t+1) = Q(t) - G(t) \tag{1.112}$$

Since the quantity of the commodity stored by the government at period $t+1$ cannot exceed $Q_{\max} > 0$ (the storage capability), i.e., $Q(t+1) \le Q_{\max}$, it follows from (1.112) that the following constraint for the quantity of the commodity released to the market by the government at period $t+1$ must be obeyed:

$$Q(t) - Q_{\max} \le G(t) \quad \text{for all } t \tag{1.113}$$

We assume a linear strictly decreasing demand function

$$D(t) = a - bP(t) \quad a, b > 0 \tag{1.114}$$

and an S-shaped monotonic piecewise linear supply function:

$$S(t+1) = g\big(P^e(t+1)\big) \tag{1.115}$$

$$g(P) = \max\big\{0; \min\{S_{\max}; -c + dP\}\big\} \quad c, d, S_{\max} > 0 \tag{1.116}$$

where $P^e(t+1)$ is the expected price for the period $t+1$, and $S_{\max}$ is the highest supply level for the commodity. The total quantity of the product available in the market at period $t+1$ is given by $S(t+1) + G(t)$, and therefore the following constraint must be obeyed in addition to the previous constraints:

$$G(t) \ge -g\big(P^e(t+1)\big) \quad \text{for all } t \tag{1.117}$$

In a completely competitive market, market equilibrium is attained instantaneously:

$$D(t+1) = S(t+1) + G(t) \tag{1.118}$$

Finally, we assume the so-called naïve (or myopic) expectation,

$$P^e(t+1) = P(t) \tag{1.119}$$

which is met frequently in textbooks.

Taking into account (1.111)–(1.119), the unique solution of (1.118) gives the following discrete-time control system:

$$\begin{aligned}
&P(t+1) = b^{-1}\big(a - G(t) - g\big(P(t)\big)\big) \\
&Q(t+1) = Q(t) - G(t) \\
&\max\big\{Q(t) - Q_{\max}; -g\big(P(t)\big)\big\} \le G(t) \le Q(t)
\end{aligned} \tag{1.120}$$

We assume that $a > Q_{\max} + S_{\max}$ and $bc < ad$ (conditions for the viability of the commodity). It is convenient to introduce the dimensionless variables

$$x_1(t) := ba^{-1}P(t-1) - x_{eq} \qquad x_2(t) = a^{-1}Q(t)$$

$$G(t) = \min\{Q(t);\ \max\{au(t);\ Q(t) - Q_{\max};\ -\max\{0;\ \min\{S_{\max};\ dP(t) - c\}\}\}\} \tag{1.121}$$

and the dimensionless constants $r := db^{-1} > 0$, $c_1 := \frac{bc}{ad} < 1$, $c_2 := \frac{bS_{\max}}{ad} < \frac{b}{d}$, $c_3 = a^{-1}Q_{\max}$, and

$$x_{eq} = \begin{cases} \frac{1+rc_1}{1+r} & \text{if } c_1 + c_2 > 1 - c_2 r \\ 1 - c_2 r & \text{if } c_1 + c_2 \le 1 - c_2 r \end{cases} \tag{1.122}$$

Thus the control system (1.120) expressed in state space form and in dimensionless coordinates is given by the following discrete-time control system:

$$\begin{aligned} x_1(t+1) &= f_1\big(x_1(t)\big) - f_2\big(x_1(t), x_2(t), u(t)\big) \\ x_2(t+1) &= L\big(x_2(t)\big) - f_2\big(x_1(t), x_2(t), u(t)\big) \\ x(t) &= \big(x_1(t), x_2(t)\big) \in \Re^2,\ u(t) \in \Re \end{aligned} \tag{1.123}$$

where

$$\begin{aligned} f_1(x) &:= 1 - x_{eq} - r\max\{0;\ \min\big(c_2;\ P(x + x_{eq}) - c_1\big)\} \\ f_2(x, y, u) &:= \min\{L(y);\ \max\{u;\ L(y) - c_3;\ f_1(x) + x_{eq} - 1\}\} \\ P(x) &:= \max\{0;\ \min(1; x)\} \\ L(y) &:= \max\{0;\ \min(c_3; y)\} \end{aligned}$$

Notice that $f_1(0) = f_2(0,0,0) = 0$. Define $\mathcal{X} = \Re^2$, $\mathcal{Y} = \Re$, and the output map $H(t,x,u) := x_1$. Hypotheses (L1), (L2), (L3) thus hold for system (1.123).

We denote by $\phi(t, t_0, x_0, u, d) := x(t)$ the solution of (1.110) with initial condition $x(t_0) = x_0 \in \mathcal{X}$, corresponding to inputs $(u,d) \in M_U \times M_D$. It should be clear that, by the previous definitions, we have defined the system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with $A_\phi \subseteq \Re^+ \times \Re^+ \times \mathcal{X} \times M_U \times M_D$ given by the set of points $(t, t_0, x_0, u, d) \in \Re^+ \times \Re^+ \times \mathcal{X} \times M_U \times M_D$, where $t \in [t_0, +\infty)$ and $\pi(t_0, x_0, u, d) = [t_0, +\infty)$ (notice that the classical semigroup property holds).

Notice that if $f(t+T, d, x, u) = f(t, d, x, u)$, $H(t+T, x, u) = H(t, x, u)$ and $\pi = \{i\frac{T}{l}\}_{i=0}^{\infty}$ for certain $T > 0$ and certain integer $l > 0$ and for all $(t, d, u, x) \in \pi \times D \times U \times \mathcal{X}$, then system (1.110) is T-periodic.

It is worth noting the following important fact for system (1.110):

Lemma 1.3 *System* (1.110) *under Hypothesis* (L1) *is Robustly Forward Complete* (RFC) *from the input* $u \in M_U$.

Concerning the proof of Lemma 1.3, we consider arbitrary $R \geq 0$, $T \in \Re^+$ and then define recursively the sequence of sets in $\mathcal{X}$ by $A(k) := A(k-1) \cup f((\pi \cap [0, 2T]) \times D \times A(k-1) \times B_U[0, R])$ for $k = 1, \ldots, l$, where $l \in Z^+$ satisfies $\tau_l = p_\pi(2T)$, $B_U[0, R] := \{u \in U; \|u\|_{\mathcal{U}} \leq R\}$, and $A(0) := \{x \in \mathcal{X}; \|x\|_{\mathcal{X}} \leq R\}$, which are bounded (by virtue of Hypothesis (L1)), and finally notice that

$$\big\{\phi(t+s, t_0, x_0, u, d);\ \|x_0\|_{\mathcal{X}} \leq R, t_0 \in [0, T], s \in [0, T], d \in M_D,\\ u \in \mathcal{M}\big(B_U[0, R]\big) \cap M_U\big\} \subseteq A(l)$$

We end this section by showing the following fact.

Lemma 1.4 $0 \in \mathcal{X}$ *is a robust equilibrium point from the input* $u \in M_U$ *for system* (1.110) *under Hypothesis* (L1).

The proof of Lemma 1.4 relies on the following result.

Lemma 1.5 *Consider system* (1.110) *under Hypothesis* (L1). *For all* $i, N \in Z^+$ *and* $\varepsilon > 0$, *there exists* $\delta(i, N, \varepsilon) > 0$ *such that the following implication holds*:

$$\begin{aligned} &\big\|x(\tau_i)\big\|_{\mathcal{X}} + \sup_{t \geq 0}\big\|u(t)\big\|_{\mathcal{U}} \leq \delta(i, N, \varepsilon)\\ &\quad\Rightarrow\quad \sup\big\{\big\|x(\tau_{i+s})\big\|_{\mathcal{X}} : d \in M_D, s \in \{0, \ldots, N\}\big\} \leq \varepsilon \end{aligned} \tag{1.124}$$

Proof We prove the Lemma 1.5 by induction on $N \in Z^+$. Notice that implication (1.124) holds for $N = 0$ (select $\delta(i, 0, \varepsilon) = \varepsilon > 0$). Next, suppose that implication (1.124) holds for certain $N \in Z^+$. Define

$$\delta(i, N+1, \varepsilon) := \min\left\{\delta(i, N, \varepsilon), \delta\left(i, N, \frac{1}{\beta(\tau_{i+N})}a^{-1}\left(\frac{\varepsilon}{2}\right)\right)\right\} \tag{1.125}$$

where $a \in K_\infty$ and $\beta \in K^+$ are the functions involved in Hypothesis (L1). By definition (1.125) and implication (1.124) it follows that

$$\begin{aligned} &\big\|x(\tau_i)\big\|_{\mathcal{X}} + \sup_{t \geq 0}\big\|u(t)\big\|_{\mathcal{U}} \leq \delta(i, N+1, \varepsilon)\\ &\quad\Rightarrow\quad \sup\big\{\big\|x(\tau_{i+s})\big\|_{\mathcal{X}} : d \in M_D, s \in \{0, \ldots, N\}\big\} \leq \varepsilon \end{aligned} \tag{1.126}$$

$$\begin{aligned} &\big\|x(\tau_i)\big\|_{\mathcal{X}} + \sup_{t \geq 0}\big\|u(t)\big\|_{\mathcal{U}} \leq \delta(i, N+1, \varepsilon)\\ &\quad\Rightarrow\quad \big\|x(\tau_{i+N})\big\|_{\mathcal{X}} \leq \frac{1}{\beta(\tau_{i+N})}a^{-1}\left(\frac{\varepsilon}{2}\right) \end{aligned} \tag{1.127}$$

Inequality (1.127), in conjunction with Hypothesis (L1) and the fact that $\|u(\tau_{i+N})\|_{\mathcal{U}} \leq \frac{1}{\beta(\tau_{i+N})}a^{-1}(\frac{\varepsilon}{2})$ (which holds if $\|x(\tau_i)\|_{\mathcal{X}} + \sup_{t \geq 0}\|u(t)\|_{\mathcal{U}} \leq \delta(i, N+1, \varepsilon)$; see definition (1.125)), implies that

$$\begin{aligned} &\big\|x(\tau_i)\big\|_{\mathcal{X}} + \sup_{t \geq 0}\big\|u(t)\big\|_{\mathcal{U}} \leq \delta(i, N+1, \varepsilon)\\ &\quad\Rightarrow\quad \sup\big\{\big\|x(\tau_{i+N+1})\big\|_{\mathcal{X}} : d \in M_D\big\} \leq \varepsilon \end{aligned} \tag{1.128}$$

Combining (1.126) and (1.128), we conclude that implication (1.124) holds for $N+1$. The proof is complete. □

We are now ready to prove Lemma 1.4.

Proof of Lemma 1.4 It suffices to show that for all $\varepsilon > 0$, $N \in Z^+$, and $l \in Z^+$ with $N \geq l$, there exists $\delta := \delta(\varepsilon, N, l) \in (0, \varepsilon]$ such that

$$\|x_0\|_{\mathcal{X}} + \sup_{t \geq 0} \|u(t)\|_{\mathcal{U}} \leq \delta, \quad t_0 \in \{\tau_0, \ldots, \tau_l\}$$
$$\Rightarrow \quad \sup\{\|\phi(\tau_i, t_0, x_0, u, d)\|_{\mathcal{X}}; \tau_i \in \pi \cap [t_0, \tau_N], d \in M_D\} \leq \varepsilon$$

The above implication holds with $\delta := \delta(\varepsilon, N, l) = \min_{i=0,\ldots,l} \delta(i, N, \varepsilon)$, where $\delta(i, N, \varepsilon) > 0$ is provided by Lemma 1.5. The proof is complete. □

1.8 Bibliographical and Historical Notes

(1) Abstract definitions of control systems with outputs were presented in [21, 46]. A definition similar in spirit to the definitions given in [21, 46] was later presented in [22]. A qualitatively different definition of a control system (which is adopted in the present text) was given in [26, 27]. The difference between Definition 1.1 of a control system with outputs and Definition 2.1 in [22] lies in property 4 (semigroup property). In the above definition we do not require $\pi(t_0, x_0, u, d) = [t_0, t_{\max})$ (in contrast with Definition 2.1 in [22]). This modification allows us to study important classes of systems, which were excluded by Definition 2.1 in [22].

(2) It should be emphasized that there are systems that do not satisfy the weak semigroup property (e.g., systems described by integrodifferential equations studied in [32], such as $\dot{x}(t) = -x(t) + \int_{t_0}^{t} \sin(tx(s))\, ds$, $x(t) \in \Re$ with initial condition $x(t_0) = x_0 \in \Re$).

(3) Initial-value problems of the form (1.18), (1.19), (1.20) arise in electrical, thermal, and hydraulic engineering (see, for instance, the model of combined heat and electricity generation and other models reported in [35, 42] concerning lossless transmission lines with electrical circuits and turbines under water-hammer conditions).

(4) Other concepts of weak solutions for linear neutral functional differential equations were given in [5, 17]. In recent works control-theoretic aspects for linear neutral functional differential equations are studied (see [15, 16]).

(5) It should be noticed that recent contributions in the literature study systems of coupled RFDEs and FDEs of the form (1.18), (1.19), (1.20) per se (see, for instance, [35, 38, 39] and references therein). The difference between [35, 38, 39] and the recent work in [31] is that the matching condition is not assumed to hold in [31]. This is the viewpoint adopted in the present work.

(6) Sampled-data feedback (1.62) has been considered in [7, 13, 25, 34, 44, 47]. Particularly, Theorem 9.3.1 in [13] provides links to the classical results in [7, 47]. Moreover, control systems under a hybrid feedback law with asynchronous switching rules (as given in [43]) can be modeled as systems of the form (1.57).

(7) Notice that there is a difference between Definition 1.6 and Definition 7.2.3 in [46]: we do not exclude interconnections of control systems that may have finite escape time. Moreover, we can allow $\Sigma_2 = (\mathcal{X}_2, Y_2, M_{S_1 \times U}, M_D, \tilde{\phi}_2, \pi_2, H_2)$ to be just a continuous map from $\Re^+ \times \mathcal{Y}_1$ into $\mathcal{Y}_2$ (a static map—this is allowed as well by Definition 7.2.3 in [46]). Of course, usually the continuity of a static map is not enough to guarantee that there is an interconnection of two subsystems. More specifically, in order to guarantee the uniqueness of the map $\phi = (\phi_1, \phi_2) : A_\phi \to \mathcal{X}_1 \times \mathcal{X}_2$, other regularity properties must be satisfied as well depending on the nature of the overall system. The well-posedness of the interconnection of two finite-dimensional systems described by ordinary differential equations is addressed in [20]; see also [9] for the description of feedback systems using input–output approaches.

(8) Discrete-time systems on normed linear spaces were studied in [23, 24]. Discrete-time systems with disturbances were studied in [18, 19].

(9) The chemostat model (1.4) with variable yield coefficients has been studied recently (see [50, 51]) and has been proposed for the justification of experimental results. The delayed chemostat model (1.11) has been studied in [29].

(10) The uncertain dynamic model for the Cournot oligopoly game (1.53) was studied in [28]. See also [1, 4, 6, 10, 12, 36, 37] for other studies for the dynamic Cournot oligopoly game.

(11) The discrete-time model (1.123) for the cobweb model with stocks was proposed in [2].

(12) The study of numerical schemes for ordinary differential equations by means of hybrid systems was proposed recently in [26, 30]. Usually, numerical schemes for ordinary differential equations were studied by means of discrete-time systems (see, for instance, [33, 48]).

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Chapter 2
Internal Stability: Notions and Characterizations

2.1 Introduction

This chapter is devoted to the analysis of internal global stability notions used in mathematical control and systems theory. The stability notions presented are developed in the system-theoretic framework described in the previous chapter so that one can obtain a wide perspective of the role of stability in various classes of deterministic systems.

The stability notions developed in this chapter are referred to as "internal" because these notions are "uniform" with respect to the effect of external inputs (e.g., disturbances). Therefore, the notions are applicable to both certain and uncertain systems, and can be regarded as direct extensions of the conventional stability notions for systems with no external inputs, i.e., for the disturbance-free case.

A large part of this chapter is also devoted to the presentation of methods of proving stability. For testing global stability properties, there exist at least seven different methods in the literature, as shown below:

(1) The first method is based upon *analytical solutions*. It attempts to obtain basic estimates for the solutions of the system by actually solving the differential (or difference) equations (or inequalities).
(2) The second method is based upon *transformation techniques*. With this method basic estimates for the solutions of the system are derived by transforming the system into a different system with special properties.
(3) The third method is based on *Lyapunov functions and functionals*. With this method basic estimates for the solutions are derived by means of one (or many) Lyapunov functional(s) and comparison lemmas.
(4) The fourth method is based on *small-gain techniques*. With this method basic estimates for the solutions of the system are derived by means of small-gain arguments.
(5) The fifth method is based on *Matrosov functions*.
(6) The sixth method is based on *fixed point theorems*.
(7) Finally, one has the method based upon *dissipative systems theory*.

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Due to the intimate connection of small-gain results to external stability notions (i.e., stability notions which are *not* uniform with respect to the effect of external inputs), the small-gain methods of proving stability will be described separately in the following chapters. However, the first three methods are explained in detail in the present chapter. Finally, it should be emphasized that the different methods of proving stability can be, and often are, combined. For example, one can use Lyapunov functionals or analytical solutions to obtain basic estimates for the solutions and use small-gain arguments for the stability proof.

In what follows, $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ will be a control system with the BIC property, $U = \{0\}$, and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Moreover, $u_0 \in M_U$ will be the identically zero input, i.e., $u_0(t) = 0 \in U$ for all $t \geq 0$. For the output map $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$, we assume that either $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ is continuous or that there exists a partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\Re^+$ with diameter $r > 0$ such that $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ satisfies Hypothesis (L2) in Sect. 1.7 of the previous chapter.

2.2 Definitions of Robust Global Asymptotic Output Stability (RGAOS)

Every stability notion must be defined for systems with solutions that are defined for all times t greater than the initial time t_0. Consequently, the following definition plays an important role to the stability notions of the present chapter.

Definition 2.1 We say that a system Σ is *Robustly Forward Complete (RFC)* if it has the BIC property and for all $R \geq 0$ and $T \geq 0$, it holds that

$$\sup\left\{ \left\| \phi(t_0+s, t_0, x_0, u_0, d) \right\|_{\mathcal{X}}; s \in [0, T], \|x_0\|_{\mathcal{X}} \leq R, t_0 \in [0, T], d \in M_D \right\} < +\infty$$

We next provide the definitions of Robust Global Asymptotic Output Stability (RGAOS) and Uniform Robust Global Asymptotic Output Stability (URGAOS).

Definition 2.2 We say that Σ is *Robustly Globally Asymptotically Output Stable* (*RGAOS*) if Σ is RFC and the following properties hold:

P1. Σ is *Robustly Lagrange Output Stable*, i.e., for all $\varepsilon > 0$ and $T \geq 0$, it holds that

$$\sup\left\{ \left\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big) \right\|_{\mathcal{Y}}; t \geq t_0, \|x_0\|_{\mathcal{X}} \leq \varepsilon, t_0 \in [0, T], d \in M_D \right\} < +\infty$$

(Robust Lagrange Output Stability).

P2. Σ is *Robustly Lyapunov Output Stable*, i.e., for all $\varepsilon > 0$ and $T \geq 0$, there exists $\delta := \delta(\varepsilon, T) > 0$ such that

$$\|x_0\|_{\mathcal{X}} \le \delta \quad t_0 \in [0, T]$$
$$\Rightarrow \quad \big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \le \varepsilon \quad \forall t \ge t_0,\ \forall d \in M_D.$$

(Robust Lyapunov Output Stability).

P3. Σ satisfies the *Robust Output Attractivity Property*, i.e., for all $\varepsilon > 0$, $T \ge 0$, and $R \ge 0$, there exists $\tau := \tau(\varepsilon, T, R) \ge 0$ such that

$$\|x_0\|_{\mathcal{X}} \le R \quad t_0 \in [0, T]$$
$$\Rightarrow \quad \big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \le \varepsilon \quad \forall t \ge t_0 + \tau, \forall d \in M_D.$$

Moreover, if $H(t, x, u) \equiv x$, then we say that Σ is *Robustly Globally Asymptotically Stable* (*RGAS*).

Definition 2.3 We say that Σ is *Uniformly Robustly Globally Asymptotically Output Stable* (*URGAOS*) if Σ is RFC and the following properties hold:

P1. Σ is *Uniformly Robustly Lagrange Output Stable*, i.e., for every $\varepsilon > 0$, it holds that

$$\sup\big\{\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}}; t \ge t_0, \|x_0\|_{\mathcal{X}} \le \varepsilon, t_0 \ge 0, d \in M_D\big\}$$
$$< +\infty$$

(Uniform Robust Lagrange Output Stability).

P2. Σ is *Uniformly Robustly Lyapunov Output Stable*, i.e., for every $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon) > 0$ such that

$$\|x_0\|_{\mathcal{X}} \le \delta \quad t_0 \ge 0$$
$$\Rightarrow \quad \big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \le \varepsilon \quad \text{for all } t \ge t_0 \text{ and } d \in M_D$$

(Uniform Robust Lyapunov Output Stability).

P3. Σ satisfies the *Uniform Robust Output Attractivity Property*, i.e., for all $\varepsilon > 0$ and $R \ge 0$, there exists $\tau := \tau(\varepsilon, R) \ge 0$ such that

$$\|x_0\|_{\mathcal{X}} \le R \quad t_0 \ge 0$$
$$\Rightarrow \quad \big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \le \varepsilon \quad \forall t \ge t_0 + \tau, \forall d \in M_D.$$

Moreover, if $H(t, x, u) \equiv x$, then we say that Σ is *Uniformly Robustly Globally Asymptotically Stable* (*URGAS*).

The following lemma shows that RFC and property P3 in Definition 2.2 are sufficient for RGAOS. In other words, Robust Lagrange and Lyapunov Output Stability are consequences of the Robust Output Attractivity property.

Lemma 2.1 *Suppose that Σ is Robustly Forward Complete (RFC) and satisfies the Robust Output Attractivity Property (property* P3 *of Definition* 2.2*). Then Σ is RGAOS.*

The proof of Lemma 2.1 is based on the following fact, which exploits the properties of the output map.

Fact I *For all $T \geq 0$ and $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon, T) > 0$ such that*

$$\sup\left\{ \left\| H(t,x,0) \right\|_{\mathcal{Y}} : \|x\|_{\mathcal{X}} \leq \delta, t \in [0,T] \right\} \leq \varepsilon.$$

Proof of Fact I Let $T \geq 0$ and $\varepsilon > 0$. We consider the following cases:

Case 1: $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ is continuous.

In this case, the continuity of the map $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ and the fact that $H(t,0,0) = 0$ for all $t \geq 0$ imply that for every $t \in [0,T]$, there exists $\tilde{\delta} := \tilde{\delta}(t,\varepsilon) > 0$ such that

$$\sup\left\{ \left\| H(\tau,x,0) \right\|_{\mathcal{Y}} : \|x\|_{\mathcal{X}} \leq \tilde{\delta}, \tau \geq 0, |\tau - t| \leq \tilde{\delta} \right\} \leq \varepsilon \tag{2.1}$$

Due to the fact that the interval $[0,T]$ is compact, there exists a finite number of times $t_i \in [0,T]$, $i = 1,\ldots,N$, with the property $[0,T] \subseteq \bigcup_{i=1,\ldots,N}[\max(0, t_i - \tilde{\delta}(t_i,\varepsilon)), t_i + \tilde{\delta}(t_i,\varepsilon)]$. By virtue of (2.1), the selection $\delta(\varepsilon,T) = \min_{i=1,\ldots,N} \tilde{\delta}(t_i,\varepsilon) > 0$ guarantees that the statement of Fact I holds.

Case 2: There exists a partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\Re^+$ with diameter $r > 0$ such that $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ satisfies Hypothesis (L2) in Sect. 1.7 of the previous chapter.

In this case, notice that there exists a finite set $\{\tau_0,\ldots,\tau_N\} = \pi \cap [0,T]$ with $\tau_{N+1} > T$. Using the fact that for each fixed $\tau_i \in \pi$, the mapping $\mathcal{X} \times U \ni (x,u) \to H(\tau_i,x,u)$ is continuous with $H(\tau_i,0,0) = 0$, it follows that for every $i = 0,\ldots,N$, there exists $\tilde{\delta}_i := \tilde{\delta}_i(\varepsilon) > 0$ such that

$$\sup\left\{ \left\| H(\tau_i,x,0) \right\|_{\mathcal{Y}} : \|x\|_{\mathcal{X}} \leq \tilde{\delta}_i \right\} \leq \varepsilon \tag{2.2}$$

By virtue of (2.2) in conjunction with the fact that, for every $(\tau_i,x,u) \in \pi \times \mathcal{X} \times U$ and $t \in [\tau_i,\tau_{i+1})$, it holds that $H(t,x,u) = H(\tau_i,x,u)$, the selection $\delta(\varepsilon,T) = \min_{i=0,\ldots,N} \tilde{\delta}_i(\varepsilon) > 0$ guarantees that Fact I holds. The proof is complete. □

We are now in a position to provide the proof of Lemma 2.1.

Proof of Lemma 2.1 It suffices to show that Σ is Robustly Lagrange and Lyapunov Output Stable, i.e., it satisfies properties P1 and P2 of Definition 2.2. First, we show that Σ is Robustly Lagrange Output Stable, by showing that $a(T,s) < +\infty$ for all $T \geq 0$, $s \geq 0$, where

$$a(T,s) := \sup\Big\{ \big\| H\big(t, \phi(t,t_0,x_0,u_0,d), 0\big) \big\|_{\mathcal{Y}} : \\ d \in M_D, t \in [t_0, +\infty), \|x_0\|_{\mathcal{X}} \le s, t_0 \in [0,T] \Big\}$$

By virtue of Robust Output Attractivity Property we have, for every $\varepsilon > 0$,

$$a(T,s) \le \varepsilon + \sup\Big\{ \big\| H\big(t, \phi(t,t_0,x_0,u_0,d), 0\big) \big\|_{\mathcal{Y}} : \\ d \in M_D, t \in \big[t_0, t_0 + \tau(\varepsilon,T,s)\big], \|x_0\|_{\mathcal{X}} \le s, t_0 \in [0,T] \Big\}$$

where $\tau := \tau(\varepsilon,T,s) \ge 0$ is the time involved in the Robust Output Attractivity Property of Definition 2.2. Also, by virtue of Robust Forward Completeness (which implies that the set $\{\phi(t,t_0,x_0,u_0,d); t \in [t_0, t_0+\tau], \|x_0\|_{\mathcal{X}} \le s, t_0 \in [0,T], d \in M_D\}$ is bounded) and since $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ maps bounded sets of $\Re^+ \times \mathcal{X} \times U$ into bounded sets of $\mathcal{Y}$, we obtain

$$\sup\Big\{ \big\| H\big(t, \phi(t,t_0,x_0,u_0,d), 0\big) \big\|_{\mathcal{Y}} : \\ d \in M_D, t \in \big[t_0, t_0 + \tau(\varepsilon,T,s)\big], \|x_0\|_{\mathcal{X}} \le s, t_0 \in [0,T] \Big\} < +\infty$$

Combining the above inequalities, we obtain that $a(T,s) < +\infty$ for all $T \ge 0$, $s \ge 0$, or equivalently that Σ is Robustly Lagrange Output Stable.

Next, we show that Σ is Robustly Lyapunov Output Stable. Consider arbitrary $\varepsilon > 0$ and $T \ge 0$. By virtue of the Robust Output Attractivity Property, there exists $\tau := \tau(\varepsilon,T) \ge 0$ such that, whenever $\|x_0\|_{\mathcal{X}} \le \varepsilon$ and $t_0 \in [0,T]$, it holds that

$$\big\| H\big(t, \phi(t,t_0,x_0,u_0,d), 0\big) \big\|_{\mathcal{Y}} \le \varepsilon \quad \forall t \in [t_0+\tau, +\infty), \forall d \in M_D. \tag{2.3}$$

Moreover, by Fact I, there exists $\tilde{\delta} := \tilde{\delta}(\varepsilon,T) > 0$ such that

$$\sup\Big\{ \big\| H(t,x,0) \big\|_{\mathcal{Y}} : \|x\|_{\mathcal{X}} \le \tilde{\delta}, t \in \big[0, T + \tau(\varepsilon,T)\big] \Big\} \le \varepsilon \tag{2.4}$$

where $\tau := \tau(\varepsilon,T) \ge 0$ is the time involved in (2.3). Finally, since $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ for Σ, it follows that, for all $\varepsilon > 0$ and $T \ge 0$, there exists $\delta' > 0$ such that

$$\sup\Big\{ \big\| \phi(\tau,t_0,x,u_0,d) \big\|_{\mathcal{X}} ; d \in M_D, \tau \in \big[t_0, t_0+\tau(\varepsilon,T)\big], t_0 \in [0,T] \Big\} \le \tilde{\delta}(\varepsilon,T)$$

provided that $\|x\|_{\mathcal{X}} \le \delta'$, where $\tau := \tau(\varepsilon,T) \ge 0$ is the time involved in (2.3), and $\tilde{\delta} := \tilde{\delta}(\varepsilon,T) > 0$ is the positive number involved in (2.4). The above inequality in conjunction with (2.4) gives

$$\sup\Big\{ \big\| H\big(\tau, \phi(\tau,t_0,x,u_0,d), 0\big) \big\|_{\mathcal{Y}} ; d \in M_D, \tau \in \big[t_0, t_0+\tau(\varepsilon,T)\big], t_0 \in [0,T] \Big\} \le \varepsilon$$

provided that $\|x\|_{\mathcal{X}} \le \delta'$.

It is clear from (2.3) and the above inequality that the Robust Lyapunov Output Stability property is satisfied for $\delta(\varepsilon,T) = \min\{\varepsilon, \delta'\}$. The proof is complete. □

The following lemma must be compared to Lemma 1.1, p. 131 in [17], and Proposition 3.2 in [19]. It shows that, for periodic systems, RGAOS is equivalent to URGAOS.

Lemma 2.2 *Suppose that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic. If Σ is RGAOS, then Σ is URGAOS.*

Proof The proof is a direct consequence of Definitions 2.2, 2.3 and from the following observation: if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic, then for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, it holds that $H(t, \phi(t, t_0, x_0, u, d), u(t)) = H(t - kT, \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d), (P_{kT}u)(t - kT))$ and $\phi(t, t_0, x_0, u, d) = \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d)$, where $k := [t_0/T]$ denotes the integer part of t_0/T, and the inputs $P_{kT}u \in M_U$, $P_{kT}d \in M_D$ are defined in Definition 1.2. □

Remark 2.1 Notice that Property P3 of Definitions 2.2, 2.3 is not equivalent to $\lim_{t\to+\infty} \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} = 0$ for all $(t_0, x_0, d) \in (\Re^+, \mathcal{X}, M_D)$ (the so-called "weak attractivity property"). In [16] (pp. 191–194) an example of an autonomous finite-dimensional system described by ODEs is reported, where a robust equilibrium point satisfies the weak attractivity property, but it is not URGAS, namely the planar system

$$\begin{cases} \dot{x} = \frac{x^2(y-x)+y^5}{(x^2+y^2)(1+(x^2+y^2)^2)} \\ \dot{y} = \frac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)} \end{cases}$$

where the right-hand sides are defined to be zero at $x = y = 0$ (and consequently the right-hand sides are locally Lipschitz functions). The above system satisfies Hypotheses (H1–4) in Sect. 1.2 of the previous chapter and is RFC. Hence, by virtue of Lemma 2.1, it follows that the above system cannot satisfy Property P3.

Remark 2.2 Obviously, if a system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, I)$ is RGAS, where I denotes the identity mapping (i.e., $I(t, x) := x$ and $\mathcal{X} = \mathcal{Y}$) and the output function is time-invariant, i.e., $H(t, x) \equiv H(x)$, then the system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is RGAOS. However, this conclusion does not hold if the output function actually depends on time. For example, consider the system with identity output map,

$$\dot{x} = -x \quad x \in \Re$$

It is clear that the above system is URGAS. However, if we define the output

$$Y = H(t, x) = \exp(4t)x$$

then it is obvious that the above system is not RGAOS.

Remark 2.3 The need to impose the Robust Lagrange Stability Property and the importance of this property are discussed in [57]. Particularly, in [57] the authors present an example of a scalar time-varying system described by ODEs with identity output map, where properties P2, P3 of Definition 2.3 hold, and the system is not URGAS. Therefore, nonuniform notions of stability are needed for the study of time-varying systems.

We stop at this point in order to emphasize the last phrase of the above comment. Since nonuniform stability notions are seldom discussed in details in most books, we present an example which illustrates the need for such notions in nonperiodic time-varying systems.

Example 2.2.1 Consider the planar, linear, and time-varying system described by the ODEs

$$\begin{aligned} \dot{x}_1(t) &= \exp(t)x_2(t) \\ \dot{x}_2(t) &= -\exp(-t)x_1(t) - 3x_2(t) \\ x(t) &= \bigl(x_1(t), x_2(t)\bigr) \in \Re^2 \end{aligned} \tag{2.5}$$

The solution of system (2.5) with initial condition $x(t_0) = (x_{10}, x_{20})^T \in \Re^2$ is given by the following equations for all $t \geq t_0$:

$$\begin{aligned} x_1(t) &= x_{10}\bigl[1 + (t - t_0)\bigr]\exp\bigl(-(t - t_0)\bigr) + x_{20}(t - t_0)\exp(t_0)\exp\bigl(-(t - t_0)\bigr) \\ x_2(t) &= -x_{10}(t - t_0)\exp(-t_0)\exp\bigl(-2(t - t_0)\bigr) \\ &\quad + x_{20}\bigl[1 - (t - t_0)\bigr]\exp\bigl(-2(t - t_0)\bigr) \end{aligned} \tag{2.6}$$

If the output Y of system (2.5) is the state component x_2, then one can show the URGAOS property. Indeed, in this case there exists a constant $M > 0$ such that the solution of (2.5) satisfies $|x_2(t)| \leq M|x_0|\exp(-(t - t_0))$ for all $t \geq t_0$.

However, if the output Y of system (2.5) is the state component x_1, then one cannot show URGAOS. To prove this, we notice that the *Uniform Robust Output Attractivity Property* does not hold. If the Uniform Robust Output Attractivity Property were valid, then for all $\varepsilon > 0$ and $R \geq 0$, we would obtain the existence of $\tau := \tau(\varepsilon, R) \geq 0$ such that for every $t_0 \geq 0$, $x_{20} \in \Re$ with $|x_{20}| \leq R$ and $t - t_0 \geq \tau$, the following inequality would hold:

$$\bigl|x_1(t)\bigr| = |x_{20}|(t - t_0)\exp(t_0)\exp\bigl(-(t - t_0)\bigr) \leq \varepsilon \tag{2.7}$$

Notice that we have considered the solution of (2.5) with $x_{10} = 0$. Clearly, (2.7) holds for all $t \geq t_0$ if $R\exp(t_0 - 1) \leq \varepsilon$ and for all $t \geq t_0 + A(\frac{\varepsilon}{R}\exp(-t_0))$ if $R\exp(t_0 - 1) > \varepsilon$, where $A := A(\frac{\varepsilon}{R}\exp(-t_0)) > 1$ is the unique solution of the equation $A\exp(-A) = \frac{\varepsilon}{R}\exp(-t_0)$. Notice that $A(\frac{\varepsilon}{R}\exp(-t_0)) > \ln(\frac{R}{\varepsilon}\exp(t_0))$ and therefore the assumption of Uniform Robust Output Attractivity Property leads to contradiction, because we must have $\tau(\varepsilon, R) \geq A(\frac{\varepsilon}{R}\exp(-t_0))$.

Finally, it should be noted that the Robust Output Attractivity Property holds if the output Y of system (2.5) is the state component x_1. In this case, we can prove RGAOS using (2.6) (but not URGAOS). Thus, nonuniform stability notions arise naturally in the study of the simple system (2.5).

2.3 *KL* Characterizations

This section is devoted to the proof of two basic results, which provide characterizations of the RGAOS and URGAOS in terms of *KL* functions.

Theorem 2.1 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Then the following statements are equivalent*:

(i) *Σ is RGAOS.*
(ii) *There exist functions $\mu, \beta \in K^+$ and $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, we have*

$$\begin{aligned} &\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big) \big\|_{\mathcal{Y}} + \mu(t) \big\| \phi(t, t_0, x_0, u_0, d) \big\|_{\mathcal{X}} \\ &\quad \le \sigma\big(\beta(t_0) \|x_0\|_{\mathcal{X}}, t - t_0\big) \quad \text{for all } t \ge t_0. \end{aligned} \tag{2.8}$$

(iii) *There exist functions $\gamma, \beta \in K^+$, $a \in K_\infty$, $\sigma \in KL$ and a constant $R \ge 0$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, we have*:

$$\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big) \big\|_{\mathcal{Y}} \le \sigma\big(\beta(t_0)\big(\|x_0\|_{\mathcal{X}} + R\big), t - t_0\big) \quad \forall t \ge t_0 \tag{2.9}$$

$$\big\| \phi(t, t_0, x_0, u_0, d) \big\|_{\mathcal{X}} \le \gamma(t) a\big(\beta(t_0)\big(\|x_0\|_{\mathcal{X}} + R\big)\big) \quad \forall t \ge t_0. \tag{2.10}$$

Theorem 2.2 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Then the following statements are equivalent*:

(i) *Σ is URGAOS.*
(ii) *Σ is RFC, and there exists a function $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, we have*

$$\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big) \big\|_{\mathcal{Y}} \le \sigma\big(\|x_0\|_{\mathcal{X}}, t - t_0\big) \quad \text{for all } t \ge t_0. \tag{2.11}$$

For the proof of Theorems 2.1 and 2.2, we need some technical lemmas that illustrate useful properties of the functions of classes K_∞ and K^+.

Lemma 2.3 *Let $a : \Re^+ \times \Re^+ \to \Re^+$ be a function with $a(\cdot, 0) = 0$ that satisfies the following properties*:

1. *for each fixed $t \ge 0$, the mapping $a(t, \cdot)$ is nondecreasing*,
2. *for each fixed $s \ge 0$, the mapping $a(\cdot, s)$ is nondecreasing*,
3. $\lim_{s \to 0^+} a(t, s) = 0$ *for all $t \ge 0$.*

Then there exists a pair of functions $\zeta \in K_\infty$ and $\gamma \in K^+$ such that

$$a(t,s) \le \zeta\big(\gamma(t)s\big) \quad \textit{for all } (t,s) \in \big(\Re^+\big)^2 \tag{2.12}$$

Proof of Lemma 2.3 Without loss of generality we may assume that $a \in C^0(\Re^+ \times \Re^+)$. Indeed, otherwise we may consider the function

$$\hat{a}(t,s) := \begin{cases} \frac{1}{s}\int_s^{2s}\int_t^{t+1} a(\tau,\xi)\,d\tau\,d\xi & \text{for } s>0 \\ 0 & \text{for } s=0 \end{cases}$$

which by virtue of the inequality $a(t,s) \le \hat{a}(t,s) \le a(t+1,2s)$ is in $C^0(\Re^+ \times \Re^+)$ and satisfies $\hat{a}(\cdot,0)=0$. Notice that $\hat{a}$ has the same Properties 1, 2, 3 of our statement with a.

By invoking Property 3, for every $t \ge 0$, there exists $\delta(t)$ in $(0,1)$ such that

$$s \le \delta(t) \quad \Rightarrow \quad a(t,s) \le \frac{1}{t+1} \tag{2.13}$$

Define the following function:

$$\tilde{\eta}(t) := \big([t]+1-t\big)\delta\big([t]+1\big) + \big(t-[t]\big)\delta\big([t]+2\big) \quad t \ge 0 \tag{2.14}$$

where $[t]$ denotes the integer part of $t \ge 0$. Notice that by definition (2.14) it follows that $\tilde{\eta}(k) = \delta(k+1)$ and $\lim_{t\to(k+1)^-}\tilde{\eta}(t) = \delta(k+2)$ for all $k \in Z^+$, which implies that $\tilde{\eta}$ is continuous. Moreover, definition (2.14) gives $0 < \tilde{\eta}(t) \le \max\{\delta([t]+1);\delta([t]+2)\}$ for all $t \ge 0$, which in conjunction with (2.13) implies:

$$s \le \tilde{\eta}(t) \quad \Rightarrow \quad \left.\begin{array}{l} s \le \delta([t]+1) \text{ or} \\ s \le \delta([t]+2) \end{array}\right\} \quad \Rightarrow \quad \left.\begin{array}{l} a([t]+1,s) \le \frac{1}{[t]+2} \text{ or} \\ a([t]+2,s) \le \frac{1}{[t]+3} \end{array}\right\}$$

The above inequality, combined with the fact that $t \le [t]+1 \le [t]+2$ and Property 2 for a, gives:

$$s \le \tilde{\eta}(t) \quad \Rightarrow \quad a(t,s) \le \frac{1}{t+1} \tag{2.15}$$

Next, define the following function:

$$\eta(t) := \exp(-t)\min_{0\le\tau\le t}\tilde{\eta}(\tau) \tag{2.16}$$

which is a C^0 strictly decreasing function $\eta : \Re^+ \to (0,+\infty)$ with $\lim_{t\to+\infty}\eta(t) = 0$ and such that

$$s \le \eta(t) \quad \Rightarrow \quad a(t,s) \le \frac{1}{t+1} \tag{2.17}$$

Let μ be the inverse function of η defined on $(0, \eta(0)]$ being nonnegative, continuous, and strictly decreasing with $\lim_{t\to 0^+} \mu(t) = +\infty$. Define

$$\tilde{\mu}(s) := \begin{cases} \mu(s) & \text{if } s \in (0, \eta(0)] \\ 0 & \text{if } s > \eta(0) \end{cases} \tag{2.18}$$

It turns out that $\tilde{\mu} : (0, +\infty) \to \Re^+$ is nonincreasing, continuous, nonnegative, and satisfies $\lim_{t\to 0^+} \tilde{\mu}(t) = +\infty$. Additionally, define

$$\beta(s) := s + \begin{cases} 0 & \text{if } s = 0 \\ \sup_{0<\tau\le s} a(\tilde{\mu}(\tau), \tau) & \text{if } s > 0 \end{cases} \tag{2.19}$$

We next show that $\beta \in K_\infty$. Indeed, by definition (2.19) it follows that $\beta(0) = 0$ and β is strictly increasing with $\lim_{s\to+\infty} \beta(s) = +\infty$. The continuity of β on $(0, +\infty)$ follows from the fact that both a and $\tilde{\mu}$ are C^0 on $(0, +\infty)$. Furthermore, notice that (2.17) and (2.18) imply

$$a\big(\tilde{\mu}(\tau), \tau\big) \le \frac{1}{\tilde{\mu}(\tau) + 1} \quad \text{for all } \tau \in \big(0, \eta(0)\big]. \tag{2.20}$$

Since $\lim_{s\to 0^+} \tilde{\mu}(s) = +\infty$, it follows from (2.20) that $\lim_{s\to 0^+} \beta(s) = 0$, and this establishes the continuity of β at zero. Let $\zeta(s) := a(s, s) + \beta(s)$. Obviously, $\zeta(\cdot)$ is of class K_∞. Moreover, when $s \ge t$, by virtue of Property 2 for a, it holds that $a(t, s) \le a(s, s) \le \zeta(s)$, which implies

$$\sup_{s\ge t>0} \frac{\zeta^{-1}(a(t, s))}{s} \le 1 \tag{2.21}$$

Also, when $0 < s \le \eta(t)$, it follows from (2.18) that $\tilde{\mu}(s) \ge t$, and hence, by Property 2 for a and (2.19), $a(t, s) \le a(\tilde{\mu}(s), s) \le \zeta(s)$. The latter implies that

$$\sup_{0<s\le\eta(t)} \frac{\zeta^{-1}(a(t, s))}{s} \le 1 \tag{2.22}$$

Using Property 1, (2.21), and (2.22), we get

$$\sup_{s>0} \frac{\zeta^{-1}(a(t, s))}{s} \le 1 + \sup_{\eta(t)\le s\le t} \frac{\zeta^{-1}(a(t, s))}{s} \le 1 + \frac{\zeta^{-1}(a(t, t))}{\eta(t)} \tag{2.23}$$

Let, finally, γ be any function of class K^+ which satisfies

$$\gamma(t) \ge \frac{\zeta^{-1}(a(t, t))}{\eta(t)} + 1 \quad \text{for all } t \ge 0 \tag{2.24}$$

The desired (2.12) is a consequence of (2.23) and (2.24). □

Lemma 2.4 *Let $a : \Re^+ \to \Re^+$ be a locally bounded function with $\lim_{s\to 0^+} a(s) = a(0) = 0$. Then there exists a function $\zeta \in K_\infty$ such that*

$$a(s) \le \zeta(s) \quad \textit{for all } s \ge 0 \tag{2.25}$$

Proof of Lemma 2.4 Define $\tilde{a}(s) := \sup_{0\le\tau\le s} a(\tau)$ and notice that $\tilde{a}$ is nondecreasing with $\lim_{s\to 0^+} \tilde{a}(s) = \tilde{a}(0) = 0$. Define $\zeta(s) := s + \frac{1}{s}\int_s^{2s} \tilde{a}(\xi)\, d\xi$ for $s > 0$ and $\zeta(0) := 0$. The desired (2.25) is a consequence of previous definitions. □

The following three technical lemmas provide essential estimates of the output and state behavior for RGAOS systems.

Lemma 2.5 *Suppose that the control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with outputs is RGAOS. Then there exist functions $\sigma \in KL$ and $\beta \in K^+$ such that the following estimate holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \ge t_0$:*

$$\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \le \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big) \tag{2.26}$$

Proof of Lemma 2.5 Let $\xi, T \ge 0$, $s \ge 0$, and define:

$$a(T, s) := \sup\big\{\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} : d \in M_D, t \ge t_0, \\ \|x_0\|_{\mathcal{X}} \le s, t_0 \in [0, T]\big\} \tag{2.27}$$

$$M(\xi, T, s) := \sup\big\{\big\| H\big(t_0 + \xi, \phi(t_0 + \xi, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} : d \in M_D, \\ \|x_0\|_{\mathcal{X}} \le s, t_0 \in [0, T]\big\} \tag{2.28}$$

First notice that by virtue of Robust Lagrange Output Stability a is well defined, i.e., $a(T, s) < +\infty$ for all $T \ge 0$, $s \ge 0$. Furthermore, notice that M is well defined, since by definitions (2.27), (2.28) the following inequality is satisfied for all $\xi, T \ge 0$ and $s \ge 0$:

$$M(\xi, T, s) \le a(T, s) \tag{2.29}$$

Notice that, a satisfies all hypotheses of the Lemma 2.3, namely, for each fixed $s \ge 0$, $a(\cdot, s)$ is nondecreasing, and for each fixed $T \ge 0$, $a(T, \cdot)$ is nondecreasing and satisfies $a(\cdot, 0) = 0$. Furthermore, Robust Lyapunov Output Stability asserts that for every $T \ge 0$, $\lim_{s\to 0^+} a(T, s) = 0$. It turns out from Lemma 2.3 that there exist functions $\zeta_1 \in K_\infty$ and $\gamma \in K^+$ such that

$$a(T, s) \le \zeta_1\big(\gamma(T)s\big) \quad \text{for all } (T, s) \in \big(\Re^+\big)^2 \tag{2.30}$$

Without loss of generality, we may assume that $\gamma \in K^+$ is nondecreasing. Inequality (2.30) in conjunction with (2.29) implies

$$M(\xi, T, s) \le \zeta_1\big(\gamma(T)s\big) \quad \text{for all } (\xi, T, s) \in \big(\Re^+\big)^3 \tag{2.31}$$

Moreover, the Robust Output Attractivity property guarantees that for all $\varepsilon > 0$, $T \geq 0$, and $R \geq 0$, there exists $\tau = \tau(\varepsilon, T, R) \geq 0$ such that

$$M(\xi, T, s) \leq \varepsilon \quad \text{for all } \xi \geq \tau(\varepsilon, T, R) \text{ and } 0 \leq s \leq R \tag{2.32}$$

Let

$$g(s) := \sqrt{s} + s^2 \tag{2.33}$$

and let p be a nondecreasing function of class K^+ with $p(0) = 1$ and

$$\lim_{t \to +\infty} p(t) = +\infty \tag{2.34}$$

Define

$$\mu(\xi) := \sup\left\{ \frac{M(\xi, T, s)}{p(T) g(\zeta_1(\gamma(T)s))} : T \geq 0, s > 0 \right\} \tag{2.35}$$

Obviously, by (2.31) and (2.35) the function $\mu : \Re^+ \to \Re^+$ is well defined and satisfies $\mu(\cdot) \leq 1$. We show that $\lim_{\xi \to +\infty} \mu(\xi) = 0$, or, equivalently, we establish that for any given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) \geq 0$ such that

$$\mu(\xi) \leq \varepsilon \quad \text{for } \xi \geq \delta(\varepsilon) \tag{2.36}$$

Notice first, that for any given $\varepsilon > 0$, there exist constants $a := a(\varepsilon)$ and $b := b(\varepsilon)$ with $0 < a < b$ such that

$$x \notin (a, b) \quad \Rightarrow \quad \frac{x}{\sqrt{x} + x^2} \leq \varepsilon \tag{2.37}$$

We next recall (2.34), which asserts that, for the above ε for which (2.37) holds, there exists $c := c(\varepsilon) \geq 0$ such that $p(T) \geq \frac{1}{\varepsilon}$ for all $T \geq c$. This by virtue of (2.37) and (2.31) yields

$$\frac{M(\xi, T, s)}{p(T) g(\zeta_1(\gamma(T)s))} \leq \varepsilon \quad \text{for all } \xi \geq 0, \text{ when } T \geq c \text{ or } \zeta_1\big(\gamma(T)s\big) \notin (a, b) \tag{2.38}$$

Hence, in order to establish (2.36), it remains to consider the case

$$a \leq \zeta_1\big(\gamma(T)s\big) \leq b \quad \text{and} \quad 0 \leq T \leq c \tag{2.39}$$

Since, for each fixed $(\xi, s) \in (\Re^+)^2$, the mappings $M(\xi, \cdot, s)$, $M(\xi, T, \cdot)$, $\gamma(\cdot)$, and $p(\cdot)$ are nondecreasing, we have

$$\frac{M(\xi, T, s)}{p(T) g(\zeta_1(\gamma(T)s))} \leq \frac{M\Big(\xi, c, \frac{\zeta_1^{-1}(b)}{\gamma(0)}\Big)}{g(a)} \tag{2.40}$$

provided that (2.39) holds. By using (2.32) with

$$\varepsilon := \varepsilon g(a) \quad T := c \quad R := \frac{\zeta_1^{-1}(b)}{\gamma(0)}$$

it follows that

$$M\left(\xi, c, \frac{\zeta_1^{-1}(b)}{\gamma(0)}\right) \le \varepsilon g(a) \quad \text{for } \xi \ge \delta(\varepsilon) := \tau\left(\varepsilon g(a), c, \frac{\zeta_1^{-1}(b)}{\gamma(0)}\right) \tag{2.41}$$

By taking into account (2.38), (2.40), (2.41), and definition (2.35) of $\mu(\cdot)$ it follows that (2.36) holds with $\delta = \delta(\varepsilon)$ as selected in (2.41). Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{\xi\to+\infty} \mu(\xi) = 0$. Define

$$\tilde{\mu}(t) := \sup_{\tau \ge t} \mu(\tau) \quad \text{for } t \ge 0; \qquad \tilde{\mu}(t) := \sup_{\tau \ge 0} \mu(\tau) \quad \text{for } t < 0 \tag{2.42}$$

Clearly, $\tilde{\mu}(\cdot)$ is a nonincreasing function with $\lim_{\xi\to+\infty} \tilde{\mu}(\xi) = 0$ and $\tilde{\mu}(t) \ge \mu(t)$ for all $t \ge 0$. Next define, for $t \ge 0$,

$$\bar{\mu}(t) := \exp(-t) + \int_{t-1}^{t} \tilde{\mu}(w)\, dw \tag{2.43}$$

Notice that $\bar{\mu} : \Re^+ \to (0, +\infty)$ is a continuous strictly decreasing function such that $\bar{\mu}(\xi) \ge \mu(\xi)$ for all $\xi \ge 0$ and $\lim_{\xi\to+\infty} \bar{\mu}(\xi) = 0$. Thus, by recalling definition (2.35) we obtain

$$M(\xi, T, s) \le \bar{\mu}(\xi)\theta(T, s) \quad \text{for all } (T, s) \in \left(\Re^+\right)^2 \text{ and } \xi \ge 0 \tag{2.44}$$

where $\theta(T, s) := p(T)g(\zeta_1(\gamma(T)s))$. Clearly, θ satisfies all hypotheses of Lemma 2.3, and therefore there exist $\zeta \in K_\infty$ and $\beta \in K^+$ such that

$$\theta(T, s) \le \zeta\left(\beta(T)s\right) \quad \text{for all } (T, s) \in \left(\Re^+\right)^2 \tag{2.45}$$

The desired (2.26) is a consequence of (2.28), (2.44), and (2.45) with $\sigma(s, t) := \bar{\mu}(t)\zeta(s)$ (which is a *KL* function). □

Lemma 2.6 *Suppose that the control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *with outputs is URGAOS. Then there exists a function* $\sigma \in KL$ *such that the following estimate holds for all* $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ *and* $t \ge t_0$:

$$\left\| H\left(t, \phi(t, t_0, x_0, u_0, d), 0\right)\right\|_{\mathcal{Y}} \le \sigma\left(\|x_0\|_{\mathcal{X}}, t - t_0\right) \tag{2.46}$$

Proof of Lemma 2.6 Let $\xi \ge 0$, $s \ge 0$, and define

$$a(s) := \sup\left\{\left\| H\left(t, \phi(t, t_0, x_0, u_0, d), 0\right)\right\|_{\mathcal{Y}} : d \in M_D, t \ge t_0, \|x_0\|_{\mathcal{X}} \le s, t_0 \ge 0\right\} \tag{2.47}$$

$$M(\xi, s) := \sup\left\{\left\| H\left(t_0 + \xi, \phi(t_0 + \xi, t_0, x_0, u_0, d), 0\right)\right\|_{\mathcal{Y}} : d \in M_D, \|x_0\|_{\mathcal{X}} \le s, t_0 \ge 0\right\} \tag{2.48}$$

First notice that by virtue of Uniform Robust Lagrange Output Stability, a is well defined, i.e., $a(s) < +\infty$ for every $s \ge 0$. Furthermore, notice that M is well defined, since by definitions (2.47), (2.48) the following inequality is satisfied for all $\xi \geq 0$ and $s \geq 0$:

$$M(\xi, s) \leq a(s) \tag{2.49}$$

Notice that a is nondecreasing (and consequently locally bounded). Moreover, Uniform Robust Lyapunov Output Stability asserts that $\lim_{s\to 0^+} a(s) = a(0) = 0$. It turns out from Lemma 2.4 that there exists a function $\zeta_1 \in K_\infty$ such that

$$a(s) \leq \zeta_1(s) \quad \text{for all } s \geq 0 \tag{2.50}$$

Inequality (2.50), in conjunction with (2.49), implies

$$M(\xi, s) \leq \zeta_1(s) \quad \text{for all } (\xi, s) \in \left(\Re^+\right)^2 \tag{2.51}$$

Moreover, the Uniform Robust Output Attractivity property guarantees that for all $\varepsilon > 0$ and $R \geq 0$, there exists $\tau = \tau(\varepsilon, R) \geq 0$ such that

$$M(\xi, s) \leq \varepsilon \quad \text{for all } \xi \geq \tau(\varepsilon, R) \text{ and } 0 \leq s \leq R \tag{2.52}$$

Let

$$g(s) := \sqrt{s} + s^2 \tag{2.53}$$

and define

$$\mu(\xi) := \sup\left\{ \frac{M(\xi, s)}{g(\zeta_1(s))} : s > 0 \right\} \tag{2.54}$$

Working exactly as in the proof of Lemma 2.5 and using (2.51), (2.52), (2.53), and definition (2.54), we may establish the existence of a continuous strictly decreasing function $\bar{\mu} : \Re^+ \to (0, +\infty)$ with $\bar{\mu}(\xi) \geq \mu(\xi)$ for all $\xi \geq 0$ and $\lim_{\xi\to+\infty} \bar{\mu}(\xi) = 0$. Thus, by recalling definition (2.54) we obtain

$$M(\xi, s) \leq \bar{\mu}(\xi)\theta(s) \quad \text{for all } \xi, s \geq 0 \tag{2.55}$$

where $\theta(s) := g(\zeta_1(s))$. The desired (2.46) is a consequence of (2.48) and (2.55) with $\sigma(s, t) := \bar{\mu}(t)\theta(s)$ (which is a *KL* function). □

Finally, the following lemma provides an estimate for the transition map, which turns out to be a necessary and sufficient condition for Robust Forward Completeness for systems which are not necessarily RGAOS.

Lemma 2.7 *Consider a control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *with the BIC property,* $U = \{0\}$, *and for which* $0 \in \mathcal{X}$ *is a robust equilibrium point from the input* $u \in M_U$. Σ *is RFC from the input u if and only if there exist functions* $\mu, \beta \in K^+$ *and* $\sigma \in KL$ *such that for every* $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, *we have*

$$\mu(t)\left\|\phi(t, t_0, x_0, u_0, d)\right\|_{\mathcal{X}} \leq \sigma\left(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\right) \quad \text{for all } t \geq t_0 \tag{2.56}$$

Proof of Lemma 2.7 Suppose first that Σ is RFC from the input u. Define

$$\omega(T,s) := \sup\bigl\{\bigl\|\phi(t_0+h,t_0,x_0,u_0,d)\bigr\|_{\mathcal{X}};\ \|x_0\|_{\mathcal{X}} \le s,\ t_0 \in [0,T], h \in [0,T], d \in M_D\bigr\} \tag{2.57}$$

Notice that by virtue of Robust Forward Completeness from the input u, the function $\omega : \Re^+ \times \Re^+ \to \Re^+$ is finite valued and for each fixed $t \ge 0$, the mappings $\omega(t,\cdot)$ and $\omega(\cdot,t)$ are nondecreasing. Moreover, definition (2.57) implies that for every $(t_0,x_0,d) \in \Re^+ \times \mathcal{X} \times M_D$, the transition map satisfies

$$\bigl\|\phi(t,t_0,x_0,u_0,d)\bigr\|_{\mathcal{X}} \le \omega\bigl(t,\|x_0\|_{\mathcal{X}}\bigr) \quad \text{for all } t \ge t_0 \tag{2.58}$$

Since for each fixed $t \ge 0$, the mappings $\omega(t,\cdot)$ and $\omega(\cdot,t)$ are nondecreasing, inequality (2.58) implies

$$\bigl\|\phi(t,t_0,x_0,u_0,d)\bigr\|_{\mathcal{X}} \le \omega(t,t) + \omega\bigl(\|x_0\|_{\mathcal{X}},\|x_0\|_{\mathcal{X}}\bigr) \quad \text{for all } t \ge t_0 \tag{2.59}$$

Let $\gamma \in K^+$ be a nondecreasing continuous function that satisfies $1+\omega(s,s) \le \gamma(s)$ for all $s \ge 0$ (e.g., $\gamma(s) := 1 + \int_s^{s+1} \omega(\xi,\xi)\,d\xi$) and define

$$a(s) := s + \begin{cases} s(1+\gamma(0)) & \text{for } 0 \le s < 1 \\ 1+\gamma(s-1) & \text{for } s \ge 1 \end{cases} \quad \text{and} \quad \mu(t) := \frac{\exp(-t)}{\gamma(t)}$$

with $a \in K_\infty$ and $\mu \in K^+$. It follows that the following inequalities hold for all $s,t \ge 0$:

$$\omega(t,t) + \omega(s,s) \le \gamma(t) + \gamma(s) \le \gamma(t)\bigl(1+\gamma(s)\bigr) \le \gamma(t)a(1+s) \tag{2.60}$$

Inequalities (2.59) and (2.60), in conjunction with (2.58) and definition of $\mu \in K^+$ above, imply that for every $(t_0,x_0,d) \in \Re^+ \times \mathcal{X} \times M_D$, the transition map satisfies

$$\mu(t)\bigl\|\phi(t,t_0,x_0,u_0,d)\bigr\|_{\mathcal{X}} \le \exp(-t)a\bigl(\|x_0\|_{\mathcal{X}}+1\bigr) \quad \text{for all } t \ge t_0 \tag{2.61}$$

Consider the control system $\Sigma' := (\mathcal{X},\mathcal{X},M_U,M_D,\phi,\pi,\tilde{H})$ with $U = \{0\}$, where

$$\tilde{H}(t,x,0) := \mu(t)x \tag{2.62}$$

It can be immediately verified that $\Sigma' := (\mathcal{X},\mathcal{X},M_U,M_D,\phi,\pi,\tilde{H})$ is a control system with outputs and the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point. Moreover, by virtue of (2.61) and Lemma 2.1, it follows that Σ' is RGAOS. Consequently, Lemma 2.5 guarantees the existence of functions $\sigma \in KL$ and $\beta \in K^+$ such that (2.56) holds for all $(t_0,x_0,d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \ge t_0$.

Conversely, suppose that (2.56) holds for the transition map of Σ. Clearly, by virtue of the BIC property, for each $(t_0,x_0,d) \in \Re^+ \times \mathcal{X} \times M_D$, there exists a maximal existence time, say, $t_{\max} \in \Re^+ \cup \{+\infty\}$, such that $[t_0,t_{\max}) \times \{(t_0,x_0,u_0,d)\} \subseteq A_\phi$ and for all $t \ge t_{\max}$, it holds that $(t,t_0,x_0,u_0,d) \notin A_\phi$. In addition, if $t_{\max} <$

$+\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} > M$. Suppose that $t_{\max} < +\infty$ for some $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$. Then there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} > \sigma(\beta(t_0)\|x_0\|_{\mathcal{X}}, 0) \max_{t\in[0,t_{\max}]} \frac{1}{\mu(t)}$. On the other hand, since (2.56) holds, we obtain

$$\left\|\phi(t, t_0, x_0, u_0, d)\right\|_{\mathcal{X}} \le \sigma\left(\beta(t_0)\|x_0\|_{\mathcal{X}}, 0\right) \max_{t\in[0,t_{\max}]} \frac{1}{\mu(t)}$$

which is clearly a contradiction. Thus, since (2.56) holds, we must have $t_{\max} = +\infty$ for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$. Moreover, using (2.56) for all $\varepsilon \ge 0$ and $T \ge 0$, we obtain

$$\sup\left\{\left\|\phi(t_0 + s, t_0, x_0, u_0, d)\right\|_{\mathcal{X}}; s \in [0, T], \|x_0\|_{\mathcal{X}} \le \varepsilon, t_0 \in [0, T], d \in M_D\right\}$$
$$\le \sigma\left(\varepsilon \max_{t\in[0,T]} \beta(t), 0\right) \max_{t\in[0,2T]} \frac{1}{\mu(t)} < +\infty$$

i.e., the property of Robust Forward Completeness from the input u is satisfied. □

We are now in a position to prove the main results of the section.

Proof of Theorem 2.1 The implication (ii) ⇒ (iii) is obvious. The implication (iii) ⇒ (i) follows immediately by applying the results of Lemma 2.1 and the definition of RFC. To be more precise, notice that inequality (2.10) implies that Σ is RFC and (by virtue of the properties of the KL functions) estimate (2.9) implies that Σ satisfies the Robust Output Attractivity Property. Consequently, since $0 \in \mathcal{X}$ is a robust equilibrium point, it follows by virtue of Lemma 2.1 that Σ is RGAOS.

Next we prove implication (i) ⇒ (ii). Suppose that Σ is RGAOS. Then Lemmas 2.5 and 2.7 guarantee that there exist functions $\bar{\sigma} \in KL$, $\tilde{\sigma} \in KL$, and $\bar{\beta}, \tilde{\beta}, \mu \in K^+$ such that the following estimates hold for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \ge t_0$:

$$\left\|H\left(t, \phi(t, t_0, x_0, u_0, d), 0\right)\right\|_{\mathcal{Y}} \le \bar{\sigma}\left(\bar{\beta}(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\right) \tag{2.63}$$

$$\mu(t)\left\|\phi(t, t_0, x_0, u_0, d)\right\|_{\mathcal{X}} \le \tilde{\sigma}\left(\tilde{\beta}(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\right) \tag{2.64}$$

Estimates (2.63) and (2.64) imply (2.8) for $\sigma(s,t) := \tilde{\sigma}(s,t) + \bar{\sigma}(s,t)$ and $\beta(t) := \max\{\bar{\beta}(t), \tilde{\beta}(t)\}$. The proof is complete. □

Proof of Theorem 2.2 The implication (i) ⇒ (ii) is a direct consequence of Lemma 2.6. The implication (ii) ⇒ (i) follows directly from the properties of KL functions. □

2.4 Transformations Preserving RGAOS

The method of verifying RGAOS using transformations is used frequently in Mathematical Control Theory and Stability Theory. Roughly speaking, we want to verify

RGAOS for a system Σ which is the transformation of another system Σ' (in the sense described in Sect. 1.6 of Chap. 1). Suppose that we can establish that Σ' is RGAOS (using another method, e.g., by solving the differential (or difference) equations (or inequalities)). Then RGAOS for Σ is guaranteed under some additional technical assumptions that are presented below.

We start with the following elementary observation, which is a direct consequence of Definitions 1.4, 1.7, and 1.8.

Lemma 2.8 *Consider a control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *which is RFC from the input* u. *Let* $\Phi : \Re^+ \times \mathcal{X} \to \mathcal{X}$ *be a change of coordinates, and* $q : \Re^+ \times V \to U$ *a transformation of* V *onto* U. *Then system* $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ *defined by* (1.106), (1.107), (1.108) *is a control system which is RFC from the input* $v \in M_V$.

Making use of the results of Lemma 1.2, Theorems 2.1, and 2.2, we are in a position to show the following propositions.

Proposition 2.1 *Consider a control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *with* $U = \{0\}$, *which satisfies the classical semigroup property and the BIC property, and for which* $0 \in \mathcal{X}$ *is a robust equilibrium point from the input* $u \in M_U$. *Suppose that* Σ *is RGAOS. Let* $\Phi : \Re^+ \times \mathcal{X} \to \mathcal{X}$ *be a change of coordinates. Then,* $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ *defined by* (1.106), (1.107), (1.108) *with* $V = U = \{0\}$ *is RGAOS. Moreover, if* Σ *is URGAOS and there exists* $a \in K_\infty$ *such that*

$$\left\| \Phi^{-1}(t, x) \right\|_{\mathcal{X}} \le a\left(\|x\|_{\mathcal{X}}\right) \quad \text{for all } (t, x) \in \Re^+ \times \mathcal{X} \tag{2.65}$$

then system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ *defined by* (1.106), (1.107), (1.108) *with* $V = U = \{0\}$ *is URGAOS.*

Proof By virtue of Lemma 1.2, $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ is a deterministic control system which satisfies the classical semigroup property and the BIC property. Moreover, $0 \in \mathcal{X}$ is a robust equilibrium point from the input u for Σ'. By virtue of Lemma 2.8, system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ is RFC from the input u.

Finally, notice that by virtue of Lemma 2.5, there exist functions $\sigma \in KL$ and $\beta \in K^+$ such that estimate (2.26) holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \ge t_0$. Definitions (1.106) and (1.108), in conjunction with (2.20), imply that the following estimate holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \ge t_0$:

$$\begin{aligned} \left\| H'\left(t, \phi'(t, t_0, x_0, u_0, d), 0\right) \right\|_{\mathcal{Y}} &= \left\| H\left(t, \phi\left(t, t_0, \Phi^{-1}(t_0, x_0), u_0, d\right), 0\right) \right\|_{\mathcal{Y}} \\ &\le \sigma\left(\beta(t_0) \left\| \Phi^{-1}(t_0, x_0) \right\|_{\mathcal{X}}, t - t_0\right) \end{aligned} \tag{2.66}$$

Since $\Phi : \Re^+ \times \mathcal{X} \to \mathcal{X}$ is a change of coordinates, there exists $a \in K_\infty$ and $\gamma \in K^+$ such that $\|\Phi^{-1}(t, x)\|_{\mathcal{X}} \le a(\gamma(t)\|x\|_{\mathcal{X}})$ for all $(t, x) \in \Re^+ \times \mathcal{X}$. Consequently, we have

$$\left\| H'\big(t, \phi'(t, t_0, x_0, u_0, d), 0\big) \right\|_{\mathcal{Y}} \le \sigma\big(\beta(t_0) a\big(\gamma(t_0)\|x_0\|_{\mathcal{X}}\big), t - t_0\big) \tag{2.67}$$

The above inequality shows that properties P1, P2, and P3 of Definition 2.2 hold, and consequently $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ is RGAOS. Moreover, if Σ is URGAOS and inequality (2.65) holds, then we obtain

$$\left\| H'\big(t, \phi'(t, t_0, x_0, u_0, d), 0\big) \right\|_{\mathcal{Y}} \le \tilde{\sigma}\big(\|x_0\|_{\mathcal{X}}, t - t_0\big) \tag{2.68}$$

where $\tilde{\sigma}(s, t) := \sigma(a(s), t)$ is a *KL* function. In this case, Theorem 2.2 guarantees that $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ is URGAOS. The proof is complete. □

Example 2.4.1 Transformation methods were used in the early stages of Stability Theory to derive conditions for asymptotic stability of linear systems described by ODEs, i.e.,

$$\begin{aligned} &\dot{x} = Ax \quad x \in \Re^n \\ &Y = x \end{aligned} \tag{2.69}$$

where the matrix $A \in \Re^{n\times n}$ is constant. The Jordan theorem on canonical representation of matrices guarantees the existence of a nonsingular matrix $P \in \Re^{n\times n}$ such that

$$PAP^{-1} = \tilde{A} = \begin{bmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & J_{r-1} & 0 \\ 0 & 0 & \dots & 0 & J_r \end{bmatrix} \tag{2.70}$$

where $J_1, \dots, J_r$ are the so-called Jordan blocks corresponding to the eigenvalues $\lambda_k = a_k + i\beta_k$ $(k = 1, \dots, n)$ of the matrix $A \in \Re^{n\times n}$ (here i denotes the imaginary unit). Particularly, blocks J_k which correspond to real eigenvalues $\lambda_k = a_k \in \Re$ have the form

$$J_k = [a_k] \quad \text{or} \quad J_k = \begin{bmatrix} a_k & \gamma_k & \dots & 0 & 0 \\ 0 & a_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_k & \gamma_k \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix} \quad \text{with } \gamma_k \neq 0 \tag{2.71}$$

and blocks J_k which correspond to complex eigenvalues $\lambda_k = a_k + i\beta_k$ have the form

$$J_k = \begin{bmatrix} a_k & \beta_k \\ -\beta_k & a_k \end{bmatrix} \quad \text{or} \quad J_k = \begin{bmatrix} K_k & L_k & \dots & 0 & 0 \\ 0 & K_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & K_k & L_k \\ 0 & 0 & \dots & 0 & L_k \end{bmatrix} \tag{2.72}$$

with

$$K_k = \begin{bmatrix} a_k & \beta_k \\ -\beta_k & a_k \end{bmatrix} \qquad L_k = \begin{bmatrix} \gamma_k & 0 \\ 0 & \gamma_k \end{bmatrix}$$

Consider the linear system

$$\begin{aligned} \dot{z}_i &= J_i z_i \quad i = 1, \dots, r \\ Y &= P^{-1} z \\ z &= \left(z_1', \dots, z_r'\right)' \in \Re^n \end{aligned} \tag{2.73}$$

where $J_1, \dots, J_r$ are the Jordan blocks of $A \in \Re^{n\times n}$ that appear in (2.70), and $P \in \Re^{n\times n}$ is the nonsingular matrix involved in (2.70). It should be clear from (2.70) that system (2.69) is the transformation of system (2.73) corresponding to the change of coordinates $\Phi(t, z) := P^{-1}z$. On the other hand, system (2.73) is the transformation of system (2.69) corresponding to the change of coordinates $\Phi(t, x) := Px$. Consequently, by virtue of Proposition 2.1, system (2.73) is URGAOS if and only if system (2.69) is URGAS.

The reader should notice that the solution of system (2.73) can be found explicitly (due to the simple structure of the Jordan blocks given in (2.71) and (2.72)). The formulae for the solution of the solution of system (2.73) show that if J_k is one of the blocks described in (2.71) and (2.72) and $a_k < 0$, then the solution of $\dot{z}_k = J_k z_k$ satisfies $|z_k(t)| \le M_k \exp(-\sigma_k(t - t_0))|z_k(t_0)|$ for all $t \ge t_0$, where $M_k \ge 1$ and $\sigma_k \in (0, -a_k]$ are appropriate constants. Consequently, if all eigenvalues of the matrix $A \in \Re^{n\times n}$ have negative real parts, then we have, for all $t \ge t_0$,

$$\left|Y(t)\right| \le \exp\bigl(-\sigma(t - t_0)\bigr)\left|z(t_0)\right|\left|P^{-1}\right| \sum_{k=1}^{r} M_k \tag{2.74}$$

where $\sigma := \min\{\sigma_k, k = 1, \dots, r\}$. Estimate (2.74) implies that system (2.73) is URGAOS. Moreover, if J_k is one of the blocks described in (2.71) and (2.72) and $a_k \ge 0$, then the solution of $\dot{z}_k = J_k z_k$ does not converge to zero for all initial conditions. Thus if one of the real parts of the eigenvalues of $A \in \Re^{n\times n}$ is nonnegative, then system (2.73) is not URGAOS.

Thus, we have shown that system (2.69) is RGAS if and only if all eigenvalues of the matrix $A \in \Re^{n\times n}$ have negative real parts.

2.5 Differential Inequalities and Comparison Lemmas

The purpose of this section is to present basic tools that enable us to handle differential inequalities. Differential inequalities can be used to obtain estimates of the state of a control system. The derived estimates can be further utilized for the proof of RGAOS (this is essentially the first method of proving stability, the method of analytical solutions) or can be combined with the method of Lyapunov functionals (as will be shown in the following section).

We start with two technical lemmas which show how one can obtain inequalities for continuous functions using differential inequalities.

Lemma 2.9 *Let* $y : [t_0, t_1) \to \Re$ *be an absolutely continuous function, where* $t_0 < t_1 \le +\infty$, *and suppose that the following implication holds for certain constants* $a > b$:

$$\text{“If } b < y(t) < a \text{ for some } t \in [t_0, t_1) \text{ and } \dot{y}(t) \text{ exists, then } \dot{y}(t) \ge 0.\text{”}$$

Then, the following implication holds:

$$\text{“If } y(t_0) \ge a, \text{ then } y(t) \ge a \text{ for all } t \in [t_0, t_1).\text{”}$$

Proof The proof is made by contradiction. Suppose that there exists $t \in (t_0, t_1)$ such that $y(t) < a$. Without loss of generality we may assume that $y(\tau) > b$ for all $\tau \in [t_0, t)$, because if the set $B := \{\tau \in [t_0, t] : y(\tau) \le b\}$ is nonempty, then we may define $\bar{t} := \inf B$, which satisfies $\bar{t} > t_0$ and $y(\tau) > b$ for all $\tau \in [t_0, \bar{t})$, and replace $t \in (t_0, t_1)$ by $\bar{t} \in (t_0, t_1)$.

Since $y(t_0) \ge a$, by continuity of $y : [t_0, t_1) \to \Re$ it follows that the set $A := \{\tau \in [t_0, t) : y(\tau) = a\}$ is nonempty and closed. Let $T := \sup A$ and notice that by virtue of continuity of $y : [t_0, t_1) \to \Re$ and the fact $y(t) < a$, we get $T < t$. Consequently, we have $y(\tau) \ne a$ for all $\tau \in (T, t]$ and $y(T) = a$. More specifically, we have $y(\tau) < a$ for all $\tau \in (T, t]$ (since the existence of $\tau \in (T, t]$ with $y(\tau) > a$ would imply the existence of certain $\xi \in (\tau, t)$ with $y(\xi) = a$ and thus contradicting the definition of $T := \sup A$). Consequently, the assumed implication gives $\dot{y}(\tau) \ge 0$ for almost all $\tau \in (T, t)$.

Since $y(t) = y(T) + \int_T^t \dot{y}(s)\,ds$, we obtain $y(t) \ge y(T) = a$, which contradicts the hypothesis $y(t) < a$. The proof is complete. □

Lemma 2.10 *Let* $y : [t_0, t_1) \to \Re$ *be a continuous function, where* $t_0 < t_1 \le +\infty$, *and suppose that the following implication holds for certain constant* $a \in \Re$:

$$\text{“If } y(t) = a \text{ for some } t \in [t_0, t_1), \text{ then } D^+ y(t) := \limsup_{h \to 0^+} \frac{y(t+h) - y(t)}{h} > 0.\text{”}$$

Then, the following implication holds:

$$\text{“If } y(t_0) \ge a, \text{ then } y(t) \ge a \text{ for all } t \in [t_0, t_1).\text{”}$$

Proof The proof is made by contradiction. Suppose that there exists $t \in (t_0, t_1)$ such that $y(t) < a$. Since $y(t_0) \geq a$, by continuity of $y : [t_0, t_1) \to \Re$ it follows that the set $A := \{\tau \in [t_0, t) : y(\tau) = a\}$ is nonempty and closed. Let $T := \sup A$ and notice that by virtue of continuity of $y : [t_0, t_1) \to \Re$ and the fact $y(t) < a$, we get $T < t$. Consequently, we have $y(\tau) \neq a$ for all $\tau \in (T, t]$ and $y(T) = a$. More specifically, we have $y(\tau) < a$ for all $\tau \in (T, t]$ (since the existence of $\tau \in (T, t]$ with $y(\tau) > a$ would imply the existence of certain $\xi \in (\tau, t)$ with $y(\xi) = a$ and thus contradicting the definition of $T := \sup A$).

Thus, for sufficiently small $h > 0$, we obtain $\frac{y(T+h)-y(T)}{h} < 0$. Taking limits, we get $D^+ y(T) := \limsup_{h \to 0^+} \frac{y(T+h)-y(T)}{h} \leq 0$, contradicting the hypothesis that $D^+ y(T) := \limsup_{h \to 0^+} \frac{y(T+h)-y(T)}{h} > 0$. The proof is complete. □

We continue with a technical lemma which provides estimates involving *KL* functions. The reader should recall that a function μ is of class $\mathcal{E}$ if $\mu \in C^0(\Re^+; \Re^+)$ with $\lim_{t \to +\infty} \mu(t) = 0$ and $\int_0^{+\infty} \mu(t)\, dt < +\infty$.

Lemma 2.11 *For every pair of functions $\rho : \Re^+ \to \Re^+$ being C^0 positive definite and μ being of class $\mathcal{E}$ with $\mu(t) > 0$ for all $t \geq 0$, there exists a KL function $\sigma : (\Re^+)^2 \to \Re^+$ which satisfies the following property*:

(P) *If $c \in [0, 1]$ is a constant, $t_0 \geq 0$, and $y : [t_0, +\infty) \to \Re^+$ is an absolutely continuous function that satisfies the following differential inequality for almost all $t \geq t_0$*:

$$\dot{y}(t) \leq -\rho\big(y(t)\big) + c\mu(t) \tag{2.75}$$

then the following estimate holds:

$$y(t) \leq \sigma\left(y(t_0) + c\int_{t_0}^{+\infty} \mu(s)\, ds, t - t_0\right) \quad \textit{for all } t \geq t_0. \tag{2.76}$$

Proof Let $\mathrm{G}(\rho, \mu, c, t_0)$ be the set of all absolutely continuous functions $y : [t_0, +\infty) \to \Re^+$ which satisfy differential inequality (2.75) a.e. for $t \geq t_0 \geq 0$ and $c \in [0, 1]$.

Define

$$g(s) := \sqrt{s} + s^2 \tag{2.77}$$

$$v(\xi) := \sup\left\{ \frac{y(t_0 + \xi)}{g(y(t_0) + c\int_{t_0}^{+\infty} \mu(s)\, ds)}; y \in \mathrm{G}(\rho, \mu, c, t_0), c \in (0, 1], t_0 \geq 0 \right\} \tag{2.78}$$

We will next show that $v(t) \leq 1$ for all $t \geq 0$ and $\lim_{t \to +\infty} v(t) = 0$. Let $\bar{v} : \Re^+ \to [0, +\infty)$ be a nonincreasing C^0 function with $v(t) \leq \bar{v}(t)$ for all $t \geq 0$ and such that $\lim_{t \to +\infty} \bar{v}(t) = 0$ (for example, take $\tilde{v}(t) := \sup_{s \geq t} v(s)$ for $t \geq 0$ and $\tilde{v}(t) := \tilde{v}(0)$ for $t < 0$, which is a nonincreasing function with $v(t) \leq \tilde{v}(t)$ for all $t \geq 0$

and $\lim_{t\to+\infty} \tilde{v}(t) = 0$, and define $\bar{v}(t) := \int_{t-1}^{t} \tilde{v}(s)\,ds$ for $t \geq 0$). Define the *KL* function

$$\sigma(s,t) := g(s)\bar{v}(t) \tag{2.79}$$

Then, definitions (2.78) and (2.79) imply that (2.76) is fulfilled for all $y \in \mathrm{G}(\rho,\mu,c,t_0)$ with $c \in (0,1]$ and $t_0 \geq 0$. Finally, the case $c = 0$ is implied by the continuity of $\sigma(s,t) := g(s)\bar{v}(t)$.

First, notice that (2.75) yields

$$y(t) \leq y(t_0) + c\int_{t_0}^{t} \mu(s)\,ds \quad \text{for all } t \geq t_0 \tag{2.80}$$

for all $y \in \mathrm{G}(\rho,\mu,c,t_0)$. Since $\lim_{t\to+\infty} \mu(t) = 0$ for any constants $r \geq \varepsilon > 0$, there exists a time $\tau := \tau(\varepsilon,r) \geq 0$ such that

$$t \geq \tau \quad \Rightarrow \quad \mu(t) \leq \min\left\{\frac{1}{2}\rho(s);\ \frac{\varepsilon}{2} \leq s \leq r\right\} \tag{2.81}$$

Let $0 < \varepsilon < r$. We now show the following implication for all $y \in \mathrm{G}(\rho,\mu,c,t_0)$:

$$\text{If } t_1 \geq \tau(\varepsilon,r) \text{ and } y(t_1) \leq \varepsilon, \quad \text{then } y(t) \leq \varepsilon,\ \forall t \geq t_1 \tag{2.82}$$

To see this, notice that, when $r \geq y(t) \geq \frac{\varepsilon}{2}$ and $t \geq \tau(\varepsilon,r)$, then by (2.75) and (2.81) we have

$$\dot{y}(t) \leq -\rho\big(y(t)\big) + \mu(t) \leq -\frac{1}{2}\rho\big(y(t)\big) < 0 \tag{2.83}$$

Lemma 2.9 and the above inequality show that implication (2.82) holds. We next establish that, if we define

$$T(\varepsilon,r) := \tau(\varepsilon,r) + \frac{2r}{\min_{\varepsilon\leq s\leq r} \rho(s)} \tag{2.84}$$

then the following is fulfilled for all $R \geq 0$ and $\varepsilon > 0$ with $\varepsilon < R + M$:

$$\text{For all } t \geq t_0 + T(\varepsilon, R+M) \text{ and } y \in \mathrm{G}(\rho,\mu,c,t_0) \quad \text{with } y(t_0) \leq R \Rightarrow y(t) \leq \varepsilon \tag{2.85}$$

where $M := \int_0^{+\infty} \mu(t)\,dt > 0$. Indeed, otherwise, by implication (2.82), there would exist $R \geq 0$ and $\varepsilon > 0$ with $\varepsilon < R + M$, $y \in \mathrm{G}(\rho,\mu,c,t_0)$ with $y(t_0) \leq R$, and $t \geq t_0 + T(\varepsilon, R+M)$ such that $y(t) > \varepsilon$. Since $t \geq \tau(\varepsilon, R+M)$, we would have, by virtue of (2.80) and (2.82) with $r = R + M$,

$$y(t) > \varepsilon \quad \text{for all } t \in \big[t_0 + \tau(\varepsilon, R+M), t_0 + T(\varepsilon, R+M)\big] \tag{2.86}$$

On the other hand, by (2.75), (2.80), (2.81), and (2.86) it follows that

$$\dot{y}(t) \leq -\frac{1}{2}\min_{\varepsilon\leq s\leq R+M} \rho(s), \quad \text{for all } t \in \big[t_0 + \tau(\varepsilon, R+M), t_0 + T(\varepsilon, R+M)\big] \tag{2.87}$$

It turns out from (2.86) and (2.87) that

$$\varepsilon < y(t) \leq R + M - \frac{1}{2}(t - t_0 - \tau) \min_{\varepsilon \leq s \leq R+M} \rho(s) \tag{2.88}$$

for all $t \in [t_0 + \tau(\varepsilon, R + M), t_0 + T(\varepsilon, R + M)]$.

Using (2.88) and taking into account definition (2.84) of $T(\cdot)$, we get $\varepsilon < y(t_0 + T) \leq 0$, which is a contradiction. This establishes (2.85).

Implication (2.82), inequality (2.80), and property (2.85) guarantee that the following attractivity property holds:

$$\begin{gathered}\text{For all } (\varepsilon, R, t_0, c) \in (0, +\infty) \times \Re^+ \times \Re^+ \times [0, 1],\ y \in \mathrm{G}(\rho, \mu, c, t_0) \\ \text{with } y(t_0) \leq R \text{ and } t \geq t_0 + T(\varepsilon, R + M) \quad \Rightarrow \quad 0 \leq y(t) \leq \varepsilon\end{gathered} \tag{2.89}$$

We are now ready to establish that $v(t) \leq 1$ for all $t \geq 0$ and $\lim_{t\to+\infty} v(t) = 0$ for the function v defined by (2.78). Since $c > 0$ and $\mu(t) > 0$ for all $t \geq 0$, it follows that the denominator in (2.78) is strictly positive and (2.77), (2.80) imply that $v(t) \leq 1$ for all $t \geq 0$. Let $\varepsilon > 0$, and let $a := a(\varepsilon)$, $b := b(\varepsilon)$ be a pair of constants with $0 < a < b$ and being defined in such a way that $x \notin [a, b] \Rightarrow \frac{x}{\sqrt{x+x^2}} < \varepsilon$. Then, by (2.77) and (2.80), it follows that

$$\frac{y(t)}{g\big(y(t_0) + c\int_{t_0}^{+\infty} \mu(s)\,ds\big)} < \varepsilon \quad \text{for all } t \geq t_0,\ y \in \mathrm{G}(\rho, \mu, c, t_0)$$

provided that either $y(t_0) + c\int_{t_0}^{+\infty} \mu(s)\,ds < a$ or $y(t_0) + c\int_{t_0}^{+\infty} \mu(s)\,ds > b$.

It remains to consider the case $a \leq y(t_0) + c\int_{t_0}^{+\infty} \mu(s)\,ds \leq b$. By (2.89) we get

$$\frac{y(t)}{g\big(y(t_0) + c\int_{t_0}^{+\infty} \mu(s)\,ds\big)} \leq \frac{y(t)}{g(a)} \leq \varepsilon \quad \text{for all } t \geq t_0 + T\big(\varepsilon g(a), b + M\big) \tag{2.90}$$

It turns out from (2.78) and (2.90) that

$$v(t) \leq \varepsilon \quad \text{for all } t \geq T\big(\varepsilon g(a), b + M\big) \tag{2.91}$$

Since $\varepsilon > 0$ is arbitrary, (2.91) asserts that $\lim_{t\to+\infty} v(t) = 0$. The proof is thus complete. □

The following comparison principle can be applied to lower semi-continuous functions and will be useful for the derivation of estimates of values of Lyapunov functionals.

Lemma 2.12 (Comparison principle) *Consider the scalar differential equation*

$$\begin{aligned}\dot{w} &= f(t, w) \\ w(t_0) &= w_0\end{aligned} \tag{2.92}$$

where $f(t, w)$ *is continuous with*

$$\sup\left\{\frac{(x-w)(f(t,x)-f(t,w))}{|x-w|^2} : (t,x) \in S, (t,w) \in S, x \neq w\right\} < +\infty$$

for every compact $S \subset \Re^+ \times J$, *where* $J \subseteq \Re$ *is an open interval. Let* $T > t_0$ *be such that the unique solution* $w(t)$ *of the initial-value problem* (2.92) *exists and satisfies* $w(t) \in J$ *for all* $t \in [t_0, T)$. *Let* $v : [t_0, T) \to \Re$ *be a lower semi-continuous function that satisfies the differential inequality*

$$D^+ v(t) := \limsup_{h \to 0^+} \frac{v(t+h) - v(t)}{h} \leq f\big(t, v(t)\big) \quad \text{for all } t \in [t_0, T) \tag{2.93}$$

Suppose furthermore that

$$v(t_0) \leq w_0 \tag{2.94}$$

$$v(t) \in J \quad \text{for all } t \in [t_0, T) \tag{2.95}$$

and that one of the following holds:

(i) *the mapping* $f(t, \cdot)$ *is nondecreasing on* $J \subseteq \Re$ *for each fixed* $t \in [t_0, T)$;
(ii) *there exists* $\phi \in C^0(\Re^+)$ *such that* $f(t, w) \leq \phi(t)$ *for all* $(t, w) \in [t_0, T) \times J$;
(iii) *the function* $v : [t_0, T) \to \Re$ *is right-continuous.*

Then $v(t) \leq w(t)$ *for all* $t \in [t_0, T)$.

Proof It suffices to show that $v(t) \leq w(t)$ on any interval $[t_0, t_1] \subset [t_0, T)$. Consider the scalar differential equation

$$\begin{aligned} \dot{z} &= f(t, z) + \lambda \\ z(t_0) &= w_0 \end{aligned} \tag{2.96}$$

where λ is a positive constant.

Claim *On any compact interval* $[t_0, t_1] \subset [t_0, T)$ *and for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $0 < \lambda < \delta$, *then* (2.96) *has a unique solution* $z(t, \lambda)$ *defined on* $[t_0, t_1]$ *and satisfies*

$$z(t, \lambda) \in J \quad \big|z(t, \lambda) - w(t)\big| < \varepsilon \quad \text{for all } t \in [t_0, t_1] \tag{2.97}$$

Proof of Claim Since $J \subseteq \Re$ is an open interval and $[t_0, t_1]$ is compact, we can find $\rho > 0$ such that if $x \in \Re$ with $|x - w(t)| \leq \rho$ for some $t \in [t_0, t_1]$, then $x \in J$. Let $S := [t_0, t_1] \times [a, b]$, where $a := \min_{t \in [t_0, t_1]} w(t) - \rho$ and $b := \max_{t \in [t_0, t_1]} w(t) + \rho$ (notice that $S \subset [t_0, t_1] \times J$), $L := \sup\{\frac{(x-y)(f(t,x)-f(t,y))}{|x-y|^2} : (t,x) \in S, (t,y) \in S, x \neq y\} < +\infty$, and $\lambda < \delta := \min\{\frac{\varepsilon}{4}, \frac{\rho}{2}\}\exp(-\frac{L+1}{2}(t_1 - t_0))$. Clearly, there exists $\tau \in (t_0, t_1]$ such that the solution of (2.96) exists for $t \in [t_0, \tau]$ and satisfies

$(t, z(t, \lambda)) \in S$ for all $t \in [t_0, \tau]$. Define $v(t) := |z(t, \lambda) - w(t)|^2$. Using the above definitions, we get $\dot{v}(t) \leq 2(L+1)v(t) + \lambda^2$ for all $t \in [t_0, \tau]$, which directly implies that $|z(t, \lambda) - w(t)| \leq \lambda \exp(\frac{L+1}{2}(t_1 - t_0))$ for all $t \in [t_0, \tau]$. Using the fact that $\lambda < \delta := \min\{\frac{\varepsilon}{4}, \frac{\rho}{2}\} \exp(-\frac{L+1}{2}(t_1 - t_0))$ and a standard contradiction argument, we can show that $|z(t, \lambda) - w(t)| \leq \min\{\frac{\varepsilon}{4}, \frac{\rho}{2}\}$ for all $t \in [t_0, t_1]$. □

Fact I $v(t) \leq z(t, \lambda)$, *for all* $t \in [t_0, t_1)$.

Proof of Fact I This fact is shown by contradiction. Suppose that there exists $t \in (t_0, t_1)$ such that $v(t) - z(t, \lambda) > 0$. For the lower semi-continuous function $m(t) := v(t) - z(t, \lambda)$, define the set

$$A^+ := \left\{\tau \in (t_0, t_1) : m(\tau) > 0\right\} \tag{2.98}$$

which by assumption is nonempty. The lower semi-continuity of $m(t) := v(t) - z(t, \lambda)$ implies that A^+ is open. Let

$$\tilde{t} := \inf\left\{t \in A^+\right\} \tag{2.99}$$

Since A^+ is open, we conclude that $\tilde{t} \notin A^+$ or equivalently that $v(\tilde{t}) \leq z(\tilde{t}, \lambda)$. On the other hand, by definition (2.99) there exists a sequence $\{\tau_i \in A^+\}_{i=1}^{\infty}$ with $\tau_i \to \tilde{t}$. Consequently, we obtain

$$v(\tau_i) - v(\tilde{t}) \geq z(\tau_i, \lambda) - z(\tilde{t}, \lambda) \tag{2.100}$$

This implies

$$D^+ v(\tilde{t}) \geq \dot{z}(\tilde{t}, \lambda) = f\left(\tilde{t}, z(\tilde{t}, \lambda)\right) + \lambda \tag{2.101}$$

We distinguish the following cases:

(i) If the mapping $f(t, \cdot)$ is nondecreasing on $J \subseteq \Re$, then the inequality $v(\tilde{t}) \leq z(\tilde{t}, \lambda)$ implies $f(\tilde{t}, v(\tilde{t})) \leq f(\tilde{t}, z(\tilde{t}, \lambda))$. The latter inequality, combined with (2.101), implies $D^+ v(\tilde{t}) > f(\tilde{t}, v(\tilde{t}))$, which contradicts (2.93).

(ii) If there exists a continuous function $\phi : [t_0, T) \to \Re$ such that $f(t, w) \leq \phi(t)$ for all $(t, w) \in [t_0, T) \times J$, then we may define the lower semi-continuous function $\tilde{v}(t) = v(t) - \int_{t_0}^{t} \phi(s)\, ds$. This function satisfies the following differential inequality:

$$D^+ \tilde{v}(t) \leq D^+ v(t) - \phi(t) \leq f\left(t, v(t)\right) - \phi(t) \leq 0$$

Consequently, by virtue of Lemma 6.3 in [3], $\tilde{v}(t)$ is nonincreasing. This implies that $\tilde{v}(t+h) \leq \tilde{v}(t)$ for all $h \geq 0$. Moreover, the lower semi-continuity of $\tilde{v}(t)$ implies that for every $\varepsilon > 0$, the inequality $\tilde{v}(t+h) \geq \tilde{v}(t) - \varepsilon$ for sufficiently small $h \geq 0$. It follows that $\tilde{v}(t)$ is a right-continuous function on $[t_0, t_1]$. By virtue of the right-continuity and definition (2.99), we must also have $v(\tilde{t}) \geq z(\tilde{t}, \lambda)$. Thus we must have $v(\tilde{t}) = z(\tilde{t}, \lambda)$, and in this case by virtue of (2.101) we obtain $D^+ v(\tilde{t}) > f(\tilde{t}, v(\tilde{t}))$, which contradicts (2.93).

(iii) By virtue of the right-continuity of the function $v : [t_0, T) \to \Re$ and definition (2.99), we must also have $v(\tilde{t}) \geq z(\tilde{t}, \lambda)$. Thus we must have $v(\tilde{t}) = z(\tilde{t}, \lambda)$, and in this case by virtue of (2.101) we obtain $D^+ v(\tilde{t}) > f(\tilde{t}, v(\tilde{t}))$, which contradicts (2.93). □

Fact II $v(t) \leq w(t)$ *for all* $t \in [t_0, t_1)$.

Proof of Fact II Again, this claim may be shown by contradiction. Suppose that there exists $a \in (t_0, t_1)$ with $v(a) > w(a)$. Let $\varepsilon = \frac{1}{2}(v(a) - w(a)) > 0$. Furthermore, let $\lambda > 0$ be selected in such a way that (2.97) is satisfied with this particular selection of $\varepsilon > 0$. Then we obtain

$$v(a) = v(a) - w(a) + w(a) = 2\varepsilon + w(a) - z(a, \lambda) + z(a, \lambda) > \varepsilon + z(a, \lambda)$$

which contradicts Fact I. □

Finally, notice that $v(t_1) = \liminf_{t \to t_1} v(t) \leq \liminf_{t \to t_1} w(t) = w(t_1)$ and consequently the inequality $v(t) \leq w(t)$ holds for all $t \in [t_0, t_1]$. The proof is complete. □

The following lemma exploits the results of the two previous lemmas and provides a sharper estimate to a subclass of the differential inequalities considered in Lemma 2.11.

Lemma 2.13 *For every continuous positive definite function* $\rho : \Re^+ \to \Re^+$ *being locally Lipschitz on* $\Re^+ \backslash \{0\}$, *there exists a KL function* $\sigma : (\Re^+)^2 \to \Re^+$ *with* $\sigma(s, 0) = s$ *and* $\frac{\partial \sigma}{\partial t}(s, t) = -\rho(\sigma(s, t))$ *for all* $t, s \geq 0$ *which satisfies the following property*:

(P) *If* $t_1 > t_0 \geq 0$ *and* $y : [t_0, t_1] \to \Re^+$ *is an absolutely continuous function that satisfies the following differential inequality for almost all* $t \in [t_0, t_1]$:

$$\dot{y}(t) \leq -\rho\big(y(t)\big) \tag{2.102}$$

then the following estimate holds:

$$y(t) \leq \sigma\big(y(t_0), t - t_0\big) \quad \text{for all } t \in [t_0, t_1]. \tag{2.103}$$

Proof Consider the autonomous scalar differential equation

$$\begin{aligned} \dot{w}(t) &= -\rho\big(w(t)\big) \\ w(t_0) &= w_0 \end{aligned} \tag{2.104}$$

where $w_0 > 0$. We will next show that the above initial-value problem admits a solution denoted by $\zeta(t - t_0, w_0)$ defined for all $t \geq t_0$ which has the following properties:

1. either $\zeta(t-t_0,w_0)>0$ for all $t\geq t_0$, or there exists $t_{\max}>t_0$ such that $\zeta(t-t_0,w_0)>0$ for all $t\in[t_0,t_{\max})$ and $\zeta(t-t_0,w_0)=0$ for all $t\geq t_{\max}$,
2. the function $\sigma:(\Re^+)^2\to\Re^+$ defined by $\sigma(s,t):=\zeta(t,s)$ for all $t\geq 0$, $s>0$ and $\sigma(0,t):=0$ for all $t\geq 0$ is a *KL* function with $\sigma(s,0)=s$ and $\frac{\partial\sigma}{\partial t}(s,t)=-\rho(\sigma(s,t))$ for all $t,s\geq 0$.

Then property (P) is an immediate consequence of Lemma 2.12. Indeed, if $y:[t_0,t_1]\to\Re^+$ is an absolutely continuous function that satisfies differential inequality (2.102) for almost all $t\in[t_0,t_1]$, we have $y(t+h)=y(t)+\int_t^{t+h}\dot{y}(\tau)\,d\tau\leq y(t)-\int_t^{t+h}\rho(y(\tau))\,d\tau$ for all $t\in[t_0,t_1)$ and $h>0$ sufficiently small. Therefore, we get $D^+y(t):=\limsup_{h\to 0^+}\frac{y(t+h)-y(t)}{h}\leq-\rho(y(t))$ for all $t\in[t_0,t_1)$. Applying Lemma 2.12 with $J:=(0,+\infty)$ and $T:=\sup\{t\in[t_0,t_1]:y(t)>0\}$, we get (2.103) for all $t\in[t_0,T)$ (notice that if $t_{\max}<T$, where $t_{\max}>t_0$ is the time with $\zeta(t-t_0,y(t_0))>0$ for all $t\in[t_0,t_{\max})$ and $\zeta(t-t_0,y(t_0))=0$ for all $t\geq t_{\max}$, we obtain a contradiction, and consequently we must have $t_{\max}\geq T$). If $T<t_1$, by continuity and monotonicity of $y:[t_0,t_1]\to\Re^+$ we obtain $y(t)=0$ for all $t\in[T,t_1]$, which shows that (2.103) holds for all $t\in[t_0,t_1)$. Finally, by the continuity and monotonicity of $y:[t_0,t_1]\to\Re^+$ we obtain (2.103) for all $t\in[t_0,t_1]$ and for the case $y(t_0)=0$.

The theory of ordinary differential equations and the fact that $\rho:\Re^+\to\Re^+$ is locally Lipschitz on $\Re^+\backslash\{0\}$ guarantees that for all $w_0>0$, there exists $t_{\max}>t_0$ such that the unique solution of (2.104) $\zeta(t-t_0,w_0)$ is defined for $t\in[t_0,t_{\max})$ and satisfies $\zeta(t-t_0,w_0)>0$ for all $t\in[t_0,t_{\max})$. By virtue of (2.104) it follows that $0<\zeta(t-t_0,w_0)\leq w_0$ for all $t\in[t_0,t_{\max})$. Consequently, if $t_{\max}<+\infty$, we have $\lim_{t\to t_{\max}^-}\zeta(t-t_0,w_0)=0$. If $t_{\max}=+\infty$, then Lemma 2.11 implies that $\lim_{t\to+\infty}\zeta(t-t_0,w_0)=0$. Notice that the theory of ordinary differential equations guarantees that for each fixed $t\in[t_0,t_{\max})$, the mapping $w_0\to\zeta(t-t_0,w_0)$ is continuous. Moreover, for each fixed $w_0>0$, the mapping $[t_0,t_{\max})\ni t\to\zeta(t-t_0,w_0)$ is nonincreasing. The solution is continuously extended by $\zeta(t-t_0,w_0)=0$ for all $t\geq t_{\max}$. Using the above properties, it can be shown that the function $\sigma:(\Re^+)^2\to\Re^+$ defined by $\sigma(s,t):=\zeta(t,s)$ for all $t\geq 0$, $s>0$ and $\sigma(0,t):=0$ for all $t\geq 0$ is continuous and nonincreasing in $t\geq 0$ with $\lim_{t\to+\infty}\sigma(s,t)=0$ for all $s\geq 0$. The reader should also notice that the equalities $\sigma(s,0)=s$ and $\frac{\partial\sigma}{\partial t}(s,t)=-\rho(\sigma(s,t))$ hold for all $t,s\geq 0$.

The only thing that remains to be shown is that the mapping $\Re^+\ni s\to\sigma(s,t)$ is nondecreasing. Let $s_1>s_2>0$ arbitrary. Since $\zeta(0,s_1)=s_1>s_2$ and $\lim_{t\to+\infty}\zeta(t,s_1)=0<s_2$, it follows that there exists unique $\tau>0$ such that $\zeta(\tau,s_1)=s_2$. For every $t\geq 0$, the semigroup property implies $\zeta(t,s_2)=\zeta(t,\zeta(\tau,s_1))=\zeta(t+\tau,s_1)$. The fact that the mapping $\Re^+\ni t\to\zeta(t,s_1)$ is nonincreasing implies that $\zeta(t,s_2)=\zeta(t+\tau,s_1)\leq\zeta(t,s_1)$. Therefore, the mapping $\Re^+\ni s\to\zeta(t,s)$ is nondecreasing; hence the mapping $\Re^+\ni s\to\sigma(s,t)$ is nondecreasing. The proof is complete. □

The following comparison lemma provides a "fading memory" estimate for an absolutely continuous mapping.

Lemma 2.14 *For each C^0 positive definite $\rho : \Re^+ \to \Re^+$, there exists a function σ of class KL, with the following property: if $y : [t_0, t_1] \to \Re^+$ is an absolutely continuous function, $u : \Re^+ \to \Re^+$ is a locally bounded mapping, and $I \subset [t_0, t_1]$ a set of Lebesgue measure zero such that $\dot{y}(t)$ is defined on $[t_0, t_1]\backslash I$ and such that the following implication holds for all $t \in [t_0, t_1]\backslash I$:*

$$y(t) \geq u(t) \quad \Rightarrow \quad \dot{y}(t) \leq -\rho\big(y(t)\big) \tag{2.105}$$

then the following estimate holds for all $t \in [t_0, t_1]$:

$$y(t) \leq \max\Big\{\sigma\big(y(t_0), t - t_0\big), \sup_{t_0 \leq s \leq t} \sigma\big(u(s), t - s\big)\Big\} \tag{2.106}$$

Moreover, if $\rho : \Re^+ \to \Re^+$ is locally Lipschitz on $(0, +\infty)$, then $\sigma(s, 0) = s$, and $\frac{\partial \sigma}{\partial t}(s, t) = -\rho(\sigma(s, t))$ for all $t, s \geq 0$.

Proof Notice that by virtue of Lemma 2.11, for each positive definite continuous function $\rho : \Re^+ \to \Re^+$, there exists a continuous function σ of class *KL* with the following property: if $y : [t_0, t_1] \to \Re^+$ is an absolutely continuous function and $I \subset [t_0, t_1]$ a set of Lebesgue measure zero such that $\dot{y}(t)$ is defined on $[t_0, t_1]\backslash I$ and such that the following differential inequality holds for all $t \in [t_0, t_1]\backslash I$:

$$\dot{y}(t) \leq -\rho\big(y(t)\big) \tag{2.107}$$

then, the following estimate holds for all $t \in [t_0, t_1]$:

$$y(t) \leq \sigma\big(y(t_0), t - t_0\big) \tag{2.108}$$

Indeed, we may consider the absolutely continuous function $\tilde{y} : \Re^+ \to \Re^+$ defined as follows:

- for the case $y(t_1) > 0$, $\tilde{y}(t) = y(t)$ for $t \in [t_0, t_1]$, $\tilde{y}(t) = y(t_1) - (t - t_1)\max_{0 \leq w \leq y(t_1)} \rho(w)$ for $\xi \in [t_1, t_1 + \frac{y(t_1)}{\max_{0 \leq w \leq y(t_1)} \rho(w)}]$, and $\tilde{y}(t) = 0$ for $t > t_1 + \frac{y(t_1)}{\max_{0 \leq w \leq y(t_1)} \rho(w)}$,
- for the case $y(t_1) = 0$, $\tilde{y}(t) = y(t)$ for $t \in [t_0, t_1]$, and $\tilde{y}(t) = 0$ for $t > t_1$.

The reader may verify that $\tilde{y} : \Re^+ \to \Re^+$ is an absolutely continuous function which satisfies $\frac{d}{dt}\tilde{y}(t) \leq -\rho(\tilde{y}(t))$ for almost all $t \geq t_0$. Lemma 2.11 applied for $\tilde{y} : [t_0, +\infty) \to \Re^+$ implies (2.108) for all $t \in [t_0, t_1]$.

Notice that we may continuously extend σ by defining $\sigma(s, t) := \sigma(s, 0)\exp(-t)$ for $t < 0$. Moreover, notice that Lemma 2.13 guarantees that $\sigma(s, 0) = s$ and $\frac{\partial \sigma}{\partial t}(s, t) = -\rho(\sigma(s, t))$ for all $t, s \geq 0$, provided that $\rho : \Re^+ \to \Re^+$ is locally Lipschitz on $(0, +\infty)$.

Clearly, (2.106) holds for $t = t_0$ and σ the function involved in (2.108). We next show that (2.106) holds for arbitrary $t \in (t_0, t_1]$.

Let arbitrary $t \in (t_0, t_1]$ and define the functions

$$\tilde{u}(\tau) = \begin{cases} u(\tau) & \forall \tau \in [t_0, t] \\ 0 & \text{otherwise} \end{cases} \qquad \bar{u}(\tau) := \limsup_{\xi \to \tau} \tilde{u}(\xi)$$

Notice that $\bar{u}$ is upper semi-continuous on $\tau \in [t_0, t]$, and consequently the function $p(\tau) := y(\tau) - \bar{u}(\tau)$ is lower semi-continuous on $[t_0, t]$. Next define the set

$$\Lambda := \left\{ \tau \in [t_0, t] : y(\tau) \le \bar{u}(\tau) \right\} \tag{2.109}$$

We distinguish the following cases:

(1) $\Lambda = \emptyset$. In this case we have $y(\tau) > \bar{u}(\tau)$ for all $\tau \in [t_0, t]$. Since $\bar{u}(\tau) \ge u(\tau)$ for all $\tau \in [t_0, t]$, the previous inequality, in conjunction with (2.105), implies that $\dot{y}(\tau) \le -\rho(y(\tau))$ for all $\tau \in [t_0, t] \backslash I$. Thus in this case Lemma 2.11 (or Lemma 2.13) guarantees that estimate (2.108) holds.

(2) $\Lambda \ne \emptyset$ and $\xi := \sup \Lambda < t$. In this case there exists a sequence $\tau_i \le \xi$ with $\tau_i \to \xi$ and $y(\tau_i) - \bar{u}(\tau_i) \le 0$. Since the function $p(t) = y(t) - \bar{u}(t)$ is lower semi-continuous, we obtain $p(\xi) = \liminf_{\tau \to \xi} p(\tau) \le 0$, and consequently $y(\xi) \le \bar{u}(\xi)$. Moreover, notice that by definition (2.109), implication (2.105), and since $\bar{u}(\tau) \ge u(\tau)$ for all $\tau \in [t_0, t]$, the differential inequality $\dot{y}(\tau) \le -\rho(y(\tau))$ holds for all $\tau \in (\xi, t] \backslash I$. Consequently, Lemma 2.11 (or Lemma 2.13) implies $y(t) \le \sigma(y(\tau), t - \tau)$ for all $\tau \in (\xi, t]$. By virtue of the continuity of σ and y, we get

$$y(t) \le \sigma\big(y(\xi), t - \xi\big)$$

which, combined with $y(\xi) \le \bar{u}(\xi)$, directly implies

$$y(t) \le \sigma\big(\bar{u}(\xi), t - \xi\big) \le \sup_{t_0 \le s \le t} \sigma\big(\bar{u}(s), t - s\big) \tag{2.110}$$

(3) $\Lambda \ne \emptyset$ and $\xi := \sup \Lambda = t$. In this case there exists a sequence $\tau_i \le t$ with $\tau_i \to t$ and $y(\tau_i) - \bar{u}(\tau_i) \le 0$. Since the function $p(t) = y(t) - \bar{u}(t)$ is lower semi-continuous, we obtain $p(t) = \liminf_{\tau \to t} p(\tau) \le 0$, and consequently $y(t) \le \bar{u}(t)$. Moreover, since $\sigma(s, 0) \ge s$ for all $s \ge 0$, it holds that

$$y(t) \le \bar{u}(t) \le \sigma\big(\bar{u}(t), 0\big) \le \sup_{t_0 \le s \le t} \sigma\big(\bar{u}(s), t - s\big)$$

Combining all the above cases, we may conclude that

$$y(t) \le \max\left\{ \sigma\big(y(t_0), t - t_0\big), \sup_{t_0 \le s \le t} \sigma\big(\bar{u}(s), t - s\big) \right\} \tag{2.111}$$

Let $M := \sup_{t_0 \le s \le t} u(s)$. For each $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma(s, \tau - \delta) - \sigma(s, \tau) < \varepsilon$ for all $(s, \tau) \in [0, M] \times [0, t]$. Notice that since

$$\bar{u}(\tau) := \limsup_{\xi \to \tau} \tilde{u}(\xi) \quad \text{and} \quad \tilde{u}(\tau) = \begin{cases} u(\tau) & \forall \tau \in [t_0, t] \\ 0 & \text{otherwise} \end{cases}$$

it follows that $\bar{u}(s) \le \sup\{u(r) : \max(s-\delta, t_0) \le r \le \min(s+\delta, t)\}$ for all $s \in [t_0, t]$. The previous inequalities imply that

$$\begin{aligned}
\sigma\big(\bar{u}(s), t-s\big) &\le \sup\big\{\sigma\big(u(r), t-s\big) : \max(s-\delta, t_0) \le r \le \min(s+\delta, t)\big\} \\
&\le \sup\big\{\sigma\big(u(r), t-r-\delta\big) : \max(s-\delta, t_0) \le r \le \min(s+\delta, t)\big\} \\
&\le \sup\big\{\sigma\big(u(r), t-r\big) : \max(s-\delta, t_0) \le r \le \min(s+\delta, t)\big\} + \varepsilon \\
&\le \sup_{t_0 \le r \le t} \sigma\big(u(r), t-r\big) + \varepsilon
\end{aligned}$$

The above inequality, in conjunction with (2.111), implies that for each $\varepsilon > 0$, it holds that

$$y(t) \le \max\Big\{\sigma\big(y(t_0), t-t_0\big), \sup_{t_0 \le s \le t} \sigma\big(u(s), t-s\big)\Big\} + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the above estimate directly implies (2.106). The proof is complete. □

Lemma 2.15 *For every pair of functions $\rho : \Re^+ \to \Re^+$ being locally Lipschitz and positive definite and μ being of class $\mathcal{E}$ with $\mu(t) > 0$ for all $t \ge 0$, there exists a KL function $\sigma : (\Re^+)^2 \to \Re^+$ which satisfies the following property*:

(P) *If $c \in [0, 1]$ is a constant, $t_0 \ge 0$, $\gamma \in K^+$ with $\int_0^{+\infty} \gamma(t)\,dt = +\infty$, and $y : [t_0, t_1] \to \Re^+$ is an absolutely continuous function that satisfies the following differential inequality for almost all $t \in [t_0, t_1]$*:

$$\dot{y}(t) \le -\gamma(t)\rho\big(y(t)\big) + c\gamma(t)\mu\left(\int_0^t \gamma(s)\,ds\right) \tag{2.112}$$

then the following estimate holds for all $t \in [t_0, t_1]$:

$$y(t) \le \sigma\left(y(t_0) + c\int_{t_0}^{+\infty} \mu\left(\int_0^s \gamma(w)\,dw\right)\gamma(s)\,ds, \int_{t_0}^t \gamma(s)\,ds\right). \tag{2.113}$$

Proof Let $\phi(t, t_0, y_0)$ denote the solution of the scalar differential equation

$$\begin{aligned}
&\dot{y}(t) = -\gamma(t)\tilde{\rho}\big(y(t)\big) + c\gamma(t)\mu\left(\int_0^t \gamma(\tau)\,d\tau\right) \\
&y(t_0) = y_0 \in \Re
\end{aligned} \tag{2.114}$$

where $\tilde{\rho}(y) := \rho(y)$ for $y \ge 0$ and $\tilde{\rho}(y) := 0$ for $y < 0$. Notice that $\tilde{\rho} : \Re \to \Re^+$ is a locally Lipschitz function which coincides with $\rho : \Re^+ \to \Re^+$ on $\Re^+$. By performing the time-scaling $\tau = \int_0^t \gamma(s)\,ds$ it is clear that

$$\phi(t, t_0, y_0) = \zeta\left(\int_0^t \gamma(s)\,ds, \int_0^{t_0} \gamma(s)\,ds, y_0\right) \tag{2.115}$$

where $\zeta(\tau, \tau_0, y_0)$ denotes the solution of the scalar differential equation

$$\frac{d}{d\tau} y(\tau) = -\tilde{\rho}\big(y(\tau)\big) + c\mu(\tau)$$
$$y(\tau_0) = y_0 \in \Re \tag{2.116}$$

Using a contradiction argument, it can be shown that $\zeta(\tau, \tau_0, y_0)$ exists for all $\tau \geq \tau_0$ and $\zeta(\tau, \tau_0, y_0) \geq 0$ for all $\tau \geq \tau_0$ if $y_0 \geq 0$. Moreover, Lemma 2.11 (applied to the absolutely continuous function $y(\tau) := \zeta(\tau, \tau_0, y_0)$, $y_0 \geq 0$) implies that there exists a *KL* function $\sigma : (\Re^+)^2 \to \Re^+$ such that the following estimate holds for all $y_0 \geq 0$:

$$0 \leq \zeta(\tau, \tau_0, y_0) \leq \sigma\left(y_0 + c\int_{\tau_0}^{+\infty} \mu(s)\, ds, \tau - \tau_0\right) \quad \text{for all } \tau \geq \tau_0 \tag{2.117}$$

It follows from (2.115) and (2.117) that the solution $\phi(t, t_0, y_0)$ with $y_0 \geq 0$ satisfies the following estimate: for all $t \in [t_0, t_1]$,

$$0 \leq \phi(t, t_0, y_0) \leq \sigma\left(y_0 + c\int_{t_0}^{+\infty} \mu\left(\int_0^s \gamma(w)\, dw\right)\gamma(s)\, ds, \int_{t_0}^{t} \gamma(s)\, ds\right) \tag{2.118}$$

By applying Lemma 2.12 with $f(t, y) = -\gamma(t)\tilde{\rho}(y) + c\gamma(t)\mu(\int_0^t \gamma(\tau)\, d\tau)$ and $J := (-a, +\infty)$, where $a > 0$ and $T := t_1$, we obtain, for all $t \in [t_0, t_1]$,

$$0 \leq y(t) \leq \phi(t, t_0, y_0) \tag{2.119}$$

The conclusion of the lemma follows by combining inequalities (2.118) and (2.119). The proof is complete. □

Remark 2.4 It should be noted that if $\rho : \Re^+ \to \Re^+$ is a continuous positive definite function, then there exists $\tilde{\rho} : \Re^+ \to \Re^+$ which is globally Lipschitz on $\Re^+$ with unit Lipschitz constant and positive definite, and satisfies $0 < \tilde{\rho}(s) \leq \rho(s)$ for all $s > 0$. Indeed, we may define $\tilde{\rho}(s) := \inf\{\rho(y) + |y - s|;\, y \geq 0\}$, which is globally Lipschitz on $\Re^+$ with unit Lipschitz constant. To see this, first notice that $\tilde{\rho}(s) := \min\{\rho(y) + |y - s|;\, 0 \leq y \leq s + \rho(s)\}$ for all $s \geq 0$. Let $s_1, s_2 \geq 0$ with $\tilde{\rho}(s_1) := \rho(y_1) + |y_1 - s_1|$ and notice that

$$\tilde{\rho}(s_2) \leq \rho(y_1) + |y_1 - s_2| \leq \rho(y_1) + |y_1 - s_1| + |s_1 - s_2| = \tilde{\rho}(s_1) + |s_1 - s_2|$$

Consequently, Lemma 2.13 guarantees that for every continuous positive definite function $\rho : \Re^+ \to \Re^+$, there exists a *KL* function $\sigma : (\Re^+)^2 \to \Re^+$ with $\sigma(s, 0) = s$ for all $s \geq 0$ which satisfies Property (P) of Lemma 2.13. Moreover, Lemma 2.14 guarantees that for each C^0 positive definite function $\rho : \Re^+ \to \Re^+$, there exists a function σ of class *KL* with $\sigma(s, 0) = s$ for all $s \geq 0$, satisfying the property described in the statement of Lemma 2.14.

2.6 Lyapunov Functionals

One of the most important tools for establishing RGAOS for a control system is the method of Lyapunov functionals. Lyapunov functionals are, roughly speaking, nonnegative functionals of the state of the control system which tend to 0 as $t \to +\infty$.

The following theorem shows that the existence of a Lyapunov functional is a sufficient condition for RGAOS for systems that satisfy the following additional hypothesis:

(HYP) There exists a constant $r > 0$, a bounded positive continuous function $h : \Re^+ \times \Re^+ \to (0, r]$, and a partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, it holds that $\pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + r] \neq \emptyset$, where $b(t, \rho) := \min\{q_\pi(t), t + h(t, \rho)\}$.

Remark 2.5 It is worth noting the following.

1. Hypothesis (HYP) holds for many classes of systems including systems described by ordinary differential equations, systems with variable sampling partition, and systems described by retarded functional differential equations.
2. Hypothesis (HYP) guarantees the existence of $\tau \in (t_0, t_0 + r]$ such that $\tau \in \pi(t_0, x_0, u_0, d)$ for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$. Thus, for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, it follows that the set $\pi(t_0, x_0, u_0, d) \cap (t_0, +\infty)$ cannot be empty.

Theorem 2.3 (Lyapunov functionals) *Let $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ be a control system with outputs satisfying Hypothesis* (HYP) *and the BIC property. Assume that $0 \in \mathcal{X}$ is a robust equilibrium point for Σ. Suppose that there exist mappings $V : \Re^+ \times \mathcal{X} \to \Re^+$, $\beta, \gamma, \mu \in K^+$ with $\int_0^{+\infty} \gamma(t)\,dt = +\infty$, $\varphi \in \mathcal{E}$, $a_1, a_2 \in K_\infty$, and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, there exists $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + r]$, where $b(t, \rho) := \min\{q_\pi(t), t + h(t, \rho)\}$ is the function in* (HYP), *with $(\tau, t_0, x_0, u_0, d) \in A_\phi$ and the following properties*:

$$a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big)$$
$$\leq V(t_0, x_0) \leq a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) \quad \text{for all } t \in [t_0, \tau] \tag{2.120}$$
$$V\big(\tau, \phi(\tau, t_0, x_0, u_0, d)\big) \leq \eta\big(\tau, t_0, V(t_0, x_0)\big) \tag{2.121}$$

where $\eta(t, t_0, \eta_0)$ denotes the unique solution of the initial-value problem

$$\dot{\eta} = -\gamma(t)\rho(\eta) + \gamma(t)\varphi\left(\int_0^t \gamma(s)\,ds\right) \quad \eta(t_0) = \eta_0 \geq 0 \tag{2.122}$$

Then, Σ is RGAOS. Particularly, if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$ and if for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, we have, in addition, $a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}) \leq V(t_0, x_0) \leq a_2(\|x_0\|_{\mathcal{X}})$ for all $t \in [t_0, \tau]$, with $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + r]$ the time for which (2.120) *and* (2.121) *hold, then Σ is URGAOS.*

Proof By virtue of Lemma 2.1, it suffices to show that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is RFC and satisfies the Robust Output Attractivity Property (Property P3 of Definition 2.2). Notice that Lemma 2.15 implies that there exist a function $\sigma(\cdot) \in KL$ and a constant $M > 0$ such that the following inequalities are satisfied for all $t_0 \geq 0$:

$$0 \leq \eta(t, t_0, \eta_0) \leq \sigma\big(\eta_0 + M, q(t, t_0)\big), \quad \text{for all } t \geq t_0, \text{ for all } \eta_0 \geq 0 \tag{2.123}$$

where $q(t, t_0) := \int_{t_0}^{t} \gamma(s)\, ds$. Furthermore, if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, it follows from Lemma 2.11 that there exists $\sigma(\cdot) \in KL$ such that the following inequalities are satisfied for all $t_0 \geq 0$:

$$0 \leq \eta(t, t_0, \eta_0) \leq \sigma(\eta_0, t - t_0) \quad \text{for all } t \geq t_0, \text{ for all } \eta_0 \geq 0 \tag{2.124}$$

Pick arbitrary $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and let $\tau_0 = t_0$. We consider the sequence $x_i = \phi(\tau_i, \tau_{i-1}, x_{i-1}, u_0, d)$, $i \geq 1$, and $\tau_i \in \pi(\tau_{i-1}, x_{i-1}, u_0, d) \cap [b(\tau_{i-1}, \|x_{i-1}\|_{\mathcal{X}}), \tau_{i-1} + r]$, $i \geq 1$, for which (2.121) holds with (τ_{i-1}, x_{i-1}) in place of (t_0, x_0). By virtue of the weak semigroup property (Property 4 of Definition 1.1), we have $\tau_i \in \pi(t_0, x_0, u_0, d)$, $\tau_i \leq t_0 + ir$ and $x_i = \phi(\tau_i, t_0, x_0, u_0, d)$ for all $i \geq 1$. The semigroup property for $\eta(t, t_0, \eta_0)$, in conjunction with inequality (2.120) and trivial induction arguments, implies that

$$\begin{aligned} &a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big) \\ &\quad \leq V(\tau_{i-1}, x_{i-1}) \leq \eta\big(\tau_{i-1}, t_0, V(t_0, x_0)\big) \quad \forall i \geq 1, t \in [\tau_{i-1}, \tau_i] \end{aligned} \tag{2.125}$$

Combining (2.123) with (2.125) and using the fact that $\max\{t_0, t - r\} \leq \tau_{i-1}$ for $t \in [\tau_{i-1}, \tau_i]$, $i \geq 1$, we obtain, for all $i \geq 1$ and $t \in [t_0, \tau_i]$,

$$\begin{aligned} &a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big) \\ &\quad \leq \sigma\big(a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) + M, q\big(\max\{t_0, t - r\}, t_0\big)\big) \end{aligned} \tag{2.126}$$

which directly implies

$$\begin{aligned} &a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big) \\ &\quad \leq \sigma\big(a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) + M, 0\big) \quad \text{for all } i \geq 1 \text{ and } t \in [t_0, \tau_i] \end{aligned} \tag{2.127}$$

We next show that $(t, t_0, x_0, u_0, d) \in A_\phi$ for all $t \geq t_0$. By virtue of estimate (2.127) and the BIC property, it suffices to show that $\tau_i \to +\infty$. Let arbitrary $T > 0$. Since the set $[0, T + r] \times [0, \varepsilon]$ is compact, where $\varepsilon := \frac{a_1^{-1}(\sigma(a_2(\beta(t_0)\|x_0\|_{\mathcal{X}}) + M, 0))}{\min\{\mu(t); t \in [0, T + r]\}}$ and h is continuous, we have $s := \min\{h(t, \rho); (t, \rho) \in [0, T + r] \times [0, \varepsilon]\} > 0$. Consider the infinite sequence $\{y_i\}_{i=0}^{\infty}$ which satisfies $y_{i+1} = \min\{q_\pi(y_i), y_i + s\}$, $i = 1, 2, \ldots$, with $y_0 = 0$. By virtue of a standard contradiction argument we can show that $y_i \to +\infty$ (as in the proof of the Claim in Sect. 1.2.5), and consequently for every $T > 0$, there exists an integer $N > 0$ such that $y_N > T + r$. Clearly, the sequence $\{\tau_i\}_{i=0}^{\infty}$

satisfies $\tau_{i+1} \geq \min\{q_\pi(\tau_i), \tau_i + s\}$ for all integers i for which $\tau_i \leq T + r$. By virtue of Theorem 1.6.1 in [34] (Comparison principle) we have $\tau_i \geq y_i$ for all integers i for which $\tau_i \leq T + r$, which implies $\tau_N > T$.

It follows that estimates (2.126) and (2.127) hold for all $t \geq t_0$. Robust Forward Completeness is an immediate consequence of (2.127), and the Robust Output Attractivity Property (Property P3 of Definition 2.2) is an immediate consequence of estimate (2.126).

Notice that if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, and if for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, we have, in addition, that $a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}) \leq V(t_0, x_0) \leq a_2(\|x_0\|_{\mathcal{X}})$ for all $t \in [t_0, \tau]$, then using (2.124) instead of (2.123), we obtain, in addition, the following estimate for all $t \geq t_0$:

$$a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}}\big) \leq \sigma\big(a_2\big(\|x_0\|_{\mathcal{X}}\big), \max\{0, t - t_0 - r\}\big)$$

The above estimate directly implies that Σ is URGAOS. The proof is complete. □

Theorem 2.3 is a general theorem that can be applied to all deterministic systems of Definition 1.1 which satisfy property (HYP). However, the reader should note that inequalities (2.120), (2.121) are prerequisites for the application of Theorem 2.3 and they must be shown by using another result. Notice that inequalities (2.120) and (2.121) require a substantial amount of knowledge of the transition map of the control system, which is usually not available. This is a serious disadvantage of the method.

There are two ways to overcome the above disadvantage for systems which satisfy the classical semigroup property:

(1) Lyapunov's approach
(2) The "discretization approach"

Next, we describe the two approaches in an informal way.

(1) Lyapunov's approach

For systems which satisfy the classical semigroup property, the above disadvantage can be overcome in an elegant way: we let the constant $r > 0$ involved in Hypothesis (HYP) to become sufficiently small. Then inequality (2.121) becomes a differential inequality,

$$V\big(\tau, \phi(\tau, t_0, x_0, u_0, d)\big) - V(t_0, x_0) \leq \eta\big(\tau, t_0, V(t_0, x_0)\big) - V(t_0, x_0)$$

$$\limsup_{h \to 0^+} \frac{V(t_0 + h, \phi(t_0 + h, t_0, x_0, u_0, d)) - V(t_0, x_0)}{h}$$
$$\leq \limsup_{h \to 0^+} \frac{\eta(t_0 + h, t_0, V(t_0, x_0)) - V(t_0, x_0)}{h}$$

which implies

$$\limsup_{h\to 0^+} \frac{V(t_0+h, \phi(t_0+h, t_0, x_0, u_0, d)) - V(t_0, x_0)}{h}$$
$$\leq -\gamma(t_0)\rho\big(V(t_0, x_0)\big) + \gamma(t_0)\varphi\left(\int_0^{t_0} \gamma(s)\,ds\right)$$

Notice that an upper bound for the quantity

$$\limsup_{h\to 0^+} \frac{V(t_0+h, \phi(t_0+h, t_0, x_0, u_0, d)) - V(t_0, x_0)}{h}$$

can (usually) be computed without knowledge of the transition map $\phi(\tau, t_0, x_0, u_0, d)$ of the control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$. Moreover, letting $r > 0$ to become arbitrarily small and assuming continuity with respect to time of the transition map $\phi(\tau, t_0, x_0, u_0, d)$, inequality (2.120) requires no knowledge of the transition map:

$$a_1\big(\big\|H(t_0, x_0, 0)\big\|_{\mathcal{Y}} + \mu(t_0)\|x_0\|_{\mathcal{X}}\big) \leq V(t_0, x_0) \leq a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big)$$

The above thoughts lead us to the production of specialized results for two kinds of systems of Definition 1.1, systems described by ODEs and systems described by RFDEs.

(2) The "discretization approach"

The idea in this approach is that we examine the value of the Lyapunov functional only at discrete time instances. For the sequence of the chosen time instances, we can show that the Lyapunov functional decreases in value. However, we do not demand an explicit knowledge of the time instances, and we do not require that the difference between two consecutive time instances must be bounded (by r).

The discretization approach has been developed mostly for systems described by ODEs and has been applied successfully for the solution of difficult control problems. We will present specialized results for systems described by ODEs. However, the method can be applied to various systems which satisfy the classical semigroup property.

Due to the close relation of the discretization approach to the stability analysis of discrete-time systems, we will describe the method in the section devoted to discrete-time systems below. Here we describe Lyapunov's approach.

2.6.1 Control Systems Described by Ordinary Differential Equations

We start by presenting a general Lyapunov theorem for systems described by ODEs of the form (1.3) under Hypotheses (H1–4).

Let $V : \Re^+ \times \Re^n \to \Re^+$ be a locally bounded function. We define the following generalized Dini derivative for all $(t,x) \in \Re^+ \times \Re^n$, $v \in \Re^n$:

$$V^0(t,x;v) := \limsup_{h\to 0^+} \frac{V(t+h,x+hv)-V(t,x)}{h} \tag{2.128}$$

Notice that if $V : \Re^+ \times \Re^n \to \Re^+$ is locally Lipschitz, then the above derivative is locally bounded.

Theorem 2.4 *Consider system* (1.3) *under Hypotheses* (H1–4) *and suppose that it is RFC. Furthermore, suppose that there exist functions* $V : \Re^+ \times \Re^n \to \Re^+$ *being locally Lipschitz,* $\beta, \gamma \in K^+$ *with* $\int_0^{+\infty} \gamma(t)\,dt = +\infty$, $\varphi \in \mathcal{E}$, $a_1, a_2 \in K_\infty$, *and a locally Lipschitz positive definite function* $\rho : \Re^+ \to \Re^+$ *such that the following inequalities hold*:

$$a_1\big(\big|H(t,x,0)\big|\big) \le V(t,x) \le a_2\big(\beta(t)|x|\big) \quad \textit{for all } (t,x) \in \Re^+ \times \Re^n \tag{2.129}$$

$$V^0\big(t,x;f(t,x,0,d)\big) \le -\gamma(t)\rho\big(V(t,x)\big) + \gamma(t)\varphi\bigg(\int_0^t \gamma(s)\,ds\bigg)$$
$$\textit{for all } (t,x,d) \in \Re^+ \times \Re^n \times D \tag{2.130}$$

Then system (1.3) *is RGAOS. Particularly, if* $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, *and* $\beta(t) \equiv 1$, *then* (1.3) *is URGAOS. Moreover, if there exist functions* $a \in K_\infty$, $\mu \in K^+$, *and a constant* $R \ge 0$ *such that* $a(\mu(t)|x|) \le V(t,x) + R$ *for all* $(t,x) \in \Re^+ \times \Re^n$, *then the requirement that* (1.3) *is RFC is not needed.*

Proof By Lemma 2.1, it suffices to show that (1.3) is RFC and satisfies the Robust Output Attractivity Property (Property P3 of Definition 2.2).

Consider a solution $x(t)$ of (1.3) under Hypotheses (H1–4) corresponding to arbitrary $d \in M_D$ with initial condition $x(t_0) = x_0 \in \Re^n$. Clearly, the solution exists for $t \in [t_0, t_{\max})$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Notice that the mapping $t \to V(t,x(t))$ is absolutely continuous on $[t_0, t_{\max})$, and we have

$$\frac{d}{dt}V\big(t,x(t)\big) = V^0\big(t,x(t);f\big(t,x(t),0,d(t)\big)\big) \quad \text{a.e.} \quad \text{on } [t_0, t_{\max}) \tag{2.131}$$

where $V^0(t,x;v) := \limsup_{h\to 0^+} \frac{V(t+h,x+hv)-V(t,x)}{h}$. Indeed, notice that for all $t \in [t_0,t_{\max})\backslash I$, where $I \subset [t_0,t_{\max})$ is the set of zero Lebesgue measure such that either $\frac{d}{dt}V(t,x(t))$ is not defined on I, or $D^+x(t) := \lim_{h\to 0^+} h^{-1}(x(t+h)-x(t)) \ne f(t,x(t),0,d(t))$, we get, for sufficiently small $h > 0$ and any $t \in [t_0,t_{\max})\backslash I$,

$$x(t+h) - x(t) - hD^+x(t) = hy_h \tag{2.132}$$

where

$$y_h = \frac{x(t+h)-x(t)}{h} - D^+x(t)$$

Since $\lim_{h\to 0^+} \frac{x(t+h)-x(t)}{h} = D^+x(t)$, we obtain that $y_h \to 0$ as $h \to 0^+$. Since $V : \Re^+ \times \Re^n \to \Re^+$ is locally Lipschitz, we have

$$V^0\bigl(t, x(t); D^+x(t)\bigr) = \limsup_{h\to 0^+} \frac{V(t+h, x(t)+hD^+x(t)) - V(t, x(t))}{h}$$
$$= \limsup_{h\to 0^+} \frac{V(t+h, x(t)+hD^+x(t)+hy_h) - V(t, x(t))}{h}$$

The above equality, in conjunction with (2.132) and the facts that $\frac{d}{dt}V(t,x(t)) = \limsup_{h\to 0^+} \frac{V(t+h,x(t+h))-V(t,x(t))}{h}$ and $D^+x(t) = f(t, x(t), 0, d(t))$ for all $t \in [t_0, t_{\max})\backslash I$, implies (2.131).

It follows from Lemma 2.15 and inequalities (2.130) and (2.131) that there exist $\sigma \in KL$ and a constant $M > 0$ such that

$$V\bigl(t, x(t)\bigr) \le \sigma\left(V(t_0, x_0) + M, \int_{t_0}^{t} \gamma(s)\,ds\right) \quad \text{for all } t \in [t_0, t_{\max}) \tag{2.133}$$

Next, we distinguish the following cases:

1. If (1.3) is RFC, then (2.133) holds for all $t \ge t_0$, then the Robust Output Attractivity Property (Property P3 of Definition 2.2) is a direct consequence of inequalities (2.129) and (2.133) and from the fact that $\int_0^{+\infty} \gamma(t)\,dt = +\infty$.
2. If there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \ge 0$ such that $a(\mu(t)|x|) \le V(t,x) + R$ for all $(t,x) \in \Re^+ \times \Re^n$, then (2.133), in conjunction with (2.129), implies the following estimate:

$$\bigl|x(t)\bigr| \le \frac{1}{\mu(t)} a^{-1}\bigl(R + \sigma\bigl(a_2\bigl(\beta(t_0)|x_0|\bigr) + M, 0\bigr)\bigr) \quad \text{for all } t \in [t_0, t_{\max}). \tag{2.134}$$

Estimate (2.134), in conjunction with the BIC property, implies that $t_{\max} = +\infty$. Moreover, estimate (2.134), in conjunction with Definition 2.1, shows that system (1.3) is RFC, since we have, for all $r, T \ge 0$,

$$\sup\bigl\{\bigl|x(t_0+s)\bigr|; s \in [0,T], |x_0| \le r, t_0 \in [0,T], d \in M_D\bigr\}$$
$$\le \frac{1}{\min_{0\le t\le 2T} \mu(t)} a^{-1}\left(R + \sigma\left(a_2\left(r \max_{0\le t\le T} \beta(t)\right) + M, 0\right)\right) < +\infty$$

The proof is complete. □

Remark 2.6 If for all $(x,d) \in \Re^n \times D$, the mapping $t \to f(t,x,0,d)$ is continuous, then inequality (2.130) is equivalent to the following inequality:

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x,0,d) \le -\gamma(t)\rho\bigl(V(t,x)\bigr) + \gamma(t)\varphi\left(\int_0^t \gamma(s)\,ds\right) \tag{2.135}$$

for almost all $(t,x) \in \Re^+ \times \Re^n$ and all $d \in D$.

Indeed, using Corollary 8.2 in [11] (p. 95), we have, for all $v \in \Re^n$ and $(t,x) \in \Re^+ \times \Re^n$,

$$\limsup_{h\to 0^+,(\tau,y)\to(t,x)} \frac{V(\tau+h, y+hv) - V(\tau,y)}{h}$$
$$= \limsup_{(\tau,y)\to(t,x)} \left\{ \frac{\partial V}{\partial t}(\tau,y) + \frac{\partial V}{\partial x}(\tau,y)v : (\tau,y) \notin \Omega \cup \Omega_V \right\}$$

where $\Omega_V \subset \Re^+ \times \Re^n$ is the zero Lebesgue measure set such that $V : \Re^+ \times \Re^n \to \Re^+$ is not differentiable on Ω_V, and $\Omega \subset \Re^+ \times \Re^n$ is an arbitrary set of zero Lebesgue measure. Taking $v = f(t,x,0,d)$ and using the facts that $\frac{\partial V}{\partial x}(\tau,y)$ is bounded in a neighborhood of $(t,x) \in \Re^+ \times \Re^n$ and that $f(\tau,y,0,d) \to f(t,x,0,d)$ for $(\tau,y) \to (t,x)$, we get

$$\limsup_{h\to 0^+,(\tau,y)\to(t,x)} \frac{V(\tau+h, y+hf(t,x,0,d)) - V(\tau,y)}{h}$$
$$= \limsup_{(\tau,y)\to(t,x)} \left\{ \frac{\partial V}{\partial t}(\tau,y) + \frac{\partial V}{\partial x}(\tau,y)f(\tau,y,0,d) : (\tau,y) \notin \Omega \cup \Omega_V \right\}$$

The above equality, together with (2.135), implies that, for all $(t,x) \in \Re^+ \times \Re^n$, $d \in D$,

$$\limsup_{h\to 0^+,(\tau,y)\to(t,x)} \frac{V(\tau+h, y+hf(t,x,0,d)) - V(\tau,y)}{h}$$
$$\le -\gamma(t)\rho\big(V(t,x)\big) + \gamma(t)\varphi\left(\int_0^t \gamma(s)\,ds\right)$$

By means of the fact that

$$V^0\big(t,x,;f(t,x,0,d)\big) \le \limsup_{h\to 0^+,(\tau,y)\to(t,x)} \frac{V(\tau+h, y+hf(t,x,0,d)) - V(\tau,y)}{h}$$

(2.130) follows readily.

2.6.2 *Control Systems Described by Retarded Functional Differential Equations, RFDEs*

We continue by presenting a general Lyapunov theorem for systems described by RFDEs of the form (1.10) under Hypotheses (S1–4). The reader may think that all arguments presented for the case of ODEs can be repeated in this case as well. This is not true. There are many technical complications arising for systems described by RFDEs. The following list presents some complications:

1. the mapping $t \to V(t, T_r(t)x)$ is not absolutely continuous, even if the functional $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re^+$ is completely locally Lipschitz (or even if it is Fréchet differentiable),
2. it may not be required that the Lyapunov functional bounds a certain function of the norm of the output.

We next show how all the above complications can be overcome. An important class of functionals is presented next.

Definition 2.4 We say that a continuous functional $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re^+$ is "almost Lipschitz on bounded sets" if there exist nondecreasing functions $M : \Re^+ \to \Re^+$, $P : \Re^+ \to \Re^+$, and $G : \Re^+ \to [1, +\infty)$ such that for all $R \geq 0$, the following properties hold:

(P1) For every $x, y \in \{x \in C^0([-r, 0]; \Re^n); \|x\|_r \leq R\}$, it holds that

$$\left|V(t, y) - V(t, x)\right| \leq M(R)\|y - x\|_r \quad \text{for all } t \in [0, R].$$

(P2) For every absolutely continuous function $x : [-r, 0] \to \Re^n$ with $\|x\|_r \leq R$ and essentially bounded derivative, it holds that

$$\left|V(t + h, x) - V(t, x)\right| \leq hP(R)\Big(1 + \sup_{-r \leq \tau \leq 0} \left|\dot{x}(\tau)\right|\Big)$$

$$\text{for all } t \in [0, R] \text{ and } 0 \leq h \leq \frac{1}{G(R + \sup_{-r \leq \tau \leq 0} |\dot{x}(\tau)|)}.$$

Let $x \in C^0([-r, 0]; \Re^n)$. By $E_h(x; v)$, where $0 \leq h < r$ and $v \in \Re^n$, we denote the following operator:

$$E_h(x; v) := \begin{cases} x(0) + (\theta + h)v & \text{for } -h < \theta \leq 0 \\ x(\theta + h) & \text{for } -r \leq \theta \leq -h \end{cases} \tag{2.136}$$

For mappings $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re$ that are almost Lipschitz on bounded sets, we will use the following Dini derivative defined for all $(t, x, v) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times \Re^n$:

$$V^0(t, x; v) := \limsup_{h \to 0^+} \frac{V(t + h, E_h(x; v)) - V(t, x)}{h}$$

The reason for introducing the above derivative becomes apparent by the following lemma.

Lemma 2.16 *Let $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re$ be an almost Lipschitz on bounded sets functional, and let $x \in C^0([t_0 - r, t_{\max}); \Re^n)$ be a solution of* (1.10) *under Hypotheses* (S1–4) *corresponding to certain $(d, u) \in M_D \times M_U$, where*

$t_{\max} \in (t_0, +\infty]$ *is the maximal existence time of the solution. Then it holds that*

$$\limsup_{h\to 0^+} h^{-1}\big(V\big(t+h, T_r(t+h)x\big) - V\big(t, T_r(t)x\big)\big) \le V^0\big(t, T_r(t)x; D^+x(t)\big) \tag{2.137}$$

for almost all $t \in [t_0, t_{\max})$, *where* $D^+x(t) = \lim_{h\to 0^+} h^{-1}(x(t+h) - x(t))$.

Proof It suffices to show that (2.137) holds for all $t \in [t_0, t_{\max})\backslash I$, where $I \subset [t_0, t_{\max})$ is the set of zero Lebesgue measure such that $D^+x(t) = \lim_{h\to 0^+} h^{-1}(x(t+h) - x(t))$ is not defined on I. Let $h > 0$ and $t \in [t_0, t_{\max})\backslash I$. We define

$$T_r(t+h)x - E_h\big(T_r(t)x; D^+x(t)\big) = hy_h \tag{2.138}$$

where

$$y_h = h^{-1}\begin{cases} x(t+h+\theta) - x(t) - (\theta+h)D^+x(t) & \text{for } -h < \theta \le 0 \\ 0 & \text{for } -r \le \theta \le -h \end{cases}$$

and notice that $y_h \in C^0([-r, 0]; \Re^n)$ (as a difference of continuous functions, see (2.138) above). Equivalently, y_h satisfies

$$y_h := \begin{cases} \frac{\theta+h}{h}\big(\frac{x(t+\theta+h)-x(t)}{\theta+h} - D^+x(t)\big) & \text{for } -h < \theta \le 0 \\ 0 & \text{for } -r \le \theta \le -h \end{cases}$$

with

$$\|y_h\|_r \le \sup\left\{\left|\frac{x(t+s) - x(t)}{s} - D^+x(t)\right|; 0 < s \le h\right\}.$$

Since $\lim_{h\to 0^+} \frac{x(t+h)-x(t)}{h} = D^+x(t)$, we obtain that $y_h \to 0$ as $h \to 0^+$. Since $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re$ is almost Lipschitz on bounded sets and since $y_h \to 0$ as $h \to 0^+$, it follows that

$$\begin{aligned} &\limsup_{h\to 0^+} h^{-1}\big(V\big(t+h, E_h\big(T_r(t)x; D^+x(t)\big) + hy_h\big) - V\big(t, T_r(t)x\big)\big) \\ &\quad = V^0\big(t, T_r(t)x; D^+x(t)\big) \end{aligned}$$

Moreover, definition (2.138) and the above equality imply

$$\begin{aligned} &\limsup_{h\to 0^+} h^{-1}\big(V\big(t+h, T_r(t+h)x\big) - V\big(t, T_r(t)x\big)\big) \\ &\quad = \limsup_{h\to 0^+} h^{-1}\big(V\big(t+h, E_h\big(T_r(t)x; D^+x(t)\big) + hy_h\big) - V\big(t, T_r(t)x\big)\big) \\ &\quad = V^0\big(t, T_r(t)x; D^+x(t)\big) \end{aligned}$$

The proof is complete. □

The reason for introducing the class of almost Lipschitz on bounded sets functionals is the following result, which shows that the mapping $t \to V(t, T_r(t)x)$ is absolutely continuous for a special class of initial conditions.

Lemma 2.17 *Let $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re$ be a functional which is almost Lipschitz on bounded sets, and let $x \in C^0([t_0 - r, t_{\max}); \Re^n)$ be a solution of* (1.10) *under Hypotheses* (S1–4) *corresponding to certain $(d, u) \in M_D \times M_U$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \Re^n)$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Then for every $T \in (t_0, t_{\max})$, the mapping $[t_0, T] \ni t \to V(t, T_r(t)x)$ is absolutely continuous.*

Proof It suffices to show that for all $T \in (t_0, t_{\max})$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{k=1}^{N} |V(b_k, T_r(b_k)x) - V(a_k, T_r(a_k)x)| < \varepsilon$ for every finite collection of pairwise disjoint intervals $[a_k, b_k] \subset [t_0, T]$ $(k = 1, \ldots, N)$ with $\sum_{k=1}^{N}(b_k - a_k) < \delta$. Let $T \in (t_0, t_{\max})$ and $\varepsilon > 0$ (arbitrary). Since the solution $x \in C^0([t_0 - r, T]; \Re^n)$ of (1.10) under Hypotheses (S1–4) corresponding to certain $(d, u) \in M_D \times M_U$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \Re^n)$ is bounded on $[t_0 - r, T]$, there exists $R_1 > 0$ such that $\sup_{t_0 \le \tau \le T} \|T_r(\tau)x\|_r \le R_1$. Moreover, by Hypothesis (S2) and since $T_r(t_0)x = x_0 \in C^1([-r, 0]; \Re^n)$, there exists $R_2 > 0$ such that $\sup_{t_0 - r \le \tau \le T} |\dot{x}(\tau)| \le R_2$. The previous observations, in conjunction with properties (P1)–(P2) of Definition 2.4 imply, for every interval $[a, b] \subset [t_0, T]$ with $b - a \le \frac{1}{G(R_1 + R_2)}$,

$$\begin{aligned}&\bigl|V\bigl(b, T_r(b)x\bigr) - V\bigl(a, T_r(a)x\bigr)\bigr|\\&\quad\le (b - a)P(R_1)(1 + R_2) + M(R_1)\bigl\|T_r(b)x - T_r(a)x\bigr\|_r\end{aligned}$$

In addition, the estimate $\sup_{t_0 - r \le \tau \le T} |\dot{x}(\tau)| \le R_2$ implies $\|T_r(b)x - T_r(a)x\|_r \le (b - a)R_2$ for every interval $[a, b] \subset [t_0, T]$. Consequently, we obtain, for every interval $[a, b] \subset [t_0, T]$ with $b - a \le \frac{1}{G(R_1 + R_2)}$,

$$\bigl|V\bigl(b, T_r(b)x\bigr) - V\bigl(a, T_r(a)x\bigr)\bigr| \le (b - a)\bigl[P(R_1)(1 + R_2) + M(R_1)R_2\bigr]$$

The previous inequality implies that for every finite collection of pairwise disjoint intervals $[a_k, b_k] \subset [t_0, T]$ $(k = 1, \ldots, N)$ with $\sum_{k=1}^{N}(b_k - a_k) < \delta$, where $\delta = \frac{1}{2}\min\{\frac{1}{G(R_1 + R_2)}; \frac{\varepsilon}{P(R_1)(1 + R_2) + M(R_1)R_2}\} > 0$, it holds that

$$\sum_{k=1}^{N}\bigl|V\bigl(b_k, T_r(b_k)x\bigr) - V\bigl(a_k, T_r(a_k)x\bigr)\bigr| < \varepsilon.$$

The proof is complete. □

The following technical lemma shows that, without loss of generality, we can restrict our attention to continuously differentiable initial conditions.

Lemma 2.18 *Consider system* (1.3) *under Hypotheses* (S1–4) *and suppose that there exist mappings* $\beta_1 : \Re^+ \times C^0([-r,0];\Re^n) \to \Re$ *and* $\beta_2 : \Re^+ \times \Re^+ \times C^0([-r,0];\Re^n) \times A \to \Re$, *where* $A \subseteq M_D \times M_U$, *with the following properties*:

(i) *for each* $(t, t_0, d, u) \in \Re^+ \times \Re^+ \times A$, *the mappings* $x \to \beta_1(t,x)$, $x \to \beta_2(t, t_0, x, d, u)$ *are continuous*,

(ii) *there exists a continuous function* $M : \Re^+ \times \Re^+ \to \Re^+$ *such that*

$$\sup\Big\{\beta_2(t_0+\xi, t_0, x_0, d, u); \sup_{t\geq 0}\big|u(\tau)\big| \leq s, \xi \in [0,T], x_0 \in C^0\big([-r,0];\Re^n\big),$$
$$\|x_0\|_r \leq s, t_0 \in [0,T], (d,u) \in A\Big\} \leq M(T,s),$$

(iii) *for every* $(t_0, x_0, d, u) \in \Re^+ \times C^1([-r,0];\Re^n) \times A$, *the solution* $x(t)$ *of* (1.10) *with initial condition* $T_r(t_0)x = x_0$ *corresponding to input* $(d,u) \in A$ *satisfies*

$$\beta_1\big(t, T_r(t)x\big) \leq \beta_2(t, t_0, x_0, d, u) \quad \text{for all } t \geq t_0. \tag{2.139}$$

Moreover, suppose that one of the following properties holds:

(iv)

$$c(T,s) := \sup\Big\{\big\|T_r(t_0+\xi)x\big\|_r; \sup_{t\geq 0}\big|u(\tau)\big| \leq s, \xi \in [0,T], x_0 \in C^0\big([-r,0];\Re^n\big),$$
$$\|x_0\|_r \leq s, t_0 \in [0,T], (d,u) \in A\Big\} < +\infty,$$

(v) *there exist functions* $a \in K_\infty$, $\mu \in K^+$ *and a constant* $R \geq 0$ *such that* $a(\mu(t)|x(0)|) \leq \beta_1(t,x) + R$ *for all* $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$.

Then for all $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r,0];\Re^n) \times A$, *the solution* $x(t)$ *of* (1.10) *with initial condition* $T_r(t_0)x = x_0$ *corresponding to input* $(d,u) \in A$ *exists for all* $t \geq t_0$ *and satisfies* (2.139).

Proof We distinguish the following cases:

(a) Property (iv) holds.

The proof will be made by contradiction. Suppose on the contrary that there exist $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r,0];\Re^n) \times A$ and $t_1 > t_0$ such that the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d,u) \in A$ satisfies

$$\beta_1\big(t_1, T_r(t_1)x\big) > \beta_2(t_1, t_0, x_0, d, u)$$

(1.9) and property (iv) implies that, for all $\tilde{x}_0 \in C^0([-r,0];\Re^n)$ with $\|x_0 - \tilde{x}_0\|_r \leq 1$,

$$\big\|T_r(t_1)x - T_r(t_1)\tilde{x}\big\|_r \leq \sqrt{\frac{K_2}{K_1}}\|x_0 - \tilde{x}_0\|_r \exp\big(K_1^{-1}L(t_1,\bar{c})(t_1-t_0)\big) \tag{2.140}$$

where $\tilde{x}(t)$ denotes the solution of (1.10) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $(d,u) \in A$, and $\bar{c} = 2c(t_1, \|x_0\|_r + 1 + \sup_{t_0 \le \tau \le t_1} |u(\tau)|) + \sup_{t_0 \le \tau \le t_1} |u(\tau)|$.

Let $\varepsilon := \beta_1(t_1, T_r(t_1)x) - \beta_2(t_1, t_0, x_0, u) > 0$. Using property (iv), (2.140), the fact that $C^1([-r,0];\Re^n)$ is dense in $C^0([-r,0];\Re^n)$, and the continuity of the mappings $x \to \beta_1(t_1, x)$ and $x \to \beta_2(t_1, t_0, x, d, u)$, we conclude that there exists $\tilde{x}_0 \in C^1([-r,0];\Re^n)$ such that

$$\|x_0 - \tilde{x}_0\|_r \le 1; \quad \left|\beta_2(t_1, t_0, x_0, d, u) - \beta_2(t_1, t_0, \tilde{x}_0, d, u)\right| \le \frac{\varepsilon}{2}$$

$$\left|\beta_1\big(t_1, T_r(t_1)x\big) - \beta_1\big(t_1, T_r(t_1)\tilde{x}\big)\right| \le \frac{\varepsilon}{2}$$

where $\tilde{x}(t)$ denotes the solution of (1.10) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $(d,u) \in A$. Combining property (iii) for $\tilde{x}(t)$ with the above inequalities and the definition of ε, we obtain $\beta_1(t_1, T_r(t_1)x) > \beta_1(t_1, T_r(t_1)x)$, a contradiction.

(b) Property (v) holds.

It suffices to show that property (iv) holds. Since there exist functions $a \in K_\infty$, $\mu \in K^+$, and a constant $R \ge 0$ such that $a(\mu(t)|x(0)|) \le \beta_1(t,x) + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, it follows that from property (iii) that for every $(t_0, x_0, d, u) \in \Re^+ \times C^1([-r,0];\Re^n) \times A$, the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d,u) \in A$ satisfies

$$a\big(\mu(t)\big|x(t)\big|\big) \le R + \beta_2(t, t_0, x_0, d, u) \quad \text{for all } t \ge t_0$$

Moreover, making use of property (ii) and the above inequality, we obtain that for every $(t_0, x_0, d, u) \in \Re^+ \times C^1([-r,0];\Re^n) \times A$, the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d,u) \in A$ satisfies, for all $t \ge t_0$,

$$\left\|T_r(t)x\right\|_r \le \|x_0\|_r + 1 + \frac{1}{\mu(t)} a^{-1}\Big(R + M\Big(t, \|x_0\|_r + \sup_{t_0 \le \tau \le t} \big|u(\tau)\big|\Big)\Big) \tag{2.141}$$

Notice that in order to obtain inequality (2.141), we have also used the causality argument that the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d,u) \in A$ depends only on the values of the input u on the interval $[t_0, t]$.

We claim that estimate (2.141) holds for all $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r,0];\Re^n) \times A$. Notice that this claim implies directly that property (iv) holds with

$$c(T,s) := s + 1 + \frac{1}{\min_{0 \le \tau \le 2T} \mu(\tau)} a^{-1}\Big(R + \max_{0 \le x \le 2s, 0 \le \tau \le 2T} M(\tau, x)\Big)$$

The proof of the claim will be made by contradiction. Suppose on the contrary that there exists $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r,0];\Re^n) \times A$ and $t_1 > t_0$ such that the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d,u) \in A$ satisfies, for all $t \geq t_0$,

$$\left\| T_r(t_1)x \right\|_r > \|x_0\|_r + 1 + \frac{1}{\mu(t_1)} a^{-1}\Big(R + M\Big(t_1, \|x_0\|_r + \sup_{t_0 \leq \tau \leq t_1} \left|u(\tau)\right|\Big)\Big) \quad (2.142)$$

Let $B := \sup_{t_0 \leq \tau \leq t_1} \|T_r(\tau)x\| < +\infty$. Using (1.9) and (2.141), it follows that (2.140) holds for all

$$\tilde{x}_0 \in C^1\big([-r,0]; \Re^n\big) \quad \text{with } \left\|x_0 - \tilde{x}_0\right\|_r \leq 1$$

and

$$\bar{c} = B + \|x_0\|_r + 2 + \max\left\{ \frac{a^{-1}(R + M(t,s))}{\mu(t)}; 0 \leq s \leq \|x_0\|_r + 1 \right.$$
$$\left. + \sup_{t_0 \leq \tau \leq t_1} \left|u(\tau)\right|, t_0 \leq t \leq t_1 \right\} + \sup_{t_0 \leq \tau \leq t_1} \left|u(\tau)\right|$$

where $\tilde{x}(t)$ denotes the solution of (1.10) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $(d,u) \in A$. Let

$$\varepsilon := \left\| T_r(t_1)x \right\|_r - \|x_0\|_r - 1 - \frac{1}{\mu(t_1)} a^{-1}\Big(R + M\Big(t_1, \|x_0\|_r + \sup_{t_0 \leq \tau \leq t_1} \left|u(\tau)\right|\Big)\Big) > 0$$

Using (2.140), (2.141), the denseness of $C^1([-r,0];\Re^n)$ in $C^0([-r,0];\Re^n)$, and the continuity of the mapping

$$x \to g(x) := \|x\|_r + 1 + \frac{1}{\mu(t_1)} a^{-1}\Big(R + M\Big(t_1, \|x\|_r + \sup_{t_0 \leq \tau \leq t_1} \left|u(\tau)\right|\Big)\Big)$$

we conclude that there exists $\tilde{x}_0 \in C^1([-r,0];\Re^n)$ such that

$$\left\|x_0 - \tilde{x}_0\right\|_r \leq 1 \qquad \left|g(x_0) - g\big(\tilde{x}_0\big)\right| \leq \frac{\varepsilon}{2} \qquad \left|\left\|T_r(t_1)x\right\|_r - \left\|T_r(t_1)\tilde{x}\right\|_r\right| \leq \frac{\varepsilon}{2}$$

where $\tilde{x}(t)$ denotes the solution of (1.10) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $(d,u) \in A$. Combining (2.141) for $\tilde{x}(t)$ with the above inequalities and the definition of ε, we obtain $\|T_r(t_1)x\|_r > \|T_r(t_1)x\|_r$, a contradiction. The proof is complete. □

Finally, we show how we can overcome the fact that it may not be required that the Lyapunov functional bounds a certain function of the norm of the output. The following definition introduces an important relation between output mappings. The equivalence relation defined next will be used extensively in the following sections of the present work.

Definition 2.5 Suppose that there exists a continuous mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$ with $h(t,0) = 0$ for all $t \geq -r$ and functions $a_1, a_2 \in K_\infty$ such that

$a_1(|h(t,x(0))|) \le \|H(t,x)\|_{\mathcal{Y}} \le a_2(\sup_{\theta\in[-r,0]}|h(t+\theta,x(\theta))|)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$. Then we say that $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional mapping h.

For example, the identity output mapping $H(t,x) = x \in C^0([-r,0];\Re^n)$ is equivalent to finite-dimensional mapping $h(t,x) = x \in \Re^n$.

We are now in a position to state the main Lyapunov sufficient condition for RGAOS.

Theorem 2.5 *Consider system* (1.10) *under Hypotheses* (S1–4) *and suppose that system* (1.10) *is RFC. Moreover, suppose that there exist functions* $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ *with* $\int_0^{+\infty} \gamma(t)\,dt = +\infty$, $\varphi \in \mathcal{E}$, *a positive definite locally Lipschitz function* $\rho : \Re^+ \to \Re^+$, *and a mapping* $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, *which is almost Lipschitz on bounded sets*, *such that*

$$a_1\big(\big\|H(t,x)\big\|_{\mathcal{Y}}\big) \le V(t,x) \le a_2\big(\beta(t)\|x\|_r\big)$$
$$\forall (t,x) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \tag{2.143}$$

$$V^0\big(t,x; f(t,x,0,d)\big) \le -\gamma(t)\rho\big(V(t,x)\big) + \gamma(t)\varphi\bigg(\int_0^t \gamma(s)\,ds\bigg)$$
$$\textit{for all } (t,x,d) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \times D \tag{2.144}$$

Then system (1.10) *is RGAOS. Furthermore,*

(i) *if* $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ *is equivalent to the finite-dimensional continuous mapping* $h : [-r,+\infty) \times \Re^n \to \Re^p$, *then inequality* (2.143) *can be replaced by the following inequality*:

$$a_1\big(\big|h\big(t,x(0)\big)\big|\big) \le V(t,x) \le a_2\big(\beta(t)\|x\|_r\big) \quad \forall (t,x) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \tag{2.145}$$

(ii) *if there exist functions* $a \in K_\infty$, $\mu \in K^+$, *and a constant* $R \ge 0$ *such that* $a(\mu(t)|x(0)|) \le V(t,x) + R$ *for all* $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, *then the requirement that* (1.10) *is RFC is not needed.*

Finally, if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, *and* $\beta(t) \equiv 1$, *then system* (1.10) *is URGAOS. In this case, if* $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ *is equivalent to the time-periodic finite-dimensional continuous mapping* $h : [-r,+\infty) \times \Re^n \to \Re^p$, *then inequality* (2.143) *can be replaced by inequality* (2.145).

Proof Case 1: (1.10) is RFC.

Consider a solution $x(t)$ of (1.10) under Hypotheses (S1–4) corresponding to arbitrary $d \in M_D$ and $u \equiv 0$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$. By Lemma 2.17, for every $T \in (t_0,+\infty)$, the mapping $[t_0,T] \ni t \to V(t,T_r(t)x)$ is absolutely continuous. It follows from (2.144) and Lemma 2.16 that $\frac{d}{dt}(V(t,T_r(t)x)) \le$

$-\gamma(t)\rho(V(t,T_r(t)x)) + \gamma(t)\varphi(\int_0^t \gamma(s)\,ds)$ a.e. on $[t_0,+\infty)$. The previous differential inequality, in conjunction with the comparison Lemma 2.15, shows that there exist $\sigma \in KL$ and a constant $M > 0$ such that

$$V\big(t, T_r(t)x\big) \le \sigma\left(V(t_0,x_0) + M, \int_{t_0}^{t} \gamma(s)\,ds\right) \quad \text{for all } t \ge t_0 \tag{2.146}$$

It follows from Lemma 2.18 that the solution $x(t)$ of (1.10) under Hypotheses (S1–4) corresponding to arbitrary $d \in M_D$ and $u \equiv 0$ with arbitrary initial condition $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$ satisfies (2.146) for all $t \ge t_0$. Next, we distinguish the following cases:

1. If (2.143) holds, then the Robust Output Attractivity Property (Property P3 of Definition 2.2) is a direct consequence of (2.143) and the fact that $\int_0^{+\infty} \gamma(t)\,dt = +\infty$. It follows from Lemma 2.1 that system (1.10) is RGAOS.
2. If (2.145) holds, then (2.146) implies the following estimate:

$$\big|h\big(t,x(t)\big)\big| \le a_1^{-1}\left(\sigma\left(a_2\big(\beta(t_0)\|x_0\|_r\big) + M, \int_{t_0}^{t} \gamma(s)\,ds\right)\right) \quad \text{for all } t \ge t_0.$$

Since $h : [-r,+\infty) \times \Re^n \to \Re^p$ is continuous with $h(t,0) = 0$ for all $t \ge -r$, it follows from Lemma 2.3 with $a(\tau,s) := \max\{|h(t-r,x)| : t \in [0,\tau], |x| \le s\}$ that there exist functions $\zeta \in K_\infty$ and $\delta \in K^+$ such that

$$\big|h(t-r,x)\big| \le \zeta\big(\delta(t)|x|\big) \quad \text{for all } (t,x) \in \Re^+ \times \Re^n$$

Combining the two previous inequalities yields

$$\begin{aligned}
&\sup_{\theta\in[-r,0]} \big|h\big(t+\theta, x(t+\theta)\big)\big| \\
&\quad \le \max\big\{a_1^{-1}\big(\sigma\big(a_2\big(\phi(t_0)\|x_0\|_r\big) + M, 0\big)\big), \zeta\big(\phi(t_0)\|x_0\|_r\big)\big\} \quad \text{for all } t \in [t_0, t_0+r] \\
&\sup_{\theta\in[-r,0]} \big|h\big(t+\theta, x(t+\theta)\big)\big| \\
&\quad \le a_1^{-1}\left(\sigma\left(a_2\big(\beta(t_0)\|x_0\|_r\big) + M, \int_{t_0}^{t-r} \gamma(s)\,ds\right)\right) \quad \text{for all } t \ge t_0 + r
\end{aligned}$$

where $\phi(t) := \beta(t) + \max_{0\le\tau\le t+r} \delta(\tau)$. The above estimates, in conjunction with the facts that $\int_0^{+\infty} \gamma(t)\,dt = +\infty$ and $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional mapping h (Definition 2.5), show that the Robust Output Attractivity Property (Property P3 of Definition 2.2) holds. Hence, system (1.10) is nonuniformly RGAOS.

Case 2: There exist functions $a \in K_\infty$, $\mu \in K^+$, and a constant $R \ge 0$ such that

$$a\big(\mu(t)\big|x(0)\big|\big) \le V(t,x) + R \quad \text{for all } (t,x) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \tag{2.147}$$

Consider a solution of (1.10) under Hypotheses (S1–4) corresponding to arbitrary $d \in M_D$ and $u \equiv 0$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$. By Lemma 2.17, for every $T \in (t_0, t_{\max})$, the mapping $[t_0, T] \ni t \to V(t, T_r(t)x)$ is absolutely continuous. It follows from (2.144) and Lemma 2.16 that for every $T \in (t_0, t_{\max})$, it holds that $\frac{d}{dt}(V(t, T_r(t)x)) \le -\gamma(t)\rho(V(t, T_r(t)x)) + \gamma(t)\varphi(\int_0^t \gamma(s)\,ds)$ a.e. on $[t_0, T]$. The previous differential inequality, in conjunction with the comparison Lemma 2.15, shows that there exist $\sigma \in KL$ and a constant $M > 0$ such that (2.146) holds for all $t \in [t_0, T]$.

Combining (2.143), (2.146), and (2.147), we obtain

$$\left|x(t)\right| \le \frac{1}{\mu(t)} a^{-1}\left(\sigma\left(a_2\left(\beta(t_0)\|x_0\|_r\right) + M, \int_{t_0}^t \gamma(s)\,ds\right) + R\right) \quad \text{for all } t \in [t_0, T] \tag{2.148}$$

Estimate (2.148) shows that $t_{\max} = +\infty$, and consequently estimates (2.146) and (2.148) hold for all $t \ge t_0$.

It follows from Lemma 2.18 that the solution $x(t)$ of (1.10) under Hypotheses (S1–4) corresponding to arbitrary $d \in M_D$ and $u \equiv 0$ with arbitrary initial condition $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$ satisfies (2.146) and (2.148) for all $t \ge t_0$. Therefore, system (1.10) is RFC, and the Robust Output Attractivity Property (Property P3 of Definition 2.2) is a direct consequence of (2.146) and (2.143) (or (2.145)), as in the previous case.

Finally, notice that if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, then Lemma 2.13 shows that there exists $\sigma \in KL$ such that

$$V\left(t, T_r(t)x\right) \le \sigma\left(V(t_0, x_0), t - t_0\right) \quad \text{for all } t \ge t_0 \tag{2.149}$$

instead of (2.146). Utilizing (2.143) with $\beta(t) \equiv 1$ and (2.149), we obtain (2.11). Therefore system (1.10) is URGAOS. If (2.145) holds, then (2.149) implies the following estimate:

$$\left|h\left(t, x(t)\right)\right| \le a_1^{-1}\left(\sigma\left(a_2\left(\beta(t_0)\|x_0\|_r\right), t - t_0\right)\right) \quad \text{for all } t \ge t_0$$

Since $h : [-r, +\infty) \times \Re^n \to \Re^p$ is continuous and time-periodic (say, with period $T > 0$) with $h(t, 0) = 0$ for all $t \ge -r$, it follows from Lemma 2.4 (applied to the continuous function $a(s) := \max\{|h(t - r, x)| : t \in [0, T], |x| \le s\}$) that there exists a function $\zeta \in K_\infty$ such that

$$\left|h(t - r, x)\right| \le \zeta\left(|x|\right) \quad \text{for all } (t, x) \in \Re^+ \times \Re^n$$

Combining the previous two inequalities, we obtain

$$\sup_{\theta \in [-r,0]} \left|h\left(t + \theta, x(t + \theta)\right)\right| \le \max\left\{\zeta\left(\|x_0\|_r\right), a_1^{-1}\left(\sigma\left(a_2\left(\|x_0\|_r\right), 0\right)\right)\right\}$$
$$\text{for all } t \in [t_0, t_0 + r]$$
$$\sup_{\theta \in [-r,0]} \left|h\left(t + \theta, x(t + \theta)\right)\right| \le a_1^{-1}\left(\sigma\left(a_2\left(\|x_0\|_r\right), t - t_0\right)\right)$$
$$\text{for all } t \ge t_0 + r$$

The above estimates, in conjunction with the fact $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional mapping h (Definition 2.5), show that Properties P1, P2, and P3 of Definition 2.3 hold. Hence, system (1.10) is URGAOS. The proof is complete. □

2.7 Examples of the Method of Lyapunov Functionals

In this section we present some examples of the use of Lyapunov functionals for the proof of RGAOS. In all cases we will show RGAOS without assuming any knowledge of the transition map; this is the advantage of the method of Lyapunov functionals.

Example 2.7.1 Consider the system

$$\begin{aligned}
\dot{x}_1(t) &= -2g_1(t)x_1(t) + d_1(t)g_2(t)x_1^2(t) - g_3(t)x_1^3(t) + d_2(t)b(t)\big|x_2\big(t-\tau(t)\big)\big|^p \\
\dot{x}_2(t) &= c(t)x_2(t) \\
Y(t) &= x_1(t) \in \Re \\
x(t) &= \big(x_1(t), x_2(t)\big)' \in \Re^2 \quad d = (d_1,\ d_2) \in D := [-1,1]\times[-1,1]
\end{aligned} \tag{2.150}$$

where $g_1 \in K^+$, $g_2 : \Re^+ \to \Re$, $g_3 : \Re^+ \to \Re^+$, $b : \Re^+ \to \Re$, and $c : \Re \to \Re$ are continuous functions, and $p > \frac{1}{2}$ is a constant. We consider system (2.150) under the following hypotheses:

(A1) There exist a continuous function $g \in K^+$ with $\int_0^{+\infty} g(s)\,ds = +\infty$ and a constant $M > 0$ such that $Mg(t) \le g_1(t)$ for all $t \ge 0$. Moreover, $g_2^2(t) \le 4g_1(t)g_3(t)$ for all $t \ge 0$.

(A2) The function $\tau : \Re^+ \to \Re$ is continuously differentiable, nonnegative, and bounded from above by a constant $r > 0$.

(A3) There exist constants $K, m > 0$ such that, for all $t \ge 0$,

$$\frac{|b(t)|^2}{g_1(t)} \exp\left(2p\int_0^{t-\tau(t)} c(s)\,ds\right) \le K\big(1-\dot{\tau}(t)\big)\exp\left(-mt - \int_0^t g(s)\,ds\right). \tag{2.151}$$

For example, Hypotheses (A1–3) are satisfied for $p = 2$, $\tau(t) := 2 + \frac{1}{2}\sin(t)$, $c(t) \equiv 1$, $b(t) \equiv 1$, $g_1(t) = \exp(6t)$, $g_2(t) = \exp(t)$, $g_3(t) \equiv 1$, with $g(t) \equiv 1$, $M := 1$, $K := 2$, $m := 1$. We next show that system (2.150) under Hypotheses (A1–3) is nonuniformly in time RGAOS. We consider the functional

$$\begin{aligned}
V(t,x) :=\ & \frac{1}{2}x_1^2(0) + \exp\left(-p^{-1}mt - \int_0^t \big(p^{-1}g(s) + 2c(s)\big)\,ds\right)x_2^2(0) \\
& + \frac{K}{2m}\exp\left(-mt - \int_0^t \big(g(s) + 2pc(s)\big)\,ds\right)\big|x_2(0)\big|^{2p}
\end{aligned}$$

$$+\frac{K}{2}\exp\left(-mt-\int_0^t g(s)\,ds\right)$$
$$\times\int_{-\tau(t)}^{0}\exp\left(-ms-2p\int_0^{t+s}c(\xi)\,d\xi\right)\left|x_2(s)\right|^{2p}ds \qquad (2.152)$$

Since $p>\frac{1}{2}$, it follows that the functional defined by (2.152) is almost Lipschitz on bounded sets. Moreover, inequalities (2.143) hold with $H(t,x)=x_1(0)\in\mathcal{Y}:=\Re$, $a_1(s):=2^{-1}s^2$, $a_2(s):=s^2+2^{-1}K(m^{-1}+r\exp(mr))s^{2p}$, and $\beta(t):=1+\max_{0\le\xi\le t}\exp(-\frac{m}{2p}\xi-\frac{1}{2p}\int_0^{\xi}g(s)\,ds)\max_{-\tau(\xi)\le s\le 0}\exp(-\int_0^{\xi+s}c(w)\,dw)$. Furthermore, we obtain, for all $(t,x,d)\in\Re^+\times C^0([-r,0];\Re^2)\times D$,

$$\begin{aligned}
&V^0\big(t,x;-2g_1(t)x_1(0)+d_1g_2(t)x_1^2(0)-g_3(t)x_1^3(0)+d_2b(t)\left|x_2\big(-\tau(t)\big)\right|^p,\\
&\quad c(t)x_2(0)\big)\\
&\le -2g_1(t)x_1^2(0)+\left|g_2(t)\right|\left|x_1(0)\right|^3-g_3(t)x_1^4(0)+\left|b(t)\right|\left|x_1(0)\right|\left|x_2\big(-\tau(t)\big)\right|^p\\
&\quad -p^{-1}\big(m+g(t)\big)\exp\left(-p^{-1}mt-\int_0^t\big(p^{-1}g(s)+2c(s)\big)\,ds\right)x_2^2(0)\\
&\quad -g(t)\frac{K}{2m}\exp\left(-mt-\int_0^t\big(g(s)+2pc(s)\big)\,ds\right)\left|x_2(0)\right|^{2p}\\
&\quad -g(t)\frac{K}{2}\exp\left(-mt-\int_0^t g(s)\,ds\right)\int_{-\tau(t)}^{0}\\
&\quad \times\exp\left(-ms-2p\int_0^{t+s}c(\xi)\,d\xi\right)\left|x_2(s)\right|^{2p}ds\\
&\quad -\frac{K}{2}\big(1-\dot\tau(t)\big)\exp\left(-m\big(t-\tau(t)\big)-\int_0^t g(s)\,ds-2p\int_0^{t-\tau(t)}c(s)\,ds\right)\\
&\quad \times\left|x_2\big(-\tau(t)\big)\right|^{2p}
\end{aligned}$$

Hypothesis (A1) implies that $-g_1(t)x_1^2+|g_2(t)||x_1|^3-g_3(t)x_1^4\le 0$ for all $(t,x_1)\in\Re^+\times\Re$. Using the inequality $|b(t)||x_1(0)||x_2(-\tau(t))|^p\le 2^{-1}g_1(t)x_1^2(0)+\frac{|b(t)|^2}{2g_1(t)}|x_2(-\tau(t))|^{2p}$ in conjunction with (2.151), we obtain:

$$\begin{aligned}
&V^0\big(t,x;-2g_1(t)x_1(0)+d_1g_2(t)x_1^2(0)-g_3(t)x_1^3(0)+d_2b(t)\left|x_2\big(-\tau(t)\big)\right|^p,\\
&\quad c(t)x_2(0)\big)\\
&\le -\frac{1}{2}g_1(t)x_1^2(0)-p^{-1}g(t)\exp\left(-p^{-1}mt-\int_0^t\big(p^{-1}g(s)+2c(s)\big)\,ds\right)x_2^2(0)\\
&\quad -g(t)\frac{K}{2m}\exp\left(-mt-\int_0^t\big(g(s)+2pc(s)\big)\,ds\right)\left|x_2(0)\right|^{2p}
\end{aligned}$$

$$-g(t)\frac{K}{2}\exp\left(-mt-\int_0^t g(s)\,ds\right)$$
$$\times\int_{-\tau(t)}^{0}\exp\left(-ms-2p\int_0^{t+s}c(\xi)\,d\xi\right)\left|x_2(s)\right|^{2p}ds$$

for all $(t,x,d)\in\Re^+\times C^0([-r,0];\Re^2)\times D$.

Since $Mg(t)\le g_1(t)$ for all $t\ge 0$, the above inequality, in conjunction with definition (2.152), gives, for all $(t,x,d)\in\Re^+\times C^0([-r,0];\Re^2)\times D$,

$$\begin{aligned}&V^0\big(t,x;-2g_1(t)x_1(0)+d_1g_2(t)x_1^2(0)-g_3(t)x_1^3(0)+d_2b(t)\left|x_2\big(-\tau(t)\big)\right|^p,\\&\quad c(t)x_2(0)\big)\\&\le-\min\{M,p^{-1},1\}g(t)V(t,x)\end{aligned}$$

Consequently, inequality (2.144) holds with

$$\rho(s):=s\quad\varphi(t)\equiv 0\quad\text{and}\quad\gamma(t):=\min\{M,p^{-1},1\}g(t)$$

If system (2.150) were RFC, we would have showed all hypotheses of Theorem 2.5. However, since the inequality $a(\mu(t)|x(0)|)\le V(t,x)+R$ holds for all $(t,x)\in\Re^+\times C^0([-r,0];\Re^2)$ with $R:=0$, $a(s):=s^2$ and

$$\mu(t):=\left(\min\left(2^{-1};\exp\left(-\frac{m}{p}t-\int_0^t\big(p^{-1}g(s)+2c(s)\big)\,ds\right)\right)\right)^{\frac{1}{2}}$$

the requirement that system (2.150) were RFC is not needed. Thus we can conclude that system (2.152) is RGAOS.

Moreover, if, in addition to Hypotheses (A1–3), the following hypothesis holds as well:

(A4) There exist constants Γ, $A>0$ such that $g(t)\ge\Gamma$ and

$$\frac{m}{2p}t+\frac{1}{2p}\int_0^t g(s)\,ds+\min_{-\tau(t)\le w\le 0}\int_0^{t+w}c(s)\,ds\ge -A\quad\text{for all } t\ge 0$$

then we can conclude that system (2.150) is URGAOS. Notice that if (A4) holds, inequalities (2.143) and (2.144) hold with $\rho(s):=\min\{M,p^{-1},1\}\Gamma s$, $H(t,x)=x_1(0)\in\mathcal{Y}:=\Re$, $a_1(s):=2^{-1}s^2$, $a_2(s):=\exp(2A)s^2+2^{-1}K(m^{-1}+r\exp(mr))\exp(2pA)s^{2p}$, $\varphi(t)\equiv 0$, $\gamma(t)\equiv 1$, and $\beta(t)\equiv 1$. Consequently, all hypotheses of Theorem 2.5 hold (without the requirement that system (2.150) is RFC).

Theorems 2.4 and 2.5 can be used for the proof of RFC (simply consider the system with zero output map $H(t,x)\equiv 0$). The following example illustrates this point.

Example 2.7.2 Consider the following system described by ODEs:

$$\begin{aligned} \dot{x} &= c_1(t,d)x + c_2(t,d)x^2 + c_3(t,d)x^3 \\ & x \in \Re, t \geq 0, d \in D \end{aligned} \tag{2.153}$$

where $D \subset \Re^m$ is a compact set, $c_i(\cdot) \in C^0(\Re^+ \times D)$, $i = 1, 2, 3$, are continuous mappings for which there exists a function $\phi \in K^+$ such that

$$2\phi(t)c_3(t,d) \leq -\left|c_2(t,d)\right| \quad \text{for all } (t,d) \in \Re^+ \times D \tag{2.154}$$

We next prove that this system is RFC. Consider the Lyapunov function candidate

$$W(t,x) := \exp\left(-\int_0^t \left(1 + 2\max_{d\in D}\left|c_1(\tau,d)\right| + \phi(\tau)\max_{d\in D}\left|c_2(\tau,d)\right|\right) d\tau\right)|x|^2 \tag{2.155}$$

Clearly, we have, for all $(t,x,d) \in \Re^+ \times \Re \times D$,

$$\begin{aligned} &\frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x)\left(c_1(t,d)x + c_2(t,d)x^2 + c_3(t,d)x^3\right) \\ &\quad = -\left(1 + 2\max_{d\in D}\left|c_1(t,d)\right| + \phi(t)\max_{d\in D}\left|c_2(t,d)\right|\right)W(t,x) \\ &\qquad + \left(2c_1(t,d)x^2 + 2c_2(t,d)x^3 + 2c_3(t,d)x^4\right) \\ &\qquad \times \exp\left(-\int_0^t \left(1 + 2\max_{d\in D}\left|c_1(\tau,d)\right| + \phi(\tau)\max_{d\in D}\left|c_2(\tau,d)\right|\right) d\tau\right) \end{aligned} \tag{2.156}$$

It follows from inequality (2.154), in conjunction with the Young inequality $2c_2(t,d)x^3 \leq \phi(t)|c_2(t,d)|x^2 + \frac{|c_2(t,d)|}{\phi(t)}x^4$, that the following inequality holds:

$$\begin{aligned} &\frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x)\left(c_1(t,d)x + c_2(t,d)x^2 + c_3(t,d)x^3\right) \leq -W(t,x) \\ &\quad \text{for all } (t,x,d) \in \Re^+ \times \Re \times D \end{aligned}$$

The reader should notice that all hypotheses of Theorem 2.4 hold with

$$\begin{aligned} &H(t,x) \equiv 0 \quad a(s) = a_2(s) := s^2 \\ &\mu(t) := \exp\left(-\frac{1}{2}\int_0^t \left(1 + 2\max_{d\in D}\left|c_1(\tau,d)\right| + \phi(\tau)\max_{d\in D}\left|c_2(\tau,d)\right|\right)d\tau\right) \\ &\beta(t) = \gamma(t) \equiv 1 \quad \rho(s) := s \quad \varphi(t) \equiv 0 \quad \text{and} \quad R := 0 \end{aligned}$$

Thus system (2.153) with zero output map is URGAOS. Particularly, this implies that system (2.153) is RFC.

Example 2.7.3 Consider system (1.7) which arises in the study of the chemostat model (1.4) of Example 1.2.1 in Chap. 1. Here we assume that $s_{\text{in}}(t) \equiv s_{\text{in}}^*$ (i.e.,

$u_2(t) \equiv 0$ from (1.6)), $D(t) \equiv D^*$ (i.e., $u_1(t) \equiv 0$ from (1.6)), $b = 0$ (i.e., $R := 1$), and $K(s) \equiv K$ (i.e., $p(x_2) \equiv 1$). Therefore the transformed chemostat model (1.7) takes the simple form

$$\begin{aligned} &\dot{x}_1 = D^* g(x_2) \\ &\dot{x}_2 = D^* \exp(-x_2)\big[M\big(1 - \exp(x_1)\big) - Mg(x_2)\exp(x_1) + 1 - \exp(x_2)\big] \\ &\quad x = (x_1, x_2) \in \Re^2 \end{aligned} \tag{2.157}$$

Moreover, we will assume that the specific growth rate $\mu(s)$ of the microbial species is a nondecreasing, locally Lipschitz, bounded function $\mu : [0, +\infty) \to [0, +\infty)$ with $\mu(0) = 0$, $\mu(s) \neq D^*$ for all $s \neq s^*$, and $\mu(s) > 0$ for all $s > 0$. Notice that the previous requirements are automatically satisfied if $\mu : [0, +\infty) \to [0, +\infty)$ is strictly increasing, but here we will avoid to make such an assumption. Consequently, the function $g : \Re \to (-1, +\infty)$ is a nondecreasing, locally Lipschitz, bounded function with $g(0) = 0$, $\lim_{x_2 \to -\infty} g(x_2) = -1$, and $g(x_2) \neq 0$ for all $x_2 \neq 0$.

We will show that system (2.157) is URGAS (although in this case the adjective "robust" does not apply since no disturbances appear in model (2.157)). To this end, we will use the Lyapunov function

$$V(x) = \exp(x_1) - x_1 - 1 + B(x_2) + Q\big(1 - \exp(x_2) + M - M\exp(x_1)\big)^2 \tag{2.158}$$

where $B(x) := M^{-1} \int_0^x \frac{g(w)}{1+g(w)} \exp(w)\, dw = \frac{1}{Ms^*} \int_{s^*}^{s^* \exp(x)} \frac{\mu(s)-D^*}{\mu(s)}\, ds$, and $Q > 0$ is a constant. We first notice that the function $V : \Re^2 \to [0, +\infty)$ is a continuously differentiable function with $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. For every $a \geq 0$, the set $\{x \in \Re^2 : V(x) \leq a\}$ is compact. Indeed, this follows from the equalities $\lim_{x_1 \to \pm\infty}(\exp(x_1) - x_1 - 1) = +\infty$ and $\lim_{x_2 \to \pm\infty} B(x_2) = +\infty$. Notice that for all $x_2 \geq 0$, it holds that $B(x_2) \geq \frac{1}{Ms^*} \int_{s^*}^{s^* \exp(x_2)} \frac{\mu(s)-D^*}{\mu_{\max}}\, ds$, where $\mu_{\max} = \sup_{s>0} \mu(s)$, which shows that $\lim_{x_2 \to +\infty} B(x_2) = +\infty$. The equality $\lim_{x_2 \to -\infty} B(x_2) = +\infty$ follows from the fact that the function $\mu : [0, +\infty) \to [0, +\infty)$ is locally Lipschitz and thus there exist $p, L > 0$ with $\mu(s) \leq Ls$ for all $0 \leq s \leq p$. Consequently, for all $x_2 \leq \ln(\frac{p}{s^*})$, we obtain the following inequalities which show that $\lim_{x_2 \to -\infty} B(x_2) = +\infty$:

$$\begin{aligned} B(x_2) &= \frac{1}{Ms^*} \int_{s^* \exp(x_2)}^{s^*} \frac{D^* - \mu(s)}{\mu(s)}\, ds \geq \frac{1}{Ms^*} \int_{s^* \exp(x_2)}^{p} \frac{D^* - \mu(s)}{\mu(s)}\, ds \\ &\geq \frac{D^*}{Ms^*} \int_{s^* \exp(x_2)}^{p} \frac{ds}{\mu(s)} - \frac{p}{Ms^*} \geq \frac{D^*}{Ms^* L} \int_{s^* \exp(x_2)}^{p} \frac{ds}{s} - \frac{p}{Ms^*} \\ &= \frac{D^*(\ln(p) - \ln(s^*) - x_2)}{Ms^* L} - \frac{p}{Ms^*} \end{aligned}$$

The following proposition helps us at this point.

Proposition 2.2 *Let $V:\Re^n \to [0,+\infty)$ be a continuous function with $V(0)=0$ and $V(x)>0$ for all $x \neq 0$. Moreover, assume that for every $a \geq 0$, the set $\{x \in \Re^n : V(x) \leq a\}$ is compact. Then there exist functions $a_1, a_2 \in K_\infty$ such that $a_1(|x|) \leq V(x) \leq a_2(|x|)$ for all $x \in \Re^n$. Furthermore, if $P:\Re^n \to [0,+\infty)$ is a continuous function with $P(0)=0$ and $P(x)>0$ for all $x \neq 0$, then there exists a locally Lipschitz positive definite function $\rho:\Re^+ \to \Re^+$ such that $\rho(V(x)) \leq P(x)$ for all $x \in \Re^n$.*

Proof of Proposition 2.2 The function $a(s) := \max\{V(x) : x \in \Re^n, |x| \leq s\}$ is a nondecreasing (and thus locally bounded) function with $\lim_{s\to 0^+} a(s) = a(0) = 0$. Therefore, Lemma 2.4 implies the existence of $a_2 \in K_\infty$ such that $a(s) \leq a_2(s)$ for all $s \geq 0$ and consequently, $V(x) \leq a_2(|x|)$ for all $x \in \Re^2$. On the other hand, since for every $a \geq 0$, the set $\{x \in \Re^2 : V(x) \leq a\}$ is compact, it follows that the function $\beta(s) := \min\{V(x) : x \in \Re^n, |x| \geq s\}$ is a well-defined, nondecreasing function with $\lim_{s\to+\infty} \beta(s) = +\infty$, $\beta(0)=0$ and $\beta(s)>0$ for all $s>0$. The reader should verify all previous statements (which require knowledge of real analysis!). The function $a_1(s) := \frac{1}{s+1}\int_0^s \beta(w)\,dw = \frac{s}{s+1}\int_0^1 \beta(\lambda s)\,d\lambda$ defined for $s \geq 0$ is a K_∞ function which satisfies all requirements.

Finally, consider $\tilde{\rho}(s) := \min\{P(x) : x \in \Re^n, 1 \leq V(x) \leq s\}$ for all $s \geq 1$ and $\tilde{\rho}(s) := \min\{P(x) : x \in \Re^n, s \leq V(x) \leq 1\}$ for all $0 \leq s \leq 1$, which is positive definite, nondecreasing on $[0,1]$ and nonincreasing on $[1,+\infty)$. Define the constant

$$a := \min\left\{\frac{1}{2}\int_0^1 \tilde{\rho}(w)\,dw, \int_1^2 \tilde{\rho}(w)\,dw\right\}$$

and the function

$$\bar{\rho}(s) := \frac{2a}{s+1}\frac{\int_0^s \tilde{\rho}(w)\,dw}{\int_0^1 \tilde{\rho}(w)\,dw} \quad \text{for } 0 \leq s \leq 1$$

and

$$\bar{\rho}(s) := a\frac{\int_s^{s+1} \tilde{\rho}(w)\,dw}{\int_1^2 \tilde{\rho}(w)\,dw} \quad \text{for } s > 1$$

The function $\bar{\rho}:\Re^+ \to \Re^+$ is continuous, positive definite, and satisfies $\bar{\rho}(s) \leq \tilde{\rho}(s)$ for all $s \geq 0$. By virtue of Remark 2.4, the function $\rho(s) := \inf\{\bar{\rho}(y) + |y-s|;\, y \geq 0\}$ satisfies all requirements. □

Therefore, by Proposition 2.2 it follows that there exist $a_1, a_2 \in K_\infty$ such that (2.129) holds with $\beta(t) \equiv 1$ and $H(t,x,0) := x$.

It is a matter of easy manipulation to show that

$$V^0\big(x; f(x)\big) = -P(x)$$

where

$$P(x) := M^{-1}\frac{g(x_2)}{1+g(x_2)}D^*\big[Mg(x_2)+\exp(x_2)-1\big]$$
$$+2D^*Q\big(1-\exp(x_2)+M-M\exp(x_1)\big)^2,$$
$$f(x) := D^*\big[g(x_2), \exp(-x_2)\big(M\big(1-\exp(x_1)\big)-Mg(x_2)\exp(x_1)+1-\exp(x_2)\big)\big]'$$

Notice that $P:\Re^2 \to [0,+\infty)$ is a continuous function with $P(0)=0$ and $P(x) > 0$ for all $x \neq 0$. It follows from Proposition 2.2 that (2.130) holds with $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, and appropriate locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$. The proof that system (2.157) is URGAS is finished with the help of Theorem 2.4.

2.8 RGAOS for Discrete-Time Systems

In this section, the method of Lyapunov functionals is developed for discrete-time systems of the form (1.110) under Hypotheses (L1–3). Particularly, we have the following technical result.

Proposition 2.3 *Consider system* (1.110) *under Hypotheses* (L1–3). *Suppose that there exist functions* $V:\pi\times\mathcal{X}\to\Re^+$, $a_1,a_2\in K_\infty$, $a_3\in C^0(\Re^+;\Re^+)$ *being positive definite with* $a_3(s)\le s$ *for all* $s\ge 0$, $\beta\in K^+$, *and* $q:\pi\to\Re^+$ *with* $\lim_{i\to+\infty} q(\tau_i)=0$ *such that, for all* $(i,x,d)\in Z^+\times\mathcal{X}\times D$,

$$a_1\big(\|H(\tau_i,x,0)\|_{\mathcal{Y}}\big)\le V(\tau_i,x)\le a_2\big(\beta(\tau_i)\|x\|_{\mathcal{X}}\big) \tag{2.159}$$

$$V\big(\tau_{i+1}, f(\tau_i,d,x,0)\big)\le V(\tau_i,x)-a_3\big(V(\tau_i,x)\big)+q(\tau_i) \tag{2.160}$$

Then system (1.110) *is RGAOS if one of the following holds*:

(i) $a_3\in K$, *or*
(ii) $\sum_{i=0}^{\infty} q(\tau_i) < +\infty$.

Moreover, if, in addition, $q\equiv 0$ *and* $\beta(t)\equiv 1$, *then system* (1.110) *is URGAOS.*

For the proof of Proposition 2.3, we need the following technical result.

Lemma 2.19 *Let* $a\in K$ *with* $a(s)\le s$ *for all* $s\ge 0$, $M\in(0,\frac{1}{2}\lim_{s\to+\infty}a(s))$, *and consider a sequence* $\{V_i\in\Re^+\}_0^\infty$ *which satisfies the following inequality*:

$$V_{i+1}\le V_i-a(V_i)+M \quad \text{for all } i\ge k_0\in Z^+ \tag{2.161}$$

Then the following inequalities hold:

$$V_i\le V_{k_0}+a^{-1}(M)+M \quad \text{for all } i\ge k_0\in Z^+ \tag{2.162}$$

$$V_i< a^{-1}(2M)+M \quad \text{for all } i\ge k_0+\frac{V_{k_0}}{M} \tag{2.163}$$

Proof of Lemma 2.19 We first prove (2.162) by induction. Notice that (2.162) holds for $i = k_0$. Suppose that (2.162) holds for some $i \in Z^+$ with $i \geq k_0$. Consider the cases:

- if $a(V_i) \geq M$, then (2.161) implies $V_{i+1} \leq V_i$, and consequently (2.162) holds for $i + 1$,
- if $a(V_i) < M$ or equivalently if $V_i < a^{-1}(M)$, then (2.161) implies $V_{i+1} \leq V_i + M < a^{-1}(M) + M$, and consequently (2.162) holds for $i + 1$.

Next, we prove the following claim: if (2.163) holds for some $i = k \in Z^+$ with $k \geq k_0$, then (2.163) holds for all $i \geq k$. Consider the cases:

- if $a(V_i) \geq M$, then (2.161) implies $V_{i+1} \leq V_i$, and consequently (2.163) holds for $i + 1$,
- if $a(V_i) < M$ or equivalently if $V_i < a^{-1}(M)$, then (2.161) implies $V_{i+1} \leq V_i + M < a^{-1}(M) + M \leq a^{-1}(2M) + M$, and consequently (2.163) holds for $i + 1$.

The proof of inequality (2.163) is made by contradiction. Suppose that there exists $k \in Z^+$ with $k \geq k_0 + \frac{V_{k_0}}{M}$ such that $V_k \geq a^{-1}(2M) + M$. By virtue of the previous claim, this implies that $V_i \geq a^{-1}(2M) + M$ for all $i = k_0, \ldots, k$. Consequently, we have $-a(V_i) + M \leq -M$ for all $i = k_0, \ldots, k$. Thus, we obtain from (2.161):

$$V_{i+1} \leq V_i - M \quad \text{for all } i = k_0, \ldots, k \tag{2.164}$$

Clearly, (2.164) implies that $V_k \leq V_{k_0} - M(k - k_0)$, and this estimate along with $k \geq k_0 + \frac{V_{k_0}}{M}$ gives $V_k \leq 0$. Clearly, this implication is in contradiction with the assumption $V_k \geq a^{-1}(2M) + M > 0$. The proof is complete. □

We are now in a position to provide the proof of Proposition 2.3.

Proof of Proposition 2.3 Notice that Lemma 1.3 implies that system (1.110) is RFC. Thus, by virtue of Lemma 2.1, it suffices to show the Robust Output Attractivity Property (property P3 of Definition 2.2). Let $\varepsilon > 0$, $R, T \geq 0$ arbitrary and consider the solution $x(t)$ of (1.110) with initial condition $x(t_0) = x_0$ with $|x_0| \leq R$, $t_0 \in [0, T]$ corresponding to $d \in M_D$ and $u \equiv 0 \in M_U$. Let $J(T) \in Z^+$ with $q_\pi(T) := \tau_{J(T)}$.

We distinguish the following cases:

Case of (i): Let $M = M(\varepsilon) \in (0, \frac{1}{2} \lim_{s \to +\infty} a_3(s))$ with $a_3^{-1}(2M) + M = a_1^{-1}(\varepsilon)$. There exists $N(\varepsilon) \in Z^+$ such that $q(\tau_i) \leq M(\varepsilon)$ for all $i \geq N(\varepsilon)$. Notice that (2.160) implies $V(\tau_{i+1}, x(\tau_{i+1})) \leq V(\tau_i, x(\tau_i)) - a_3(V(\tau_i, x(\tau_i))) + M$ for all $i \geq N(\varepsilon)$ with $\tau_i \geq \tau_s = p_\pi(t_0)$. Lemma 2.19 and the fact $a_3^{-1}(2M) + M = a_1^{-1}(\varepsilon)$ imply that $V(\tau_i, x(\tau_i)) \leq a_1^{-1}(\varepsilon)$ for all $i \geq j + \frac{V(\tau_j, x(\tau_j))}{M(\varepsilon)}$ with $j = j(\varepsilon, T) := \max\{N(\varepsilon), J(T)\}$. Moreover, (2.160 implies that $V(\tau_j, x(\tau_j)) \leq V(\tau_s, x_0) + \sum_{k=0}^{j(\varepsilon,T)} q(\tau_k)$. Using (2.159), in conjunction with previous inequalities, implies that

$$V\big(\tau_i, x(\tau_i)\big) \le a_1^{-1}(\varepsilon)$$

$$\text{for all } i \ge \tilde{N}(\varepsilon, T, R) := j(\varepsilon, T) + \frac{a_2(R \max_{k=0,\dots,J(T)} \beta(\tau_k)) + \sum_{k=0}^{j(\varepsilon,T)} q(\tau_k)}{M(\varepsilon)}$$

with $j = j(\varepsilon, T) := \max\{N(\varepsilon), J(T)\}$. Therefore, inequality (2.159) gives $\|H(t, x(t))\|_{\mathcal{Y}} \le \varepsilon$ for all $t \ge t_0 + \tau(\varepsilon, T, R)$ with $\tau(\varepsilon, T, R) := \tau_{\tilde{N}(\varepsilon,T,R)}$.

Case of (ii): Define $B := \sum_{i=0}^{\infty} q(\tau_i) < +\infty$. Notice that (2.160) implies

$$V\big(\tau_{i+1}, x(\tau_{i+1})\big) \le V\big(\tau_i, x(\tau_i)\big) + q(\tau_i) \quad \text{for all } i \in Z^+ \text{ with } \tau_i \ge p_\pi(t_0)$$

Consequently, we obtain (inductively) $V(\tau_i, x(\tau_i)) \le V(\tau_s, x_0) + B$ for all $i \in Z^+$ with $\tau_i \ge \tau_s = p_\pi(t_0)$. This implies (using (2.159)) $V(\tau_i, x(\tau_i)) \le G = G(T, R) := a_2(R \max_{k=0,\dots,J(T)} \beta(\tau_k)) + B + 1$ for all $i \in Z^+$ with $\tau_i \ge \tau_s = p_\pi(t_0)$. Also, define

$$\rho(v) := \exp(v - G) \min_{v \le y \le G} a_3(y) \tag{2.165}$$

which obviously is a strictly increasing, continuous, and positive definite function on $[0, G]$. Clearly, there exists $\tilde{\rho} \in K$ with $\tilde{\rho}(v) = \rho(v)$ for all $v \in [0, G]$ (e.g., $\tilde{\rho}(v) := \rho(G) + v - G$ for $v > G$). Notice that (2.160) and definition (2.165) implies $V(\tau_{i+1}, x(\tau_{i+1})) \le V(\tau_i, x(\tau_i)) - \tilde{\rho}(V(\tau_i, x(\tau_i))) + q(\tau_i)$ for all $i \in Z^+$ with $\tau_i \ge p_\pi(t_0)$. From this point the proof continues with the same procedure as in the previous case.

Case of $q \equiv 0$ *and* $\beta(t) \equiv 1$: Notice that (2.160) implies $V(\tau_{i+1}, x(\tau_{i+1})) \le V(\tau_i, x(\tau_i))$ for all $i \in Z^+$ with $\tau_i \ge p_\pi(t_0)$. Consequently, we obtain (inductively) $V(\tau_i, x(\tau_i)) \le V(\tau_s, x_0)$ for all $i \in Z^+$ with $\tau_i \ge \tau_s = p_\pi(t_0)$. The previous inequality, in conjunction with (2.159) with $\beta(t) \equiv 1$, implies Uniform Robust Lagrange and Lyapunov Output Stability. Thus we are left with the proof of the Uniform Robust Output Attractivity Property.

Notice that if $V(\tau_k, x(\tau_k)) \le a_1^{-1}(\varepsilon)$ for some $k \in Z^+$ with $\tau_k \ge \tau_s = p_\pi(t_0)$, then we get $V(\tau_i, x(\tau_i)) \le a_1^{-1}(\varepsilon)$ for all $i \ge k$.

Let $S(\varepsilon, R) := \{a_3(y) : y \in [a_1^{-1}(\varepsilon), a_1^{-1}(\varepsilon) + a_2(R)]\} > 0$. We claim that

$$V\big(\tau_i, x(\tau_i)\big) \le a_1^{-1}(\varepsilon) \quad \text{for all } i \ge s + \frac{a_2(R)}{S(\varepsilon, R)}$$

The proof is made by contradiction. Suppose that there exists $k \in Z^+$ with $k \ge s + \frac{a_2(R)}{S(\varepsilon,R)}$ such that $V_k > a_1^{-1}(\varepsilon)$. By virtue of the previous observation, this implies that $V_i > a_1^{-1}(\varepsilon)$ for all $i = s, \dots, k$. Consequently, we have $V_{i+1} \le V_i - S(\varepsilon, R)$ for all $i = s, \dots, k$. Thus we obtain $V_k \le V_s - S(\varepsilon, R)(k - s)$, and this estimate, in conjunction with our assumption $k \ge s + \frac{a_2(R)}{S(\varepsilon,R)} \ge s + \frac{V_s}{S(\varepsilon,R)}$, gives $V_k \le 0$. Clearly, this implication is in contradiction with the assumption $V_k > a_1^{-1}(\varepsilon)$. The reader should verify that the inequality $t \geq t_0 + r([\frac{a_2(R)}{S(\varepsilon,R)}] + 2)$ implies that $t \geq \tau_i$ with $i \geq s + \frac{a_2(R)}{S(\varepsilon,R)}$. The proof is complete. □

Example 2.8.1 Consider the nonlinear finite-dimensional discrete-time time-varying system

$$\begin{aligned}
&x_1(t+1) = d(t)x_1(t)\\
&x_2(t+1) = 2^{-t}d(t)\big|x_1(t)\big|^{\frac{1}{2}}\\
&Y(t) = H\big(t, x(t)\big) := x_2(t)\\
&\quad x(t) := \big(x_1(t), x_2(t)\big) \in \Re^2,\, t \in Z^+,\, d(t) \in [-2, 2]
\end{aligned} \tag{2.166}$$

with $\pi = Z^+$. Consider the continuous function $V(t,x) := \exp(-t)|x_1| + |x_2|$, which clearly satisfies the following inequality:

$$|Y| = |x_2| \leq V(t,x) \leq 2|x| \quad \text{for all } (t,x) \in Z^+ \times \Re^2 \tag{2.167}$$

Moreover, notice that for all $(t,x,d) \in Z^+ \times \Re^2 \times [-2,2]$, we obtain

$$\begin{aligned}
V\big(t+1, dx_1, 2^{-t}d|x_1|^{\frac{1}{2}}\big) &= \exp(-t-1)|d||x_1| + 2^{-t}|d||x_1|^{\frac{1}{2}}\\
&\leq 2e^{-1}\exp(-t)|x_1| + 2^{-t+1}|x_1|^{\frac{1}{2}}\\
&\leq \frac{2+e}{2e}\exp(-t)|x_1| + \frac{2e}{e-2}\left(\frac{e}{4}\right)^t\\
&\leq \lambda V(t,x) + q(t)
\end{aligned} \tag{2.168}$$

where $\lambda := \frac{2+e}{2e} \in (0,1)$ and $q(t) := \frac{2e}{e-2}(\frac{e}{4})^t$ with $\lim_{t\to+\infty} q(t) = 0$. By virtue of (2.167) and (2.168), it follows that statement (i) of Proposition 2.3 is satisfied with $\beta(t) \equiv 1$, $a_1(s) := s$, $a_2(s) := 2s$, and $a_3(s) := (1-\lambda)s$. We conclude that system (2.166) is RGAOS.

Example 2.8.2 Consider again the cobweb model (1.123) with buffer stocks. For the case with no government intervention, the corresponding dynamical system (1.123) with $u(t) \equiv 0$ is actually one-dimensional (the state variable $x_2(t)$ is constant and irrelevant) and is given by the difference equation

$$x_1(t+1) = h\big(x_1(t) + x_{\mathrm{eq}}\big) - x_{\mathrm{eq}} \tag{2.169}$$

where

$$h(y) = \begin{cases} 1 & \text{if } y \leq c_1\\ 1 + c_1 r - ry & \text{if } c_1 < y < c_1 + c_2\\ 1 - c_2 r & \text{if } y \geq c_1 + c_2 \end{cases} \tag{2.170}$$

We consider the following cases:

Case 1: If $c_1 + c_2 > 1 - c_2 r$ and $r < 1$, then we can show that system (1.123) with $u(t) \equiv 0$ is URGAOS.

Indeed, we can prove this fact by using Proposition 2.3 and by considering the Lyapunov function $V(x_1) := |x_1|$. It is a matter of simple but tedious calculations to show that the following inequality holds:

$$V\big(h(x_1 + x_{\rm eq}) - x_{\rm eq}\big) < V(x_1) \quad \text{for all } x \in \Re, x_1 \neq 0 \tag{2.171}$$

Define $P(x_1) := V(x_1) - V(h(x_1 + x_{\rm eq}) - x_{\rm eq})$. Clearly, inequality (2.171) implies that $P : \Re \to [0, +\infty)$ is a continuous function with $P(0) = 0$ and $P(x) > 0$ for all $x \neq 0$. By Proposition 2.2 there exists a locally Lipschitz positive definite function $a_3 : \Re^+ \to \Re^+$ such that $a_3(V(x)) \leq P(x)$ for all $x \in \Re$. Consequently, inequalities (2.159) and (2.160) hold with $a_1(s) = a_2(s) := s$, $q \equiv 0$, and $\beta(t) \equiv 1$.

Case 2: If $c_1 + c_2 \leq 1 - c_2 r$ (or $S_{\max} \leq \frac{ad-bc}{b+d}$), then we notice that $h(y) \geq c_1 + c_2$ for all $y \in \Re$. Thus, for every initial condition $x_1(0) \in \Re$, we obtain

$$x_1(t) = 0 \quad \big(\text{or } P(t) = b^{-1}(a - S_{\max})\big) \quad \text{for all } t \geq 2. \tag{2.172}$$

In this case, we can show that system (1.123) with $u(t) \equiv 0$ is URGAOS, by combining Lemmas 1.3, 2.1, and 2.2 (notice that system (1.123) is autonomous). In fact, in this case the phenomenon of finite-time stability appears as (2.172) shows (i.e., the equilibrium point is approached in finite time).

Thus, in any of the above two cases, system (1.123) with $u(t) \equiv 0$ is URGAOS. The characterization of the set of the parameters for which URGAOS occurs is sharp. Indeed, if $c_1 + c_2 > 1 - c_2 r$ and $r \geq 1$ (or $S_{\max} > \frac{ad-bc}{b+d}$ and $d \geq b$), the dynamics of system (1.123) present the well-known phenomenon of the "hog-cycles" or nontrivial period-2 solutions, as described in the economic literature. For the mildly nonlinear case (1.123), the "hog-cycles" can be given explicitly:

- If $c_1 + c_2 > 1$, a nontrivial period-2 solution is

$$x_{1,1} = 1 - x_{\rm eq}$$
$$x_{1,2} = 1 - r(1 - c_1) - x_{\rm eq} \leq c_1 - x_{\rm eq}.$$

- If $1 \geq c_1 + c_2$ and $c_1 + rc_2 \geq 1$, a nontrivial period-2 solution is

$$x_{1,1} = 1 - x_{\rm eq}$$
$$x_{1,2} = 1 - rc_2 - x_{\rm eq} \leq c_1 - x_{\rm eq}.$$

- If $c_1 + rc_2 < 1$, a nontrivial period-2 solution is

$$x_{1,1} = 1 - r(1 - c_1) + r^2 c_2 - x_{\rm eq} \geq c_1 + c_2 - x_{\rm eq}$$
$$x_{1,2} = 1 - rc_2 - x_{\rm eq} < c_1 + c_2 - x_{\rm eq}.$$

- If $r = 1$, there are infinite nontrivial period-2 solutions

$$x_{1,1} \in \left(\max\{c_1; 1 - c_2\} - x_{\mathrm{eq}}, \min\{1; c_1 + c_2\} - x_{\mathrm{eq}}\right) \quad \text{arbitrary}$$
$$x_{1,2} = 1 + c_1 - x_{1,1} - 2x_{\mathrm{eq}}.$$

Thus, we have performed a complete parametric stability analysis for system (1.123).

As suggested previously, the stability analysis for discrete-time systems is strongly related to the discretization approach for Lyapunov stability analysis of systems described by ODEs.

The following result is the main result for the discretization approach for systems described by ODEs.

Proposition 2.4 *Consider system* (1.3) *under Hypotheses* (H1–4) *and suppose that there exist a function* $V \in C^0(\Re^+ \times \Re^n; \Re^+)$ *and functions* $a_1, a_2 \in K_\infty$, *and* $\beta, \mu \in K^+$ *satisfying* (2.129) *and the following inequality*:

$$a_1\left(\mu(t)|x|\right) \le V(t, x) \quad \text{for all } (t, x) \in \Re^+ \times \Re^n \tag{2.173}$$

Moreover, suppose that there exist a locally bounded function $T : (0, +\infty) \to (0, +\infty)$, *a positive definite function* $q \in C^0(\Re^+; \Re^+)$, *and a function* $a \in K_\infty$ *such that for all* $t_0 \ge 0$, $x_0 \in \Re^n \backslash \{0\}$, *and* $d \in M_D$, *the solution* $x(t, t_0, x_0; d)$ *of* (1.3) *with initial condition* $x(t_0, t_0, x_0; d) = x_0$ *corresponding to* $d \in M_D$ *exists on* $[t_0, t_0 + T(V(t_0, x_0))]$ *and satisfies the following inequalities*:

$$V\left(t, x(t, t_0, x_0; d)\right) \le a\left(V(t_0, x_0)\right) \quad \text{for all } t \in \left[t_0, t_0 + T\left(V(t_0, x_0)\right)\right] \tag{2.174}$$

$$\min_{t \in [t_0, t_0 + T(V(t_0, x_0))]} V\left(t, x(t, t_0, x_0; d)\right) \le V(t_0, x_0) - q\left(V(t_0, x_0)\right) \tag{2.175}$$

Then system (1.3) *is RGAOS. Moreover, if* $\beta(t) \equiv 1$, *then system* (1.3) *is URGAOS.*

Proof Let $\sigma \in KL$ be the function with the following property.

(P) If $\{V_i \ge 0\}_{i=0}^{\infty}$ is a sequence with $V_{i+1} \le V_i - q(V_i)$, then $V_i \le \sigma(V_0, i)$ for all $i \ge 0$.

The existence of $\sigma \in KL$ which satisfies property (P) is guaranteed by Lemma 4.3 in [19]. In order to present a complete proof, we notice that the function $\sigma \in KL$ can be constructed by following the procedure in the proof of Lemma 2.11 and defining $v(i) := \sup\{\frac{V_i}{g(V_0)} : \{V_i\}_0^\infty \in G(V_0), V_0 > 0\}$, where $g(s) = s^2 + \sqrt{s}$, and $G(V_0)$ denotes the set of all sequences $\{V_i \ge 0\}_{i=0}^{\infty}$ with $V_{i+1} \le V_i - q(V_i)$ for all $i \ge 0$. Then we show that $v(i) \to 0$ and the function $\sigma \in KL$ can be defined as $\sigma(s, i) := g(s)v(i)$ for all $s > 0$, $i \in Z^+$ and $\sigma(0, i) := 0$ for all $i \in Z^+$.

Define $T(0)=1$. Let $t_0 \geq 0$, $x_0 \in \Re^n$, and $d \in M_D$ arbitrary and define the following sequences:

$$\begin{aligned} x_{i+1} &= x(\tau_i + s_i, \tau_i, x_i; d) \qquad T_i = T\big(V(\tau_i, x_i)\big) \\ \tau_{i+1} &= \tau_i + s_i \qquad V_i = V(\tau_i, x_i) \quad i \geq 0 \end{aligned} \tag{2.176}$$

with $\tau_0 = t_0$, where $s_i \in [0, T_i]$ satisfies

$$V\big(x(\tau_i + s_i, \tau_i, x_i; d)\big) = \min_{s \in [0, T_i]} V\big(x(\tau_i + s, \tau_i, x_i; d)\big) \tag{2.177}$$

for the case $x_i \neq 0$ and $s_i = T_i = 1$ for the case $x_i = 0$. Notice that by virtue of the semigroup property we obtain that $x_i = x(\tau_i, t_0, x_0; d)$.

Inequality (2.175) and definitions (2.176), (2.177) imply that

$$V_{i+1} \leq V_i - q(V_i) \tag{2.178}$$

for the case $x_i \neq 0$. For the case $x_i = 0$, by the uniqueness of solution of (1.3) we have $x_{i+1} = 0$, and consequently inequality (2.178) holds as well in this case. Therefore, property (P) guarantees that

$$V_i \leq \sigma(V_0, i) \quad \text{for all } i \geq 0 \tag{2.179}$$

where $\sigma \in KL$ is the function involved in property (P).

Inequality (2.174), definitions (2.176) and the semigroup property guarantee that $V(x(t, t_0, x_0; d)) = V(x(t, \tau_i, x_i; d)) \leq a(V_i)$ for all $t \in [\tau_i, \tau_{i+1}]$ for the case $x_i \neq 0$. By the uniqueness of solution of (1.3), it follows that $V(x(t, t_0, x_0; d)) = V(x(t, \tau_i, x_i; d)) \leq a(V_i)$ for all $t \in [\tau_i, \tau_{i+1}]$ for the case $x_i = 0$ as well. Since $\{V_i \geq 0\}_{i=0}^{\infty}$ is nonincreasing (a consequence of (2.178)), we obtain

$$V\big(x(t, t_0, x_0; d)\big) \leq a\big(V(t_0, x_0)\big) \quad \text{for all } t \in [t_0, \sup \tau_i) \tag{2.180}$$

Next, we show that

$$V\big(x(t, t_0, x_0; d)\big) \leq a\big(V(t_0, x_0)\big) \quad \text{for all } t \geq 0 \tag{2.181}$$

It should be noticed that Robust Forward Completeness, Robust Lyapunov and Lagrange stability follow directly from inequality (2.181) in conjunction with (2.129) and (2.173). Moreover, if $\beta(t) \equiv 1$, then Uniform Robust Lyapunov and Uniform Lagrange stability follows directly from inequality (2.181).

For the proof of inequality (2.181), we distinguish two cases:

Case 1: $\sup \tau_i < +\infty$.

By virtue of inequality (2.179) we obtain that $\lim V_i = 0$ and consequently $\lim_{t \to (\sup \tau_i)^-} V(x(t, t_0, x_0; d)) = 0$. This implies that $\lim_{t \to (\sup \tau_i)^-} x(t, t_0, x_0; d) = 0$, which implies $x(t, t_0, x_0; d) = 0$ for all $t \geq \sup \tau_i$. Therefore inequality (2.181) is a consequence of (2.180) and the fact that $V(x(t, t_0, x_0; d)) = 0$ for all $t \geq \sup \tau_i$.

Case 2: $\sup \tau_i = +\infty$.

In this case inequality (2.181) is a direct consequence of inequality (2.180).

We next show Robust Attractivity. Let $\varepsilon > 0$, $S \geq 0$, $R \geq 0$, $(t_0, x_0) \in \Re^+ \times \Re^n$ with $t_0 \in [0, S]$, $|x_0| \leq R$, and $d \in M_D$ be arbitrary. By virtue of (2.129), (2.178), and the semigroup property it follows that if $V_i \leq a^{-1}(a_1(\varepsilon))$ for some $i \geq 0$, then we have $|H(t, x(t, t_0, x_0; d), 0)| \leq \varepsilon$ for all $t \geq \tau_i$. Define $J := \min\{i \geq 0 : V_i \leq a^{-1}(a_1(\varepsilon))\}$. Let $N_\varepsilon(S, R) \in Z^+$ such that $\sigma(a_2(R \max_{0 \leq t \leq S} \beta(t)), N_\varepsilon(S, R)) \leq a^{-1}(a_1(\varepsilon))$ and notice that inequalities (2.129) and (2.179) imply that $J \leq N_\varepsilon(S, R)$.

Next suppose that $J \geq 1$. Since $V_i \geq a^{-1}(a_1(\varepsilon))$ for all $i \leq J - 1$ (a consequence of definition $J := \min\{i \geq 0 : V_i \leq a^{-1}(a_1(\varepsilon))\}$), we get from (2.176) and the facts that $\{V_i \geq 0\}_{i=0}^{\infty}$ is nonincreasing, $V(t_0, x_0) \leq a_2(R \max_{0 \leq t \leq S} \beta(t))$, and that

$$\tau_{i+1} = \tau_i + s_i \leq \tau_i + T_i \leq \tau_i + \tilde{T}_\varepsilon(S, R) \quad \text{for all } i \leq J - 1$$

where

$$\tilde{T}_\varepsilon(S, R) := \sup\Big\{T(V) : a^{-1}\big(a_1(\varepsilon)\big) \leq V \leq a^{-1}\big(a_1(\varepsilon)\big) + a_2\Big(R \max_{0 \leq t \leq S} \beta(t)\Big)\Big\}$$

Therefore $\tau_{i+1} \leq i\tilde{T}_\varepsilon(S, R)$ for all $i \leq J - 1$, and therefore inequality $J \leq N_\varepsilon(S, R)$ implies $\tau_J \leq N_\varepsilon(S, R)\tilde{T}_\varepsilon(S, R)$. It follows that $V(x(t, t_0, x_0; d)) \leq a^{-1}(a_1(\varepsilon))$ for all $t \geq N_\varepsilon(S, R)\tilde{T}_\varepsilon(S, R)$.

The above conclusion holds as well in the case $J = 0$; namely we have

$$V\big(x(t, t_0, x_0; d)\big) \leq a^{-1}\big(a_1(\varepsilon)\big) \quad \text{for all } t \geq N_\varepsilon(S, R)\tilde{T}_\varepsilon(S, R)$$

Notice that, if $\beta(t) \equiv 1$, then $N_\varepsilon(S, R) \in Z^+$ and

$$\tilde{T}_\varepsilon(S, R) := \sup\Big\{T(V) : a^{-1}\big(a_1(\varepsilon)\big) \leq V \leq a^{-1}\big(a_1(\varepsilon)\big) + a_2\Big(R \max_{0 \leq t \leq S} \beta(t)\Big)\Big\}$$

are independent of $S \geq 0$. The proof is complete. □

Remark 2.7 The reader should notice that the converse of Proposition 2.4 holds for the case of Uniform Robust Global Asymptotic Stability (URGAS), i.e., if system (1.3) is URGAS, then for every function $V \in C^0(\Re^+ \times \Re^n; \Re^+)$ which satisfies (2.129) for certain $a_1, a_2 \in K_\infty$ with $\beta(t) \equiv 1$, there exist a function $a \in K_\infty$ and a locally bounded function $T : (0, +\infty) \to (0, +\infty)$ such that for all $t_0 \geq 0$, $x_0 \in \Re^n \backslash \{0\}$, and $d \in M_D$, the solution $x(t, t_0, x_0; d)$ of (1.3) with initial condition $x(t_0, t_0, x_0; d) = x_0$ corresponding to $d \in M_D$ exists on $[0, T(V(t_0, x_0))]$ and satisfies inequalities (2.173), (2.174), and (2.175). Notice that (2.173) with $\mu(t) \equiv 1$ is a direct consequence of (2.129) and the fact that $H(t, x, 0) := x$ for all $(t, x) \in \Re^+ \times \Re^n$. Moreover, since system (1.3) is URGAS, Theorem 2.2 implies the existence of $\sigma \in KL$ such that for all $t_0 \geq 0$, $x_0 \in \Re^n$, and $d \in M_D$, the solution $x(t, t_0, x_0; d)$ of (1.3) with initial condition $x(t_0, t_0, x_0; d) = x_0$ corresponding to $d \in M_D$ satisfies (2.11) with $H(t, x, 0) := x$. Without loss of generality, we may assume that for each $s>0$, the mapping $t\to\sigma(s,t)$ is strictly decreasing (if not, replace $\sigma(s,t)$ by $\sigma(s,t)+s\exp(-t)$).

Combining (2.129) and (2.11) with $H(t,x,0):=x$, we obtain

$$V\big(x(t,t_0,x_0;d)\big)\le a_2\big(\sigma\big(a_1^{-1}\big(V(x_0)\big),t\big)\big)\quad\text{for all } t\ge 0 \tag{2.182}$$

Let $q\in(0,1)$, and let $T(s)>0$ be the unique solution of $a_2(\sigma(a_1^{-1}(s)T(s)))=(1-q)s$ for each $s>0$. It can be shown that the mapping $(0,+\infty)\ni s\to T(s)$ is bounded on every compact set $S\subset(0,+\infty)$. Therefore, by virtue of (2.182), we conclude that inequalities (2.174) and (2.175) hold with $a(s):=a_2(\sigma(a_1^{-1}(s),0))$ and $q(s):=qs$.

For the case of nonuniform RGAOS, we can present a result which is useful for the solution of important control problems (see Chap. 6).

Proposition 2.5 *Consider system* (1.3) *under Hypotheses* (H1–4). *Suppose that there exists a function $\gamma\in K^+$ satisfying*

$$\sum_{j=0}^{+\infty}\gamma(2j)<+\infty \tag{2.183}$$

$$\lim_{t\to+\infty}\gamma(t)=0 \tag{2.184}$$

It is further assumed that there exist functions $a,a_1,a_2\in K_\infty$, $\beta,\mu\in K^+$, $\rho\in C^0(\Re^+;\Re^+)$ being positive definite such that (2.129) *and* (2.173) *hold and the following properties are fulfilled for all $(t_0,x_0,d)\in\Re^+\times\Re^n\times M_D$ and $j\in Z^+$:*

$$\sup_{t\in[t_0,[t_0]+2]}V\big(t,x(t,t_0,x_0;d)\big)\le a\big(V(t_0,x_0)\big)+\gamma(t_0) \tag{2.185}$$

$$V\big(2j+2,x(2j+2,2j,x_0;d)\big)\le V(2j,x_0)-\rho\big(V(2j,x_0)\big)+\gamma(2j) \tag{2.186}$$

where $x(t,t_0,x_0;d)$ denotes the unique solution of (1.3) *with initial condition $x(t_0)=x_0$ corresponding to $d\in M_D$. Then system* (1.3) *is RGAOS.*

Proof Let $(t_0,x_0,d)\in\Re^+\times\Re^n\times M_D$, and let $j\in Z^+$ be the smallest integer which satisfies $t_0\le 2j$. Inequality (2.186) implies that, for any pair of integers $i\ge j$,

$$V\big(2i,x(2i,t_0,x_0,d)\big)\le V\big(2j,x(2j,t_0,x_0,d)\big)+\sum_{k=0}^{i}\gamma(2k) \tag{2.187}$$

Let $M:=\sum_{k=0}^{+\infty}\gamma(2k)$ and $B:=\sup_{t\ge0}\gamma(t)$. Then, by (2.129), (2.185), and (2.186), we get

$$V\big(t,x(t,t_0,x_0,d)\big)\le a\big(a\big(a_2\big(\beta(t_0)|x_0|\big)\big)+B+M\big)+B\quad\text{for all } t\ge t_0 \tag{2.188}$$

Inequality (2.188), in conjunction with (2.129), implies RFC and Robust Lagrange Output Stability. Therefore, according to Lemma 2.1, in order to establish RGAOS, it suffices to show that system (1.3) satisfies the property of Uniform Output Attractivity on compact sets of initial data. To establish this property, consider arbitrary constants $\varepsilon > 0$, $R \geq 0$, $T \geq 0$ and let $(t_0, x_0, d) \in \Re^+ \times \Re^n \times M_D$ with $t_0 \in [0, T]$ and $|x_0| \leq R$. Define $K := a(a(a_2(R \max_{t\in[0,T]} \beta(t)) + B + M) + B$. Then by (2.188) it holds that

$$V\big(t, x(t, t_0, x_0, d)\big) \leq K \quad \text{for all } t \geq t_0 \tag{2.189}$$

Also, define

$$\tilde{\rho}(s) := \min_{s \leq y \leq K} \rho(y) \tag{2.190}$$

which obviously is a nondecreasing and continuous function, and let $J \geq 0$ be an integer with $\frac{1}{2}\tilde{\rho}(K) \geq \gamma(2i)$ for all integers $i \geq J$, whose existence is guaranteed from (2.184). Next, consider the sequence

$$q_i := \inf\left\{s \in [0, K] : \frac{1}{2}\tilde{\rho}(s) \geq \gamma(2i)\right\} \quad \text{for } i \geq J \tag{2.191}$$

By virtue of (2.184) and (2.191), $q_i \to 0$. Consequently, there exists an integer $N := N(\varepsilon, K) \geq J$ such that

$$q_i + \gamma(2i) \leq S(\varepsilon) \quad \text{and} \quad \gamma(2i) \leq \frac{1}{2}a_1(\varepsilon) \quad \text{for all } i \geq N \tag{2.192}$$

where $a_1 \in K_\infty$ is the function involved in (2.129), and

$$S(\varepsilon) := a^{-1}\left(\frac{1}{2}a_1(\varepsilon)\right) \tag{2.193}$$

Notice next that (2.186) asserts that for all integers $i \geq \max(N, j)$, the following holds:

$$\begin{aligned} &V\big(2(i+1), x\big(2(i+1), t_0, x_0; d\big)\big) \\ &\quad \leq \max\left(S(\varepsilon), V\big(2i, x(2i, t_0, x_0; d)\big) - \frac{1}{2}\rho\big(V\big(2i, x(2i, t_0, x_0; d)\big)\big)\right) \end{aligned} \tag{2.194}$$

Indeed, in order to establish (2.194), we consider two cases. For the first case, assume that $V(2i, x(2i, t_0, x_0, d)) \geq q_i$. Then it follows from (2.190), (2.191), and (2.192) that $\frac{1}{2}\rho(V(2i, x(2i, t_0, x_0, d))) \geq \gamma(2i)$, and this, in conjunction with (2.186), implies (2.194). The other case is $V(2i, x(2i, t_0, x_0, d)) \leq q_i$. Then the latter, in conjunction with (2.186) and (2.192), implies again (2.194).

The following property is a consequence of (2.194):

$$\begin{aligned} &V\big(2i, x(2i, t_0, x_0, d)\big) \leq S(\varepsilon) \\ &\qquad \text{for all integers } i \geq \max(N, j) + \frac{2K}{\tilde{\rho}(S(\varepsilon))} + 1 \end{aligned} \tag{2.195}$$

To show (2.195), suppose on the contrary that there exists integer $i \geq \max(N, j) + \frac{2K}{\tilde{\rho}(S(\varepsilon))} + 1$ with $V(2i, x(2i, t_0, x_0, d)) > S(\varepsilon)$. Then, (2.194) would imply

$$V\big(2k, x(2k, t_0, x_0, d)\big) > S(\varepsilon) \quad \text{for all } k = \max(N, j),\ \max(N, j) + 1, \ldots, \max(N, j) + i \tag{2.196}$$

By (2.189), (2.190), (2.194), and (2.196) we would have

$$V\big(2(k+1), x\big(2(k+1), t_0, x_0, d\big)\big) \leq V\big(2k, x(2k, t_0, x_0, d)\big) - \frac{1}{2}\tilde{\rho}\big(S(\varepsilon)\big)$$
$$\text{for all } k = \max(N, j),\ \max(N, j) + 1, \ldots, \max(N, j) + i - 1$$

and therefore

$$V\big(2k, x(2k, t_0, x_0, d)\big) \leq K - \big(k - \max\big(N, j\big)\big)\frac{1}{2}\tilde{\rho}\big(S(\varepsilon)\big)$$
$$\text{for all } k = \max(N, j),\ \max(N, j) + 1, \ldots, \max(N, j) + i - 1$$

The previous inequality for $k = i$ gives $V(2i, x(2i, t_0, x_0, d)) \leq S(\varepsilon)$, which is a contradiction, and this establishes (2.195).

Finally, by (2.185) and (2.195) we obtain $\sup_{t \in [2i, 2i+2]} V(t, \phi(t, t_0, x_0; d)) \leq a(S(\varepsilon)) + \gamma(2i)$ for all integers $i \geq \max(N, j) + \frac{2K}{\tilde{\rho}(S(\varepsilon))} + 1$. This, in conjunction with (2.192) and (2.193), gives

$$\sup_{t \geq 2i} V\big(t, \phi(t, t_0, x_0, d)\big) \leq a_1(\varepsilon) \quad \text{for all integers } i \geq \max(N, j) + \frac{2K}{\tilde{\rho}(S(\varepsilon))} + 1 \tag{2.197}$$

Using the inequality above and (2.129), we may conclude that the property of Uniform Output Attractivity on compact sets of initial data holds for system (1.3). This completes the proof of Proposition 2.5. □

Propositions 2.4 and 2.5 show that the discretization approach extends the classical Lyapunov analysis, since it does not assume that a certain differential inequality holds for all times. However, it is important to be able to estimate accurately the value of the Lyapunov function along the trajectory of the solution. This disadvantage can be overcome in certain cases.

2.9 Bibliographical and Historical Notes

1. For proving stability by means of Fixed Point Theorems, see [7] and references therein. In our opinion this method of proving stability is flexible and in most of the cases is very similar to small-gain arguments (that are presented later in Chap. 5).

2. The notion of RFC for systems described by ODEs was introduced in [22] and was extended to general systems in [21, 26]. It should be noted that the notion of RFC implies the standard notion of forward completeness, which requires that for every initial condition the solution of the system exists for all times greater than the initial time, or equivalently, the solutions of the system do not present finite escape time. Conversely, an extension of Proposition 5.1 in [39] to the time-varying case shows that every forward complete system described by ODEs, whose dynamics are locally Lipschitz with respect to (t, x), uniformly in $d \in D$, is RFC. A Lyapunov characterization for usual forward completeness for systems described by ODEs is given in [2].
3. The notion of URGAOS for systems described by ODEs was used in [54, 55] (see also [20, 36, 56, 59]). For recent results on uniform global asymptotic stability of nonlinear and time-varying switched systems, see [38] and references therein. Particularly, for systems with identity output mapping, the notion of URGAS was used in [32, 39]. Nonuniform notions for RGAOS (and RGAS) were developed in [21–26, 29–31] in complete analogy with the uniform notions. Many results of the present chapter have appeared in [4–8, 13, 18] For systems described by RFDEs, similar notions have appeared in [4-07, [18]] (see also [33]).
4. The notion of inf-convolutions (used in Remark 2.4) is studied in [11].
5. Differential inequalities are exploited in the book [34] (where vector differential inequalities are used as well). A useful comparison lemma appears in [32]. The reader should notice that the use of differential inequalities is a basic tool for the derivation of useful results for systems (see, for example, the frequent use of various differential inequalities in [15]).
6. The problem of the absolute continuity of the Lyapunov functional for systems described by RFDEs was studied in [47, 48] (see also [24, 30, 31]). The problem of using a Dini derivative for the Lyapunov functional which can be estimated without knowledge of the solution is crucial for systems described by RFDEs: many times the derivatives used depend on the knowledge of the solution (see [17, 33, 46, 59]). We have decided to work with Dini derivatives which can be estimated without knowledge of the solution (because this is the big advantage of the Lyapunov method; see [24, 30, 31, 47, 48] as well as [13]).
7. Excellent introductions to the method of Lyapunov functions and functionals for systems described by ODEs are given in [32, 53] and for systems described by RFDEs in [17, 33]. The books [16, 33] are classical in stability studies.
8. Chemostat models, like the chemostat model (2.157) studied in Example 2.7.3, are very frequently analyzed by means of monotone system theory (see [52]). Usually, it is assumed that the specific growth rate for the microbial species is a strictly increasing, continuously differentiable, bounded function $\mu : [0, +\infty) \to [0, +\infty)$ with $\mu(0) = 0$.
9. The question of whether (U)RGAOS for a system implies the existence of a Lyapunov functional is crucial: see [3, 22, 24, 29, 39, 55, 56]. The following chapter is devoted to this important question.
10. The Lyapunov method for discrete-time systems is developed in [19, 23, 25, 35]; also see [37].

11. The phenomenon of "hog cycles" or nontrivial period-2 solutions (which means that the system is not RGAOS) for economic cobweb models like the model studied in Example 2.8.4, are known for a long time; see, for instance, [14]. The "hog cycles" have interesting economic interpretations.
12. The "discretization approach" for Lyapunov functions was described in [1] (see also [32, 44, 45, 49, 50], the Appendix in [12], and the proof of the main result in [28]). Clearly, the discretization approach does not require a Lyapunov function with a negative definite derivative. Instead, the discretization approach requires that the difference of the values of the Lyapunov function at two consecutive time instances is negative. Therefore, in this approach the Lyapunov function can even have a positive derivative in certain regions of the state space. However, the main difficulty in the application of this approach is the estimation of the difference of the values of the Lyapunov function at two consecutive time instances. The application of this approach to feedback stabilization problems gave very important results in [12] (see also recent extensions in [28]). The recent work [27] attempts the application of the discretization approach in a way that the difference can be estimated without knowledge of the solution map. The analysis in [27] shows that the discretization approach can be utilized with ideas that exploit higher derivatives of Lyapunov functions (see [9, 10, 60]) and ideas from the original theorem of Matrosov (see [51]).
13. In the present book we will not study the Matrosov methodology of proving stability (see [51]). However, it should be noted that the original result by Matrosov has been generalized recently in various directions (see [40–43, 51, 58]).

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Chapter 3
Converse Lyapunov Results

3.1 Introduction

This chapter is devoted to the answer of the following question: do Lyapunov functionals exist for a RGAOS system Σ? The previous chapter showed that one of the most important ways of proving stability is the derivation of estimates which guarantee appropriate stability properties by means of Lyapunov functionals. The objective of this chapter is to show that such Lyapunov functionals always exist. For future reference, the results demonstrating the existence of Lyapunov functionals are referred to as converse Lyapunov results.

Under minimal regularity requirements, it will also be shown that the Lyapunov functionals have desired regularity properties. This is crucial for systems described by ODEs or systems described by RFDEs, since one can prove stability properties without knowledge of the transition map of the system.

In what follows, $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ denotes a control system with the BIC property, $U = \{0\}$, and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Let $u_0 \in M_U$ denote the identically zero input, i.e., $u_0(t) = 0 \in U$ for all $t \geq 0$. Moreover, we assume that Hypothesis (HYP) of Sect. 2.6 holds. For the output map $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$, we assume that either $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ is continuous or that there exists a partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\Re^+$ with diameter $r > 0$ such that $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ satisfies Hypothesis (L2) in Sect. 1.7. Finally, it is assumed that for every $(t_0, x_0, d_1, d_2) \in \Re^+ \times \mathcal{X} \times M_D \times M_D$ and for every $T \in \pi(t_0, x_0, u_0, d_1) \cap (t_0, +\infty)$, there exists $d \in M_D$ such that $\phi(t, T, \phi(T, t_0, x_0, u_0, d_1), u_0, d_2) = \phi(t, t_0, x_0, u_0, d)$ for all $t \geq T$ and $\phi(t, t_0, x_0, u_0, d_1) = \phi(t, t_0, x_0, u_0, d)$ for all $t_0 \leq t \leq T$. This assumption holds:

- for systems described by ODEs under Hypotheses (H1–4),
- for systems described by RFDEs under Hypotheses (S1–4),
- for systems described by coupled RFDEs and FDEs under Hypotheses (P1–5),
- for systems described by FDEs under Hypotheses (Q1–3),

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- for systems with variable sampling partition of the form (1.57) under Hypotheses (A1–4) for which the mappings $f(t,\tau,x,x_0,u,u_0,d,d_0)$ and $R(\tau,x,x_0,u,u_0,d,d_0)$ are independent of $d \in D$ and $d_0 \in D$,
- for discrete-time systems under Hypotheses (L1–3).

3.2 Sontag's Result on *KL* functions

In this section, we present a technical result which appeared in [25] as Proposition 7 and plays an important role in the construction of Lyapunov functionals. It is stated below.

Theorem 3.1 *For each $\sigma \in KL$, there exist $a_1, a_2 \in K_\infty$ such that $a_1(\sigma(s,t)) \le e^{-t} a_2(s)$ for all $s,t \ge 0$.*

The proof of Theorem 3.1 depends on two technical lemmas, which show important properties for the classes of *KL* and K_∞ functions. The following lemma is proved in the same way as in [25].

Lemma 3.1 *For every $\sigma \in KL$, there exist $a_1, a_2 \in K_\infty$ such that*

$$\sigma(s,t) \le a_1\big(\exp(-t)\big)a_2(s) \quad \textit{for all } s,t \ge 0.$$

Proof Without loss of generality we may assume that $\sigma(s,0) \in K_\infty$. Otherwise, we may replace $\sigma \in KL$ by $\tilde{\sigma} \in KL$, where $\tilde{\sigma}(s,t) := \sigma(s,t) + s\exp(-t)$. Define

$$a(t) := \sup_{s>0} \frac{\sigma(s,t)}{\sqrt{\sigma(s,0)} + \sigma^2(s,0)} \tag{3.1}$$

First notice that $a : \Re^+ \to [0,1]$ is well defined and nonincreasing. For every $\varepsilon > 0$, there exist $0 < a(\varepsilon) \le b(\varepsilon)$ such that $\frac{x}{\sqrt{x}+x^2} \le \varepsilon$ for all $x \in (0, a(\varepsilon)]$ or $x \in [b(\varepsilon), +\infty)$. Therefore, from definition (3.1) we obtain, for all $\varepsilon > 0$,

$$\begin{aligned} a(t) &\le \max\left\{\varepsilon, \max_{a(\varepsilon)\le s\le b(\varepsilon)} \frac{\sigma(s,t)}{\sqrt{\sigma(s,0)} + \sigma^2(s,0)}\right\} \\ &\le \max\left\{\varepsilon, \frac{\sigma(b(\varepsilon),t)}{\sqrt{\sigma(a(\varepsilon),0)} + \sigma^2(a(\varepsilon),0)}\right\} \end{aligned} \tag{3.2}$$

For every $\varepsilon > 0$, there exists $T(\varepsilon) \ge 0$ such that

$$\sigma\big(b(\varepsilon),t\big) \le \varepsilon\Big[\sqrt{\sigma\big(a(\varepsilon),0\big)} + \sigma^2\big(a(\varepsilon),0\big)\Big] \quad \text{for all } t \ge T(\varepsilon)$$

Consequently, by (3.2), for all $\varepsilon > 0$, it holds

$$a(t) \le \varepsilon \quad \text{for all } t \ge T(\varepsilon). \tag{3.3}$$

Inequality (3.3) shows that $\lim_{t\to+\infty} a(t) = 0$. Definition (3.1) implies

$$\sigma(s,t) \le a(t)a_2(s) \quad \forall t \ge 0, s > 0 \tag{3.4}$$

where $a_2(s) := \sqrt{\sigma(s,0)} + \sigma^2(s,0)$. Notice that inequality (3.4) holds for $s = 0$ as well. We next define

$$p(r) := \begin{cases} 0 & \text{for } r = 0 \\ a(-\ln r) & \text{for } r \in (0,1] \\ a(0) & \text{for } r > 1 \end{cases} \tag{3.5}$$

Notice that $p : \Re^+ \to [0,1]$ is nondecreasing with $\lim_{r\to 0^+} p(r) = 0$. By Lemma 2.4, there exists $a_1 \in K_\infty$ such that $p(s) \le a_1(s)$ for all $s \ge 0$. Inequality $\sigma(s,t) \le a_1(\exp(-t))a_2(s)$ for all $s, t \ge 0$ is a direct consequence of definition (3.5) and inequality (3.4). The proof is complete. □

The following technical lemma is proved using a different way from that in [25].

Lemma 3.2 *For every $a \in K_\infty$, there exists $p \in K_\infty$ such that $a(rs) \le p(r)p(s)$ for all $r, s \ge 0$.*

Proof Define

$$\gamma(r) := \sup_{s>0} \frac{a(rs)}{a(\sqrt{s}+s^2) + \sqrt{a(s)} + a^2(s)} \quad \text{for all } r \ge 0 \tag{3.6}$$

Definition (3.6) gives $\gamma(0) = 0$. Moreover, for $r \le 1$, it holds that $a(rs) \le a(s) \le a(\sqrt{s}+s^2)$ for all $s > 0$. Consequently, we obtain $\sup_{s>0} \frac{a(rs)}{a(\sqrt{s}+s^2)+\sqrt{a(s)}+a^2(s)} \le 1 < +\infty$. For $r > 1$, we notice that $a(rs) \le a(s^2)$ for all $s \ge r$ and $a(rs) \le a(\sqrt{s})$ for all $s \le \frac{1}{r^2}$. Consequently, we have, for $r > 1$,

$$\begin{aligned} &\sup_{s>0} \frac{a(rs)}{a(\sqrt{s}+s^2) + \sqrt{a(s)} + a^2(s)} \\ &\quad \le \max\left\{1, \max_{r^{-2}\le s\le r} \frac{a(rs)}{a(\sqrt{s}+s^2) + \sqrt{a(s)} + a^2(s)}\right\} \\ &\quad \le \frac{a(r^2)}{a(r^{-1})} < +\infty \end{aligned} \tag{3.7}$$

Therefore, the function $\gamma : \Re^+ \to \Re^+$ is well defined and locally bounded. The monotonicity of $a \in K_\infty$ implies that $\gamma : \Re^+ \to \Re^+$ is nondecreasing. For all $r \le 1$ and $\varepsilon > 0$, it holds that

$$\frac{a(rs)}{a(\sqrt{s}+s^2) + \sqrt{a(s)} + a^2(s)} \le \sqrt{a(s)} \le \varepsilon \quad \text{for all } s \le a^{-1}(\varepsilon^2)$$

$$\frac{a(rs)}{a(\sqrt{s}+s^2) + \sqrt{a(s)} + a^2(s)} \le \frac{1}{a(s)} \le \varepsilon \quad \text{for all } s \ge a^{-1}(\varepsilon^{-1})$$

The above inequalities, in conjunction with definition (3.6), imply that, for all $r \le 1$ and $\varepsilon \in (0,1)$, we have

$$\gamma(r) \le \max\left\{\varepsilon, \max_{a^{-1}(\varepsilon^2)\le s\le a^{-1}(\varepsilon^{-1})} \frac{a(rs)}{a(\sqrt{s}+s^2)+\sqrt{a(s)}+a^2(s)}\right\}$$
$$\le \max\left\{\varepsilon, \frac{a(ra^{-1}(\varepsilon^{-1}))}{2\varepsilon^2}\right\} \tag{3.8}$$

It follows from (3.8) that, for all $\varepsilon \in (0,1)$ and $r \le \frac{a^{-1}(2\varepsilon^3)}{a^{-1}(\varepsilon^{-1})}$, it holds that $\gamma(r) \le \varepsilon$. This implies that $\lim_{r\to 0^+}\gamma(r) = 0$. By Lemma 2.4, there exists $\zeta \in K_\infty$ such that $\gamma(s) \le \zeta(s)$ for all $s \ge 0$. Define, for all $s \ge 0$,

$$p(s) := \max\left\{\zeta(s), a\left(\sqrt{s}+s^2\right)+\sqrt{a(s)}+a^2(s)\right\} \tag{3.9}$$

The desired inequality $a(rs) \le p(r)p(s)$ for all $r, s \ge 0$ is a consequence of definitions (3.6), (3.9) and the fact that $\gamma(s) \le \zeta(s)$ for all $s \ge 0$. The proof is complete. □

We are now in a position to prove Theorem 3.1. Again, the proof is different from that in [25].

Proof of Theorem 3.1 By Lemma 3.1, for every $\sigma \in KL$, there exist $p, q \in K_\infty$ such that $\sigma(s,t) \le p(\exp(-t))q(s)$ for all $s, t \ge 0$. Moreover, by Lemma 3.2, there exists $b \in K_\infty$ such that $rs \le p^{-1}(b(r)b(s))$ for all $r, s \ge 0$. Thus, we obtain, for all $t \ge 0$, $s > 0$,

$$p^{-1}\left(\frac{\sigma(s,t)}{q(s)}\right) \le \exp(-t) \quad \Rightarrow \quad b^{-1}\big(\sigma(s,t)\big)b^{-1}\left(\frac{1}{q(s)}\right) \le \exp(-t) \tag{3.10}$$

Define $a_1(s) := b^{-1}(s)$, $a_2(s) := \frac{1}{b^{-1}(\frac{1}{q(s)})}$ for $s > 0$, and $a_2(0) := 0$. The reader should notice that $a_1, a_2 \in K_\infty$. The desired inequality $a_1(\sigma(s,t)) \le \exp(-t)a_2(s)$ for all $s, t \ge 0$ is a direct consequence of (3.10) and previous definitions. The proof is complete. □

3.3 Construction of the Lyapunov Functional

The following results show that the existence of a Lyapunov functional is a necessary and sufficient condition for Σ to be (U)RGAOS.

Theorem 3.2 (Lyapunov functionals) *Let $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ be a control system with outputs satisfying Hypothesis* (HYP) *and the BIC property with $0 \in \mathcal{X}$ as a robust equilibrium point for Σ. System Σ is RGAOS if and only if there exist mappings $V : \Re^+ \times \mathcal{X} \to \Re^+$, $\beta, \gamma, \mu \in K^+$ with $\int_0^{+\infty}\gamma(t)\,dt = +\infty$, $\varphi \in \mathcal{E}$, $a_1, a_2 \in K_\infty$, and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, there exists a constant $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_\mathcal{X}), t_0 + r]$, where $b(t,\rho) := \min\{q_\pi(t), t + h(t,\rho)\}$ is the function involved in Hypothesis* (HYP) *with $(\tau, t_0, x_0, u_0, d) \in A_\phi$ and the following properties*:

$$a_1\big(\big\|H\big(t,\phi(t,t_0,x_0,u_0,d),0\big)\big\|_{\mathcal{Y}}+\mu(t)\big\|\phi(t,t_0,x_0,u_0,d)\big\|_{\mathcal{X}}\big) \\ \leq V(t_0,x_0)\leq a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) \quad \forall t\in[t_0,\tau] \tag{3.11}$$

$$V\big(\tau,\phi(\tau,t_0,x_0,u_0,d)\big)\leq \eta\big(\tau,t_0,V(t_0,x_0)\big) \tag{3.12}$$

where $\eta(t,t_0,\eta_0)$ denotes the unique solution of the initial-value problem

$$\dot{\eta}=-\gamma(t)\rho(\eta)+\gamma(t)\varphi\left(\int_0^t\gamma(s)\,ds\right) \qquad \eta(t_0)=\eta_0\geq 0 \tag{3.13}$$

Particularly, if system $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,\pi,H)$ is RGAOS, then there exist mappings $V:\Re^+\times\mathcal{X}\to\Re^+$, $\mu,\beta\in K^+$, and $a_1,a_2\in K_\infty$ such that, for all $(t_0,x_0,d)\in\Re^+\times\mathcal{X}\times M_D$ and $\tau\in\pi(t_0,x_0,u_0,d)$, properties (3.11), (3.12), (3.13) are satisfied with $\eta(t,t_0,s):=\exp(-(t-t_0))s$, $\rho(s):=s$, $\gamma(t)\equiv 1$, and $\varphi(t)\equiv 0$.

Proof Suppose first that $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,\pi,H)$ is RGAOS. Then by statement (ii) of Theorem 2.1, there exist functions $\mu,\beta\in K^+$ and $\sigma\in KL$ such that for every $(t_0,x_0,d)\in\Re^+\times\mathcal{X}\times M_D$, we have

$$\big\|H\big(t,\phi(t,t_0,x_0,u_0,d),0\big)\big\|_{\mathcal{Y}}+\mu(t)\big\|\phi(t,t_0,x_0,u_0,d)\big\|_{\mathcal{X}} \\ \leq\sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}},t-t_0\big) \quad \text{for all } t\geq t_0 \tag{3.14}$$

Moreover, by recalling Theorem 3.1, there exist functions a_1, a_2 of class K_∞ such that the KL function $\sigma(s,t)$ is dominated by $a_1^{-1}(\exp(-2t)a_2(s))$. Combining the previous observations with estimate (3.14), we obtain the following estimate for all $(t_0,x_0,d)\in\Re^+\times\mathcal{X}\times M_D$:

$$a_1\big(\big\|H\big(t,\phi(t,t_0,x_0,u_0,d),0\big)\big\|_{\mathcal{Y}}+\mu(t)\big\|\phi(t,t_0,x_0,u_0,d)\big\|_{\mathcal{X}}\big) \\ \leq\exp\big(-2(t-t_0)\big)a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) \quad \text{for all } t\geq t_0 \tag{3.15}$$

We define, for all $(t_0,x_0)\in\Re^+\times\mathcal{X}$,

$$V(t_0,x_0):=\sup\big\{\exp(t-t_0)a_1\big(\big\|H\big(t,\phi(t,t_0,x_0,u_0,d),0\big)\big\|_{\mathcal{Y}} \\ +\mu(t)\big\|\phi(t,t_0,x_0,u_0,d)\big\|_{\mathcal{X}}\big);\,t\geq t_0,d\in M_D\big\} \tag{3.16}$$

It is immediate to verify that definition (3.16), in conjunction with estimate (3.15), guarantees that inequality (3.11) holds for all $(t_0,x_0,d)\in\Re^+\times\mathcal{X}\times M_D$ and $\tau\in\pi(t_0,x_0,u_0,d)$. Moreover, by virtue of definition (3.16) and the hypothesis which guarantees that for every $(t_0,x_0,d_1,d_2)\in\Re^+\times\mathcal{X}\times M_D\times M_D$ and for every $T\in\pi(t_0,x_0,u_0,d_1)\cap(t_0,+\infty)$, there exists $d\in M_D$ such that $\phi(t,T,\phi(T,t_0,x_0,u_0,d_1),u_0,d_2)=\phi(t,t_0,x_0,u_0,d)$ for all $t\geq T$ and $\phi(t,t_0,x_0,u_0,d_1)=\phi(t,t_0,x_0,u_0,d)$ for all $t_0\leq t\leq T$, it follows that inequality (3.12) holds with $\eta(t,t_0,s):=\exp(-(t-t_0))s$ for all $(t_0,x_0,d)\in\Re^+\times\mathcal{X}\times M_D$ and $\tau\in\pi(t_0,x_0,u_0,d)$ (and consequently, (3.13) holds for $\rho(s):=s$, $\gamma(t)\equiv 1$, and $\varphi(t)\equiv 0$).

The rest of the proof is a consequence of Theorem 2.3. The proof is complete. □

Theorem 3.3 (Lyapunov functionals) *Let* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *be a control system with outputs satisfying Hypothesis* (HYP) *and the BIC property with* $0 \in \mathcal{X}$ *as a robust equilibrium point for* Σ. *Moreover, suppose that* Σ *is RFC. System* Σ *is URGAOS if and only if there exist mappings* $V : \Re^+ \times \mathcal{X} \to \Re^+$, $a_1, a_2 \in K_\infty$, *and a locally Lipschitz positive definite function* $\rho : \Re^+ \to \Re^+$ *such that for every* $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, *there exists* $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_\mathcal{X}), t_0 + r]$, *where* $b(t, \rho) := \min\{q_\pi(t), t + h(t, \rho)\}$ *is the function involved in Hypothesis* (HYP), *with* $(\tau, t_0, x_0, u_0, d) \in A_\phi$ *and the following properties*:

$$a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_\mathcal{Y}\big) \le V(t_0, x_0) \le a_2\big(\|x_0\|_\mathcal{X}\big) \quad \forall t \in [t_0, \tau] \tag{3.17}$$

$$V\big(\tau, \phi(\tau, t_0, x_0, u_0, d)\big) \le \eta\big(\tau, t_0, V(t_0, x_0)\big) \tag{3.18}$$

where $\eta(t, t_0, \eta_0)$ *denotes the unique solution of the initial-value problem*

$$\dot{\eta} = -\rho(\eta), \qquad \eta(t_0) = \eta_0 \ge 0. \tag{3.19}$$

Particularly, if system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *is URGAOS, then there exist mappings* $V : \Re^+ \times \mathcal{X} \to \Re^+$ *and* $a_1, a_2 \in K_\infty$ *such that for all* $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ *and* $\tau \in \pi(t_0, x_0, u_0, d)$, *properties* (3.17), (3.18), (3.19) *are satisfied with* $\eta(t, t_0, s) := \exp(-(t - t_0))s$ *and* $\rho(s) := s$.

Proof Suppose first that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is URGAOS. Then by Theorem 2.2, there exists a function $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, we have

$$\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_\mathcal{Y} \le \sigma\big(\|x_0\|_\mathcal{X}, t - t_0\big) \quad \forall t \ge t_0 \tag{3.20}$$

Moreover, by recalling Theorem 3.1, there exist functions a_1, a_2 of class K_∞ such that the KL function $\sigma(s, t)$ is dominated by $a_1^{-1}(\exp(-2t)a_2(s))$. Combining the previous observations with estimate (3.20), we obtain the following estimate for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$:

$$a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_\mathcal{Y}\big) \le \exp\big(-2(t - t_0)\big)a_2\big(\|x_0\|_\mathcal{X}\big) \quad \forall t \ge t_0 \tag{3.21}$$

We define, for all $(t_0, x_0) \in \Re^+ \times \mathcal{X}$,

$$V(t_0, x_0) := \sup\big\{\exp(t - t_0)a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_\mathcal{Y}\big) : t \ge t_0, d \in M_D\big\} \tag{3.22}$$

It is immediate to verify that definition (3.22), in conjunction with estimate (3.21), guarantees that inequality (3.17) holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $\tau \in \pi(t_0, x_0, u_0, d)$. Moreover, by virtue of definition (3.22) and the hypothesis which guarantees that for every $(t_0, x_0, d_1, d_2) \in \Re^+ \times \mathcal{X} \times M_D \times M_D$ and for every $T \in \pi(t_0, x_0, u_0, d_1) \cap (t_0, +\infty)$, there exists $d \in M_D$ such that

$$\phi\big(t, T, \phi(T, t_0, x_0, u_0, d_1), u_0, d_2\big) = \phi(t, t_0, x_0, u_0, d) \quad \text{for all } t \ge T$$
$$\phi(t, t_0, x_0, u_0, d_1) = \phi(t, t_0, x_0, u_0, d) \quad \text{for all } t_0 \le t \le T$$

it follows that inequality (3.18) holds with $\eta(t, t_0, s) := \exp(-(t - t_0))s$ for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $\tau \in \pi(t_0, x_0, u_0, d)$. Consequently, (3.19) holds for $\rho(s) := s$.

The rest of the proof is a consequence of Theorem 2.3. The proof is complete. □

Since Lemma 2.7 shows that RFC for $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is equivalent to RGAOS for system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, \tilde{H})$ with output $\tilde{H}(t, x, u) := \mu(t)x$, where $\mu \in K^+$ is the function involved in inequality (2.56), the following corollary follows readily from Theorem 3.2.

Corollary 3.1 *Suppose that the control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *with outputs satisfies Hypothesis* (HYP) *and the BIC property, and* $0 \in \mathcal{X}$ *is a robust equilibrium point for* Σ. *System* Σ *is RFC if and only if there exist mappings* $V : \Re^+ \times \mathcal{X} \to \Re^+$, $\beta, \gamma, \mu \in K^+$ *with* $\int_0^{+\infty} \gamma(t)\,dt = +\infty$, $\varphi \in \mathcal{E}$, $a_1, a_2 \in K_\infty$, *and a locally Lipschitz positive definite function* $\rho : \Re^+ \to \Re^+$ *such that for every* $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, *there exists* $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_\mathcal{X}), t_0 + r]$, *where* $b(t, \rho) := \min\{q_\pi(t), t + h(t, \rho)\}$ *is the function in* (HYP), *with* $(\tau, t_0, x_0, u_0, d) \in A_\phi$ *and the following properties*:

$$a_1\big(\mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_\mathcal{X}\big) \le V(t_0, x_0) \le a_2\big(\beta(t_0)\|x_0\|_\mathcal{X}\big) \quad \forall t \in [t_0, \tau] \tag{3.23}$$

$$V\big(\tau, \phi(\tau, t_0, x_0, u_0, d)\big) \le \eta\big(\tau, t_0, V(t_0, x_0)\big) \tag{3.24}$$

where $\eta(t, t_0, \eta_0)$ *denotes the unique solution of the initial-value problem*

$$\dot{\eta} = -\gamma(t)\rho(\eta) + \gamma(t)\varphi\left(\int_0^t \gamma(s)\,ds\right) \qquad \eta(t_0) = \eta_0 \ge 0 \tag{3.25}$$

Particularly, if system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *is RFC, then there exist mappings* $V : \Re^+ \times \mathcal{X} \to \Re^+$, $\mu, \beta \in K^+$, *and* $a_1, a_2 \in K_\infty$ *such that for every* $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ *and* $\tau \in \pi(t_0, x_0, u_0, d)$, *properties* (3.23), (3.24), (3.25) *are satisfied with* $\eta(t, t_0, s) := \exp(-(t - t_0))s$, $\rho(s) := s$, $\gamma(t) \equiv 1$, *and* $\varphi(t) \equiv 0$.

Remark 3.1 Theorems 3.2, 3.3, and Corollary 3.1 are existence-type results which do not provide ways of "finding" such a Lyapunov functional. Moreover, they do not guarantee regularity properties for the Lyapunov functionals. On the other hand, Sect. 2.6 showed that the existence of Lyapunov functionals with certain regularity properties is crucial for establishing RGAOS without knowing the evolution map of the system (for certain types of systems). Consequently, the question of the existence of a Lyapunov functional with certain regularity properties becomes important, and the following section is devoted to its answer.

3.4 Regularity Properties of the Lyapunov Functional

In this section we consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. We assume that Hypothesis (HYP) of Sect. 2.6 holds and that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is (U)RGAOS. Furthermore, we will employ one of the following hypotheses:

(REG1) For every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|H(t, \phi(t, t_0, x, u_0, d), 0) - H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} < \varepsilon$ for all $t, t_0 \in I$, $d \in M_D$, $x, x_0 \in S$ with $\|x - x_0\|_{\mathcal{X}} < \delta$, $t \geq t_0$.

(REG2) For every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$, there exists $L \geq 0$ such that $\|H(t, \phi(t, t_0, x, u_0, d), 0) - H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq L\|x - x_0\|_{\mathcal{X}}$ for all $t, t_0 \in I$, $d \in M_D$, $x, x_0 \in S$ with $t \geq t_0$.

(REG3) There exist sets $X_\lambda \subseteq \mathcal{X}$ parameterized by $\lambda \geq 0$ and a nondecreasing function $P : \Re^+ \to [1, +\infty)$ satisfying that for every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$, there exists a constant $L \geq 0$ such that $\|H(t, \phi(t, t_0, x, u_0, d), 0) - H(\tau, \phi(\tau, t_0, x, u_0, d), 0)\|_{\mathcal{Y}} + \|\phi(t, \tau, x, u_0, d) - x\|_{\mathcal{X}} \leq LP(\lambda)|t - \tau|$ for all $t, \tau, t_0 \in I$, $d \in M_D$, $x \in X_\lambda \cap S$ with $t \geq \tau \geq t_0$. Moreover, for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, it holds that $\pi(t_0, x_0, u_0, d) = [t_0, +\infty)$ (i.e., the classical semigroup property holds).

It is clear that (REG2) implies (REG1). Later in this section we will show that systems described by ODEs, systems described by RFDEs, and discrete-time systems satisfy some of the Hypotheses (REG1–3). We now proceed to the construction of a Lyapunov functional which exploits Hypotheses (REG1–3).

By Theorem 2.1 (or Theorem 2.2) there exist functions $\beta \in K^+$ and $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, we have

$$\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \leq \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big) \quad \forall t \geq t_0 \tag{3.26}$$

The reader should notice that in the case of URGAOS we have $\beta(t) \equiv 1$. Moreover, by recalling Theorem 3.1 there exist functions $\tilde{a}_1, \tilde{a}_2$ of class K_∞ such that the KL function $\sigma(s, t)$ is dominated by $\tilde{a}_1^{-1}(\exp(-2t)\tilde{a}_2(s))$. Thus, taking into account estimate (3.26), we have, for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$,

$$\tilde{a}_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d)\big)\big\|_{\mathcal{Y}}\big) \leq \exp\big(-2(t - t_0)\big)\tilde{a}_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) \tag{3.27}$$

Without loss of generality we may assume that $\tilde{a}_1 \in K_\infty$ is globally Lipschitz on $\Re^+$ with unit Lipschitz constant, namely, $|\tilde{a}_1(s_1) - \tilde{a}_1(s_2)| \leq |s_1 - s_2|$ for all $s_1, s_2 \geq 0$. To see this, notice that we can always replace $\tilde{a}_1 \in K_\infty$ by the function $\bar{a}_1(s) := \inf\{\tilde{a}_1(y) + |y - s|; y \geq 0\}$, which is of class K_∞, globally Lipschitz on $\Re^+$ with unit Lipschitz constant, and satisfies $\bar{a}_1(s) \leq \tilde{a}_1(s)$. Moreover, without loss of generality we may assume that $\beta \in K^+$ is nondecreasing.

We define for all $q \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers:

$$\begin{aligned} U_q(t, x) := \sup\big\{&\max\big\{0, \tilde{a}_1\big(\big\|H\big(\tau, \phi(\tau, t, x, u_0, d)\big)\big\|_{\mathcal{Y}}\big) - q^{-1}\big\} \\ &\exp\big((\tau - t)\big) : \tau \geq t, d \in M_D\big\} \end{aligned} \tag{3.28}$$

It should be noted that if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic, then $U_q : \Re^+ \times \mathcal{X} \to \Re^+$ is T-periodic and if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is autonomous, then $U_q : \Re^+ \times \mathcal{X} \to \Re^+$ is independent of $t \in \Re^+$. Clearly, we obtain from estimate (3.27) and definition (3.28) that

$$\max\{0, \tilde{a}_1(\|H(t,x)\|_{\mathcal{Y}}) - q^{-1}\} \le U_q(t,x) \le \tilde{a}_2(\beta(t)\|x\|_{\mathcal{X}})$$
$$\forall (t,x,q) \in \Re^+ \times \mathcal{X} \times \mathbb{N} \tag{3.29}$$

Moreover, by virtue of definition (3.28), the weak semigroup property, and the hypothesis which guarantees that for every $(t_0, x_0, d_1, d_2) \in \Re^+ \times \mathcal{X} \times M_D \times M_D$ and for every $T \in \pi(t_0, x_0, u_0, d_1) \cap (t_0, +\infty)$, there exists $d \in M_D$ such that

$$\phi(t, T, \phi(T, t_0, x_0, u_0, d_1), u_0, d_2) = \phi(t, t_0, x_0, u_0, d) \quad \forall t \ge T$$
$$\phi(t, t_0, x_0, u_0, d_1) = \phi(t, t_0, x_0, u_0, d) \quad \forall t_0 \le t \le T$$

we obtain, for all $(t,x,q) \in \Re^+ \times \mathcal{X} \times \mathbb{N}$, $d \in M_D$, and $\tau \in \pi(t,x,u_0,d)$,

$$U_q(\tau, \phi(\tau, t, x, u_0, d)) \le \exp(-(\tau - t)) U_q(t,x) \tag{3.30}$$

By estimate (3.27) it follows that for all $(q,R) \in \mathbb{N} \times \Re^+$, $\tau \ge t + \tilde{T}(R,q)$, $(t,d) \in [0,R] \times M_D$, and $x \in \mathcal{X}$ with $\|x\|_{\mathcal{X}} \le R$, it holds that

$$\tilde{a}_1(\|H(\tau, \phi(\tau, t, x, u_0, d))\|_{\mathcal{Y}}) \le \exp(-2(\tau - t)) \tilde{a}_2(\beta(t)\|x\|_{\mathcal{X}}) \le q^{-1}$$

where

$$\tilde{T}(R,q) := \frac{1}{2}\log(1 + q\tilde{a}_2(\beta(R)R)) \tag{3.31}$$

Thus, by virtue of definitions (3.28) and (3.31), we conclude that

$$U_q(t,x) = \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t, x, u_0, d))\|_{\mathcal{Y}}) - q^{-1}\} \exp((\tau - t)) :$$
$$t \le \tau \le t + \xi, d \in M_D\} \quad \forall \xi \ge \tilde{T}(\max\{t, \|x\|_r\}, q) \tag{3.32}$$

Equality (3.32) implies the following inequalities for all $t \in [0,R]$ and $(x,y) \in \mathcal{X} \times \mathcal{X}$ with $\|x\|_{\mathcal{X}} \le R$, $\|y\|_{\mathcal{X}} \le R$:

$$\begin{aligned}
&|U_q(t,y) - U_q(t,x)| \\
&= \big|\sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t, y, u_0, d))\|_{\mathcal{Y}}) - q^{-1}\} \exp((\tau - t)) : \\
&\qquad t \le \tau \le t + \bar{T}(R,q), d \in M_D\} \\
&\quad - \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t, x, u_0, d))\|_{\mathcal{Y}}) - q^{-1}\} \exp((\tau - t)) : \\
&\qquad t \le \tau \le t + \bar{T}(R,q), d \in M_D\}\big| \\
&\le \sup\{\exp((\tau - t)) |\tilde{a}_1(\|H(\tau, \phi(\tau, t, y, u_0, d))\|_{Y}) \\
&\qquad - \tilde{a}_1(\|H(\tau, \phi(\tau, t, x, u_0, d))\|_{\mathcal{Y}})| : t \le \tau \le t + \tilde{T}(R,q), d \in M_D\} \\
&\le \sup\{\exp((\tau - t)) \|H(\tau, \phi(\tau, t, y, u_0, d)) - H(\tau, \phi(\tau, t, x, u_0, d))\|_{\mathcal{Y}} : \\
&\qquad t \le \tau \le t + \tilde{T}(R,q), d \in M_D\}
\end{aligned} \tag{3.33}$$

Notice that in the above inequalities we have used the facts that the functions $\max\{0, s - q^{-1}\}$ and $\tilde{a}_1(s)$ are globally Lipschitz on $\Re^+$ with unit Lipschitz constant.

Thus we are ready to use inequality (3.33) in order to establish certain properties.

(1) If Hypothesis (REG1) holds, then for every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$ and for all $\varepsilon > 0$, $q \in \mathbb{N}$, there exists $\delta > 0$ such that $|U_q(t,y) - U_q(t,x)| < \varepsilon$ for all $t \in I$ and $x, y \in S$ with $\|x - y\|_{\mathcal{X}} < \delta$. In this case, define

$$V(t,x) := \sum_{q=1}^{\infty} 2^{-q} U_q(t,x) \tag{3.34}$$

Clearly, definitions (3.28), (3.34) and inequality (3.29) imply the existence of functions $a_1, a_2 \in K_\infty$ such that, for all $(t,x) \in \Re^+ \times \mathcal{X}$, $d \in M_D$,

$$\sup_{\tau \geq t} a_1\big(\big\|H\big(\tau, \phi(\tau, t, x, u_0, d), 0\big)\big\|_{\mathcal{Y}}\big) \leq V(t,x) \leq a_2\big(\beta(t)\|x\|_{\mathcal{X}}\big) \tag{3.35}$$

Particularly, we have $a_2(s) := \tilde{a}_2(s)$ and $a_1(s) := \sum_{q=1}^{\infty} 2^{-q} \max\{0, \tilde{a}_1(s) - q^{-1}\}$ (the reader should notice that $a_1 \in K_\infty$). Moreover, inequality (3.30), in conjunction with definition (3.34), implies that, for all $(t,x) \in \Re^+ \times \mathcal{X}$, $d \in M_D$ and $\tau \in \pi(t, x, u_0, d)$,

$$V\big(\tau, \phi(\tau, t, x, u_0, d)\big) \leq \exp\big(-(\tau - t)\big) V(t,x) \tag{3.36}$$

Furthermore, if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic, then $V : \Re^+ \times \mathcal{X} \to \Re^+$ is T-periodic, and if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is autonomous, then $V : \Re^+ \times \mathcal{X} \to \Re^+$ is independent of $t \in \Re^+$. Take arbitrary bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$ and $\varepsilon > 0$. Clearly, there exists an integer $N > 1$ such that $\tilde{a}_2(\beta(t)\|x\|_{\mathcal{X}}) 2^{-N} < \frac{\varepsilon}{4}$ for all $t \in I$, $x \in S$. It follows from (3.29) that

$$\sum_{q=N+1}^{\infty} 2^{-q} U_q(t,x) \leq \tilde{a}_2\big(\beta(t)\|x\|_{\mathcal{X}}\big) \sum_{q=N+1}^{\infty} 2^{-q} < \frac{\varepsilon}{4} \quad \text{for all } t \in I, x \in S$$

Consequently, (3.34) implies that $|V(t,x) - V(t,y)| \leq \frac{\varepsilon}{2} + \sum_{q=1}^{N} 2^{-q} |U_q(t,x) - U_q(t,y)|$ for all $t \in I$ and $x, y \in S$. Moreover, there exists $\delta > 0$ such that $|U_q(t,y) - U_q(t,x)| < \frac{\varepsilon}{2}$ for all $t \in I$, $x, y \in S$, $q = 1, \ldots, N$ with $\|x - y\|_{\mathcal{X}} < \delta$. Finally, we obtain the following property:

For every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|V(t,y) - V(t,x)| < \varepsilon$ for all $t \in I$ and $x, y \in S$ with $\|x - y\|_{\mathcal{X}} < \delta$.

(2) If Hypothesis (REG2) holds, then there exists a function $G : \Re^+ \times \mathbb{N} \to \Re^+$ with $G(q,q) \geq G(R,q)$ for all $q \geq R$ and such that for all $R \geq 0$, $t \in [0,R]$, $q \in \mathbb{N}$, and $(x,y) \in \mathcal{X} \times \mathcal{X}$ with $\|x\|_{\mathcal{X}} \leq R$, $\|y\|_{\mathcal{X}} \leq R$, it holds

$$\big|U_q(t,y) - U_q(t,x)\big| \leq G(R,q)\|y - x\|_{\mathcal{X}} \tag{3.37}$$

Particularly, if $L(R)$ is a nondecreasing, nonnegative function satisfying the property that $\|H(t, \phi(t, t_0, x, u_0, d), 0) - H(t, \phi(t, t_0, y, u_0, d), 0)\|_{\mathcal{Y}} \leq L(R)\|x - y\|_{\mathcal{X}}$ for all $t, t_0 \in [0,R]$, $d \in M_D$, $(x,y) \in \mathcal{X} \times \mathcal{X}$ with $\|x\|_{\mathcal{X}} \leq R$, $\|y\|_{\mathcal{X}} \leq R$ and $t \geq t_0$, then we can define

$$G(R,q) := L\big(R + \tilde{T}(R,q)\big) \exp\big(\tilde{T}(R,q)\big) \tag{3.38}$$

In this case, introduce

$$V(t,x) := \sum_{q=1}^{\infty} \frac{2^{-q} U_q(t,x)}{1+G(q,q)} \tag{3.39}$$

Clearly, definitions (3.28), (3.39) and inequality (3.29) imply the existence of functions $a_1, a_2 \in K_\infty$ such that (3.35) holds (particularly, we have $a_2(s) := \tilde{a}_2(s)$ and $a_1(s) := \sum_{q=1}^{\infty} 2^{-q} \frac{\max\{0, \tilde{a}_1(s) - q^{-1}\}}{1+G(q,q)}$). Moreover, inequality (3.30), in conjunction with definition (3.39), implies that (3.36) holds for all $(t,x) \in \Re^+ \times \mathcal{X}$, $d \in M_D$, and $\tau \in \pi(t,x,u_0,d)$. Furthermore, if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic, then $V : \Re^+ \times \mathcal{X} \to \Re^+$ is T-periodic, and if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is autonomous, then $V : \Re^+ \times \mathcal{X} \to \Re^+$ is independent of $t \in \Re^+$. Finally, for all $R \geq 0$, $t \in [0,R]$, and $(x,y) \in \mathcal{X} \times \mathcal{X}$ with $\|x\|_\mathcal{X} \leq R$, $\|y\|_\mathcal{X} \leq R$, it follows from (3.37) that

$$\begin{aligned}
\left|V(t,x) - V(t,y)\right| &\leq \sum_{q=1}^{\infty} 2^{-q} \frac{|U_q(t,x) - U_q(t,y)|}{1+G(q,q)} \\
&\leq \|y-x\|_\mathcal{X} \sum_{q=1}^{\infty} 2^{-q} \frac{G(R,q)}{1+G(q,q)} \\
&\leq \|y-x\|_\mathcal{X} \left(\sum_{q=1}^{[R]+1} 2^{-q} \frac{G(R,q)}{1+G(q,q)} + 1 \right)
\end{aligned}$$

Consequently, we have established the following property:

There exists a nondecreasing function $M : \Re^+ \to \Re^+$ such that for all $t \in [0,R]$ and $(x,y) \in \mathcal{X} \times \mathcal{X}$ with $\|x\|_\mathcal{X} \leq R$, $\|y\|_\mathcal{X} \leq R$, it holds

$$\left|V(t,y) - V(t,x)\right| \leq M(R) \|y-x\|_\mathcal{X} \tag{3.40}$$

(3) If Hypotheses (REG2) and (REG3) hold, then there exists a function $G : \Re^+ \times \mathbb{N} \to \Re^+$ with $G(q,q) \geq G(R,q)$ for all $q \geq R$ and such that, for all $R \geq 0$, $t \in [0,R]$, $q \in \mathbb{N}$, and $(x,y) \in \mathcal{X} \times \mathcal{X}$ with $\|x\|_\mathcal{X} \leq R$, $\|y\|_\mathcal{X} \leq R$, inequality (3.37) holds.

Let $R, \lambda \geq 0$, $q \in N$ arbitrary, $t_1, t_2 \in [0,R]$ with $t_1 \leq t_2$, and $x \in X_\lambda$ with $\|x\|_\mathcal{X} \leq R$. Clearly, we have, for all $d \in M_D$,

$$\begin{aligned}
\left|U_q(t_1,x) - U_q(t_2,x)\right| \leq{}& \left(1 - \exp\left(-(t_2-t_1)\right)\right) U_q(t_1,x) \\
&+ \left|\exp\left(-(t_2-t_1)\right) U_q(t_1,x) - U_q\left(t_2, \phi(t_2,t_1,x,u_0,d)\right)\right| \\
&+ \left|U_q\left(t_2, \phi(t_2,t_1,x,u_0,d)\right) - U_q(t_2,x)\right|
\end{aligned} \tag{3.41}$$

Since Σ is RGAOS (and consequently RFC), Lemma 2.7 implies that there exist functions $\mu \in K^+$ and $\tilde{a} \in K_\infty$ such that for every $(t_0,x_0,d) \in \Re^+ \times \mathcal{X} \times M_D$, we have

$$\left\|\phi(t,t_0,x_0,u_0,d)\right\|_\mathcal{X} \leq \mu(t) \tilde{a}\left(\|x_0\|_\mathcal{X}\right) \quad \forall t \geq t_0 \tag{3.42}$$

Without loss of generality, we may assume that $\mu \in K^+$ is nondecreasing. By virtue of (3.37), (3.30), (3.41), and (3.42), we obtain, for all $t_1, t_2 \in [0, R]$ with $t_1 \le t_2$ and $d \in M_D$,

$$\begin{aligned}\big|U_q(t_1,x) - U_q(t_2,x)\big| \le (t_2 - t_1)U_q(t_1,x) + \exp\big(-(t_2-t_1)\big)U_q(t_1,x)\\ - U_q\big(t_2, \phi(t_2,t_1,x,u_0,d)\big)\\ + G\big(\mu(R)\tilde{a}(R), q\big)L(R)P(\lambda)(t_2-t_1)\end{aligned} \tag{3.43}$$

where $L(R) \ge 0$ is a nondecreasing function satisfying $\|H(t,\phi(t,t_0,x,u_0,d),0) - H(\tau,\phi(\tau,t_0,x,u_0,d),0)\|_{\mathcal{Y}} + \|\phi(t,\tau,x,u_0,d) - x\|_{\mathcal{X}} \le L(R)P(\lambda)|t-\tau|$ for all $t,\ \tau,\ t_0 \in [0,R]$, $d \in M_D$ with $t \ge \tau \ge t_0$, and $x \in X_\lambda$ with $\|x\|_{\mathcal{X}} \le R$. Definition (3.28) implies that for every $\varepsilon > 0$, there exists $d_\varepsilon \in M_D$ with the following property:

$$\begin{aligned}U_q(t_1,x) - \varepsilon \le \sup\big\{\max\big\{0, \tilde{a}_1\big(\big\|H\big(\tau,\phi(\tau,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}}\big) - q^{-1}\big\}\\ \exp\big((\tau - t_1)\big);\ \tau \ge t_1\big\} \le U_q(t_1,x)\end{aligned} \tag{3.44}$$

Thus, using definition (3.28) and (3.44), we obtain

$$\begin{aligned}&\exp\big(-(t_2-t_1)\big)U_q(t_1,x) - U_q\big(t_2,\phi(t_2,t_1,x,u_0,d_\varepsilon)\big)\\ &\quad\le \max\big\{A_q(t_1,t_2,x),\ B_q(t_1,t_2,x)\big\} - B_q(t_1,t_2,x) + \varepsilon\exp\big(-(t_2-t_1)\big)\end{aligned} \tag{3.45}$$

where

$$\begin{aligned}A_q(t_1,t_2,x) &:= \sup\big\{\max\big\{0, \tilde{a}_1\big(\big\|H\big(\tau,\phi(\tau,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}}\big) - q^{-1}\big\}\\ &\qquad \exp\big((\tau-t_2)\big);\ t_2 \ge \tau \ge t_1\big\}\\ B_q(t_1,t_2,x) &:= \sup\big\{\max\big\{0, \tilde{a}_1\big(\big\|H\big(\tau,\phi(\tau,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}}\big) - q^{-1}\big\}\\ &\qquad \exp\big((\tau-t_2)\big);\ \tau \ge t_2\big\}\end{aligned} \tag{3.46}$$

Since the functions $\max\{0, s - q^{-1}\}$ and $\tilde{a}_1(s)$ are globally Lipschitz on $\Re^+$ with unit Lipschitz constant, we have

$$\begin{aligned}&A_q(t_1,t_2,x) - B_q(t_1,t_2,x)\\ &\quad\le \sup\big\{\max\big\{0, \tilde{a}_1\big(\big\|H\big(\tau,\phi(\tau,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}}\big) - q^{-1}\big\}\exp\big((\tau-t_2)\big);\\ &\qquad\qquad t_2 \ge \tau \ge t_1\big\} - \max\big\{0, \tilde{a}_1\big(\big\|H\big(t_2,\phi(t_2,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}}\big) - q^{-1}\big\}\\ &\quad\le \sup\big\{\max\big\{0, \tilde{a}_1\big(\big\|H\big(\tau,\phi(\tau,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}}\big) - q^{-1}\big\};\ t_2 \ge \tau \ge t_1\big\}\\ &\qquad - \max\big\{0, \tilde{a}_1\big(\big\|H\big(t_2,\phi(t_2,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}}\big) - q^{-1}\big\}\\ &\quad\le \sup\big\{\big|\tilde{a}_1\big(\big\|H\big(\tau,\phi(\tau,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}}\big) - \tilde{a}_1\big(\big\|H\big(t_2,\phi(t_2,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}}\big)\big|;\\ &\qquad\qquad t_2 \ge \tau \ge t_1\big\}\\ &\quad\le \sup\big\{\big\|H\big(\tau,\phi(\tau,t_1,x,u_0,d_\varepsilon)\big) - H\big(t_2,\phi(t_2,t_1,x,u_0,d_\varepsilon)\big)\big\|_{\mathcal{Y}};\ t_2 \ge \tau \ge t_1\big\}\end{aligned} \tag{3.47}$$

Distinguishing the cases $A_q(t_1,t_2,x) \geq B_q(t_1,t_2,x)$ and $A_q(t_1,t_2,x) \leq B_q(t_1,t_2,x)$, it follows from (3.45) and (3.47) that

$$\exp\big(-(t_2-t_1)\big)U_q(t_1,x) - U_q\big(t_2,\phi(t_2,t_1,x,u_0,d_\varepsilon)\big) \leq L(R)P(\lambda)(t_2-t_1)+\varepsilon$$

where $L(R)\geq 0$ is a nondecreasing function satisfying $\|H(t,\phi(t,t_0,x,u_0,d),0) - H(\tau,\phi(\tau,t_0,x,u_0,d),0)\|_{\mathcal{Y}} + \|\phi(t,\tau,x,u_0,d)-x\|_{\mathcal{X}} \leq L(R)P(\lambda)|t-\tau|$ for all $t,\tau,t_0\in[0,R]$, $d\in M_D$ with $t\geq\tau\geq t_0$, and $x\in X_\lambda$ with $\|x\|_{\mathcal{X}}\leq R$. Combining the above inequality with (3.43) and the right-hand side of (3.29), we obtain

$$\begin{aligned}&\big|U_q(t_1,x)-U_q(t_2,x)\big|\\&\quad\leq (t_2-t_1)P(\lambda)\big(\tilde{a}_2\big(\beta(R)R\big)+L(R)+L(R)G\big(\mu(R)\tilde{a}(R),q\big)\big)+\varepsilon\end{aligned} \tag{3.48}$$

Since (3.48) holds for all $\varepsilon>0$, $R,\lambda\geq 0$, $q\in\mathbb{N}$, $x\in X_\lambda$ with $\|x\|_{\mathcal{X}}\leq R$, and $t_1,t_2\in[0,R]$ with $t_1\leq t_2$, it follows that for all R, $\lambda\geq 0$, $q\in\mathbb{N}$, $x\in X_\lambda$ with $\|x\|_{\mathcal{X}}\leq R$, and $t_1,t_2\in[0,R]$,

$$\big|U_q(t_1,x)-U_q(t_2,x)\big|\leq|t_2-t_1|P(\lambda)\bar{G}(R,q) \tag{3.49}$$

where $\bar{G}(R,q):=\tilde{a}_2(\beta(R)R)+L(R)+L(R)G(\mu(R)\tilde{a}(R),q)$. Finally, we define

$$V(t,x):=\sum_{q=1}^{\infty}\frac{2^{-q}U_q(t,x)}{1+G(q,q)+\bar{G}(q,q)} \tag{3.50}$$

Clearly, definitions (3.28), (3.50) and inequality (3.29) imply the existence of functions $a_1,a_2\in K_\infty$ such that (3.35) holds (particularly, we have $a_2(s):=\tilde{a}_2(s)$ and $a_1(s):=\sum_{q=1}^{\infty}2^{-q}\frac{\max\{0,\tilde{a}_1(s)-q^{-1}\}}{1+G(q,q)+\bar{G}(q,q)}$). Moreover, inequality (3.30), in conjunction with definition (3.50), implies that (3.36) holds for all $(t,x)\in\Re^+\times\mathcal{X}$, $d\in M_D$, and $\tau\geq t$. Furthermore, if $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,\pi,H)$ is T-periodic, then $V:\Re^+\times\mathcal{X}\to\Re^+$ is T-periodic, and if $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,\pi,H)$ is autonomous, then $V:\Re^+\times\mathcal{X}\to\Re^+$ is independent of $t\in\Re^+$. Similarly, as in the previous case, we may establish that property (3.40) holds. Finally, by virtue of (3.49) and definition (3.50), we obtain, for all $R,\lambda\geq 0$, $x\in X_\lambda$ with $\|x\|_{\mathcal{X}}\leq R$, and $t_1,t_2\in[0,R]$,

$$\begin{aligned}\big|V(t_2,x)-V(t_1,x)\big|&\leq\sum_{q=1}^{\infty}\frac{2^{-q}|U_q(t_2,x)-U_q(t_1,x)|}{1+G(q,q)+\bar{G}(q,q)}\\&\leq|t_2-t_1|P(\lambda)\sum_{q=1}^{\infty}\frac{2^{-q}\bar{G}(R,q)}{1+G(q,q)+\bar{G}(q,q)}\\&\leq|t_2-t_1|P(\lambda)\left(\sum_{q=1}^{[R]+1}\frac{2^{-q}\bar{G}(R,q)}{1+G(q,q)+\bar{G}(q,q)}+1\right)\end{aligned}$$

Therefore, we have shown almost all conclusions of the following theorem.

Theorem 3.4 (Lyapunov functionals) *Suppose that* $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,\pi,H)$ *is a control system with outputs satisfying Hypothesis* (HYP) *and the BIC property* *with $0 \in \mathcal{X}$ as a robust equilibrium point for Σ. Moreover, suppose that system Σ is RGAOS. Then there exist mappings $V : \Re^+ \times \mathcal{X} \to \Re^+$, $\beta \in K^+$, and $a_1, a_2 \in K_\infty$ such that for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $\tau \in \pi(t_0, x_0, u_0, d)$, the following inequalities hold*:

$$a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}}\big) \le V(t_0, x_0) \le a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) \quad \forall t \in [t_0, \tau] \tag{3.51}$$

$$V\big(\tau, \phi(\tau, t_0, x_0, u_0, d)\big) \le \exp\big(-(\tau - t_0)\big)V(t_0, x_0) \tag{3.52}$$

Moreover, if system Σ is URGAOS, then (3.51) *holds with $\beta(t) \equiv 1$. Furthermore,*

(i) *If Hypothesis* (REG1) *holds, then for every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|V(t, y) - V(t, x)| < \varepsilon$ for all $t \in I$ and $x, y \in S$ with $\|x - y\|_{\mathcal{X}} < \delta$.*

(ii) *If Hypothesis* (REG2) *holds, then there exists a nondecreasing function $M : \Re^+ \to \Re^+$ such that for all $R \ge 0$, $t \in [0, R]$, and $(x, y) \in \mathcal{X} \times \mathcal{X}$ with $\|x\|_{\mathcal{X}} \le R$, $\|y\|_{\mathcal{X}} \le R$, it holds that*

$$\big|V(t, y) - V(t, x)\big| \le M(R)\|y - x\|_{\mathcal{X}}. \tag{3.53}$$

(iii) *If Hypotheses* (REG2) *and* (REG3) *hold, then there exists a nondecreasing function $M : \Re^+ \to \Re^+$ such that for all $R, \lambda \ge 0$, $t_1, t_2 \in [0, R]$, and $x \in X_\lambda \subseteq \mathcal{X}$ with $\|x\|_{\mathcal{X}} \le R$, it holds that*

$$\big|V(t_2, x) - V(t_1, x)\big| \le M(R)P(\lambda)|t_2 - t_1|. \tag{3.54}$$

Moreover, if the set $\bigcup_{\lambda \ge 0} X_\lambda$ is dense in $\mathcal{X}$, then $V : \Re^+ \times \mathcal{X} \to \Re^+$ is continuous.

Finally, if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic, then $V : \Re^+ \times \mathcal{X} \to \Re^+$ is T-periodic, and if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is autonomous, then $V : \Re^+ \times \mathcal{X} \to \Re^+$ is independent of $t \in \Re^+$.

Proof We only have to show the continuity of $V : \Re^+ \times \mathcal{X} \to \Re^+$ with respect to $t \in \Re^+$ for the case where (REG2) and (REG3) hold and under the additional hypothesis that the set $\bigcup_{\lambda \ge 0} X_\lambda$ is dense in $\mathcal{X}$. Let $(t_0, x_0) \in \Re^+ \times \mathcal{X}$, $\varepsilon > 0$, and define $R := \|x_0\|_{\mathcal{X}} + t_0 + 2$. Since the set $\bigcup_{\lambda \ge 0} X_\lambda$ is dense in $\mathcal{X}$, there exist $\lambda \ge 0$ and $x \in X_\lambda \subseteq \mathcal{X}$ such that

$$\|x - x_0\|_{\mathcal{X}} \le \min\left\{\frac{\varepsilon}{4(1 + M(R))}, 1\right\}$$

where $M : \Re^+ \to \Re^+$ is the nondecreasing function involved in (3.53). Consequently, using (3.53), (3.54), and the above inequality, we have, for all $t \in \Re^+$ with $|t - t_0| \le 1$,

$$\begin{aligned}
\big|V(t, x_0) - V(t_0, x_0)\big| &\le \big|V(t, x_0) - V(t, x)\big| + \big|V(t, x) - V(t_0, x)\big| \\
&\quad + \big|V(t_0, x) - V(t_0, x_0)\big| \\
&\le 2M(R)\|x - x_0\|_{\mathcal{X}} + M(R)P(\lambda)|t - t_0| \\
&\le \frac{\varepsilon}{2} + M(R)P(\lambda)|t - t_0|
\end{aligned}$$

It follows that $|V(t,x_0) - V(t_0,x_0)| \leq \varepsilon$ for all $t \in \Re^+$ with $|t - t_0| \leq \min\{\frac{\varepsilon}{2(1+M(R)P(\lambda))}, 1\}$. The proof is complete. □

The result of Theorem 3.4 can lead to the production of specialized results for three kinds of systems of Definition 1.1.

3.4.1 Control Systems Described by ODEs

We start by presenting a general Lyapunov theorem for systems described by ODEs of the form (1.3) under Hypotheses (H1–4). We assume that system (1.3) under Hypotheses (H1–4) is RGAOS.

Clearly, inequality (1.2) guarantees that Hypothesis (REG2) holds if in addition we further assume that $H : \Re^+ \times \Re^n \to \Re^k$ is locally Lipschitz and independent of $u \in U$. Moreover, the fact that

$$\phi(t,t_0,x,u_0,d) = x + \int_{t_0}^{t} f\big(s,\phi(s,t_0,x,u_0,d),0,d(s)\big)ds$$

implies that $|\phi(t,t_0,x,u_0,d) - x| \leq (t-t_0)\max_{t_0 \leq s \leq t} \gamma(s)a(\mu(s)\tilde{a}(|x_0|))$, where $\gamma \in K^+$ and $a \in K_\infty$ are the functions involved in Hypothesis (H4), and $\mu \in K^+$ and $\tilde{a} \in K_\infty$ are the functions involved in (3.42). This observation combined with the fact that $H : \Re^+ \times \Re^n \to \Re^k$ is locally Lipschitz implies that Hypothesis (REG3) holds as well with $X_\lambda = \Re^n$ for all $\lambda \geq 0$.

Therefore, we obtain the following:

Theorem 3.5 *Consider system* (1.3) *under Hypotheses* (*H*1–4) *and suppose that* $H : \Re^+ \times \Re^n \to \Re^k$ *is locally Lipschitz and independent of* $u \in U$. *Furthermore, suppose that* (1.3) *is RGAOS and that the following hypothesis holds*:

(H5) *There exists a countable set* $A \subset \Re^+$, *which is either finite or* $A = \{t_k; k = 1,\ldots,\infty\}$ *with* $t_{k+1} > t_k > 0$ *for all* $k = 1,2,\ldots$ *and* $\lim t_k = +\infty$, *such that the mapping* $(t,x,u,d) \in (\Re^+ \backslash A) \times \Re^n \times U \times D \to f(t,x,u,d)$ *is continuous. Moreover, for each fixed* $(t_0,x,u,d) \in \Re^+ \times \Re^n \times U \times D$, *we have* $\lim_{t \to t_0^+} f(t,x,u,d) = f(t_0,x,u,d)$.

Then, there exist functions $V : \Re^+ \times \Re^n \to \Re^+$, *which is locally Lipschitz*, $\beta \in K^+$, *and* $a_1, a_2 \in K_\infty$ *such that the following inequalities hold*:

$$a_1\big(|H(t,x)|\big) \leq V(t,x) \leq a_2\big(\beta(t)|x|\big) \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{3.55}$$

$$V^0\big(t,x;f(t,x,0,d)\big) \leq -V(t,x) \quad \forall (t,x,d) \in \Re^+ \times \Re^n \times D \tag{3.56}$$

where $V^0(t,x;v)$ *is defined in* (2.128) *for all* $v \in \Re^n$. *Moreover, if* (1.3) *is URGAOS, then* $\beta(t) \equiv 1$. *Finally, if* (1.3) *is* T*-periodic, then* $V : \Re^+ \times \mathcal{X} \to \Re^+$ *is* T*-periodic, and if* (1.3) *is autonomous, then* $V : \Re^+ \times \mathcal{X} \to \Re^+$ *is independent of* $t \in \Re^+$.

Proof By Theorem 3.4, there exist mappings $V : \Re^+ \times \Re^n \to \Re^+$, which is locally Lipschitz, $\beta \in K^+$, and $a_1, a_2 \in K_\infty$ such that for all $(t, x, d) \in \Re^+ \times \Re^n \times M_D$ and $\tau \geq t$, (3.55) holds and the following inequality holds:

$$V\big(\tau, \phi(\tau, t, x, u_0, d)\big) \leq \exp\big(-(\tau - t)\big) V(t, x) \tag{3.57}$$

The conclusion of the theorem is a direct consequence of the following fact:

Fact $\lim_{\tau \to t^+} \frac{\phi(\tau,t,x,u_0,\tilde{d})-x}{\tau-t} = f(t, x, 0, \tilde{d}(t))$ *for every continuous* $\tilde{d} \in M_D$.

Indeed, by letting $\tilde{d}(t) \equiv d \in D$ it follows from the above fact and (3.57) that (3.56) holds for all $(t, x, d) \in \Re^+ \times \Re^n \times D$.

The above fact is a direct consequence of Hypothesis (H5). It is clear that for every continuous $d \in M_D$, the composite map $f(\tau, x, 0, d(\tau))$ is continuous on $(\Re^+ \backslash A) \times \Re^n$. Applying repeatedly Theorem 2.1 (p. 43) in [9] on each one of the intervals contained in $[t, +\infty)\backslash A$, we conclude that the solution $x(\tau) = \phi(\tau, t, x, u_0, d)$ of (1.3) satisfies $\dot{x}(\tau) = f(\tau, x(\tau), 0, d(\tau))$ for all $\tau \in [t, +\infty)\backslash A$. Since the composite map $\tau \to f(\tau, x, 0, d(\tau))$ is right-continuous on $\Re^+$, by virtue of the mean value theorem, it follows that $\lim_{h \to 0^+} \frac{x(t+h)-x(t)}{h} = f(t, x, 0, d(t))$ for all $\tau \in [t, +\infty)$.

The proof is complete. □

3.4.2 Control Systems Described by RFDEs

We continue by presenting a general converse Lyapunov theorem for systems described by RFDEs of the form (1.10) under Hypotheses (S1–4). It is further assumed that (1.10) is RGAOS and that the following hypothesis holds:

(S5) The mapping $H(t, x, u)$ is independent of $u \in U$ and Lipschitz on bounded sets, in the sense that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r, 0]; \Re^n)$, there exists a constant $L_H \geq 0$ such that

$$\big\| H(t, x) - H(\tau, y) \big\|_{\mathcal{Y}} \leq L_H\big(|t - \tau| + \|x - y\|_r\big)$$
$$\forall (t, \tau) \in I \times I, \forall (x, y) \in S \times S$$

Hypothesis (S5) is equivalent to the existence of a continuous function $L_H : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $t \geq 0$, the mappings $L_H(t, \cdot)$ and $L_H(\cdot, t)$ are nondecreasing, with the following property:

$$\big\| H(t, x) - H(\tau, y) \big\|_{\mathcal{Y}} \leq L_H\big(\max\{t, \tau\}, \|x\|_r + \|y\|_r\big)\big(|t - \tau| + \|x - y\|_r\big)$$
$$\forall (t, \tau, x, y) \in \Re^+ \times \Re^+ \times C^0\big([-r, 0]; \Re^n\big) \times C^0\big([-r, 0]; \Re^n\big) \tag{3.58}$$

Inequality (1.9), in conjunction with Hypothesis (S5), implies that Hypothesis (REG2) holds. Moreover, Hypothesis (REG3) holds as well with X_λ being the set of absolutely continuous functions $x : [-r, 0] \to \Re^n$ with $\sup_{\theta \in [-r,0]} |\dot{x}(\theta)| \leq \lambda$. Indeed, the fact that

$$\phi(t,t_0,x,u_0,d) = \begin{cases} x(0)+\int_{t_0}^{t+\theta} f(s,\phi(s,t_0,x,u_0,d),0,d(s))\,ds & \text{for } 0\geq\theta\geq t_0-t \\ x(t-t_0+\theta) & \text{for } t_0-t>\theta\geq t_0-t-r \end{cases}$$

implies that $\|\phi(t,t_0,x,u_0,d)-x\|_r \leq (t-t_0)(1+\lambda)\max_{t_0\leq s\leq t}\gamma(s)a(\mu(s)\tilde{a}\times(\|x\|_r))$, where $\gamma\in K^+$ and $a\in K_\infty$ are the functions involved in Hypothesis (S2), and $\mu\in K^+$ and $\tilde{a}\in K_\infty$ are the functions involved in (3.42). This observation, combined with Hypothesis (S5), implies that Hypothesis (REG3) holds as well.

Now, we are in a position to state the following:

Theorem 3.6 *Consider system* (1.10) *under Hypotheses* (S1–5). *Suppose that* (1.10) *is RGAOS. Then, there exist mappings* $V:\Re^+\times C^0([-r,0];\Re^n)\to\Re^+$, *which is almost Lipschitz on bounded sets,* $\beta\in K^+$, *and* $a_1,a_2\in K_\infty$ *such that the following inequalities hold for* $(t,x,d)\in\Re^+\times C^0([-r,0];\Re^n)\times D$:

$$a_1\big(\|H(t,x)\|_{\mathcal{Y}}\big)\leq V(t,x)\leq a_2\big(\beta(t)\|x\|_r\big) \tag{3.59}$$

$$V^0\big(t,x;f(t,x,0,d)\big)\leq -V(t,x) \tag{3.60}$$

where $V^0(t,x;f(t,x,0,d)) := \limsup_{h\to 0^+}\frac{V(t+h,E_h(x;f(t,x,0,d)))-V(t,x)}{h}$, *and* $E_h(x;v)$ *is defined in* (2.136) *for all* $v\in\Re^n$. *Moreover, if* (1.10) *is URGAOS, then* $\beta(t)\equiv 1$. *Finally, if* (1.10) *is* T*-periodic, then* $V:\Re^+\times C^0([-r,0];\Re^n)\to\Re^+$ *is* T*-periodic, and if, additionally,* (1.10) *is autonomous, then* $V:\Re^+\times C^0([-r,0];\Re^n)\to\Re^+$ *is independent of* $t\in\Re^+$.

Proof By Theorem 3.4 there exist mappings $V:\Re^+\times C^0([-r,0];\Re^n)\to\Re^+$, which is almost Lipschitz on bounded sets, $\beta\in K^+$, and $a_1,a_2\in K_\infty$ such that for all $(t,x,d)\in\Re^+\times C^0([-r,0];\Re^n)\times M_D$ and $\tau\geq t$, (3.59) holds and the following inequality holds:

$$V\big(\tau,\phi(\tau,t,x,u_0,d)\big)\leq \exp\big(-(\tau-t)\big)V(t,x) \tag{3.61}$$

The reader should notice that the continuity of $V:\Re^+\times C^0([-r,0];\Re^n)\to\Re^+$ results from the denseness of the set $\bigcup_{\lambda\geq 0}X_\lambda$ in $C^0([-r,0];\Re^n)$. The conclusion of the theorem is a direct consequence of the following fact:

Fact $\lim_{h\to 0^+}\frac{\phi(t+h,t,x,u_0,\tilde{d})-E_h(x;f(t,x,0,\tilde{d}(t)))}{h}=0$ *for every continuous* $\tilde{d}\in M_D$.

Indeed, by letting $\tilde{d}(t)\equiv d\in D$ it follows from the above fact and (3.61) that (3.60) holds for all $(t,x,d)\in\Re^+\times C^0([-r,0];\Re^n)\times D$.

The above fact is a direct consequence of Hypothesis (S3). It is clear that for every continuous $d\in M_D$, the composite map $f(\tau,x,0,d(\tau))$ is continuous on $(\Re^+\backslash A)\times C^0([-r,0];\Re^n)$. Applying repeatedly Theorem 2.1 (p. 43) in [9] on each one of the intervals contained in $[t,+\infty)\backslash A$, we conclude that the solution $T_r(\tau)x=\phi(\tau,t,x,u_0,d)$ of (1.10) satisfies $\dot{x}(\tau)=f(\tau,T_r(\tau)x,0,d(\tau))$ for all $\tau\in[t,+\infty)\backslash A$. Since the composite map $\tau\to f(\tau,x,0,d(\tau))$ is right-continuous on $\Re^+$, by virtue of the mean value theorem, it follows that $\lim_{h\to 0^+} \frac{x(t+h)-x(t)}{h} = f(t, x, 0, d(t))$ for all $\tau \in [t, +\infty)$.

The proof is complete. □

3.4.3 Discrete-Time Systems

We consider discrete-time systems of the form (1.110) under Hypotheses (L1–3). Particularly, we will further assume that (1.110) is RGAOS and that the following additional hypothesis holds.

(L4) For all bounded sets $S \subset \mathcal{X}$, $I \subset \pi$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup\{\|f(t, d, x, 0) - f(t, d, x_0, 0)\|_{\mathcal{X}}; d \in D\} < \varepsilon$ for all $t \in I$ and $x, x_0 \in S$ with $\|x - x_0\|_{\mathcal{X}} < \delta$.

The following fact is an immediate consequence of Hypothesis (L4) for system (1.110).

Fact *System* (1.110) *satisfies the property of* continuous dependence with respect to the initial conditions, *i.e., for every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\phi(t, t_0, x, u_0, d) - \phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} < \varepsilon$ for all $t, t_0 \in I$, $d \in M_D$, $x, x_0 \in S$ with $\|x - x_0\|_{\mathcal{X}} < \delta$, and $t \geq t_0$.*

It follows that Hypothesis (REG1) holds. The following result is a direct consequence of Theorem 3.4.

Proposition 3.1 *Consider system* (1.110) *under Hypotheses* (L1–4). *Suppose that* (1.110) *is RGAOS. Then there exist mappings $V : \Re^+ \times \mathcal{X} \to \Re^+$, $a_1, a_2 \in K_\infty$, and $\beta \in K^+$ such that*

$$a_1\big(\|H(t,x)\|_{\mathcal{Y}}\big) \leq V(t,x) \leq a_2\big(\beta(t)\|x\|_{\mathcal{X}}\big) \quad \forall (t,x) \in \Re^+ \times \mathcal{X} \tag{3.62}$$

$$V\big(\tau_{i+1}, f(\tau_i, d, x, 0)\big) \leq \exp(-1) V(\tau_i, x) \quad \forall (i, x, d) \in Z^+ \times \mathcal{X} \times D \tag{3.63}$$

Moreover, if system (1.110) *is URGAOS, then* (3.62) *holds with $\beta(t) \equiv 1$. Furthermore, for every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|V(t, y) - V(t, x)| < \varepsilon$ for all $t \in I$ and $x, y \in S$ with $\|x - y\|_{\mathcal{X}} < \delta$. Finally, if, additionally,* (1.110) *is T-periodic, then $V : \Re^+ \times \mathcal{X} \to \Re^+$ is T-periodic.*

3.5 Bibliographical and Historical Notes

1. There is a large literature on converse Lyapunov theorems for the case of systems described by ODEs (see, e.g., [1–4, 7, 8, 12, 15, 18, 21–24, 26–28] and references therein). The question of obtaining a smooth Lyapunov function is

important, in particular, for design of feedback control systems. However, for the purposes of the present work, we do not have to consider smooth Lyapunov functions. In fact, we believe that most of the results obtained by working with a smooth Lyapunov function can be also obtained by using a Lipschitz continuous Lyapunov function. Therefore, Theorem 3.5 is a novel result which does *not* assume the continuity of the right-hand sides of the differential equations or Lipschitz continuity with respect to the state variables.

2. Theorem 3.2 appeared in [14]. As remarked earlier, the use of Theorem 3.2 is restricted since no regularity properties are shown for the Lyapunov functional.
3. Converse Lyapunov theorems for discrete-time systems (with disturbances) have appeared in [11, 13, 17]. Again, the question of obtaining a smooth Lyapunov function is important.
4. Theorem 3.6 has appeared in [16] under the same hypotheses. The reader should notice that there is a large literature for converse Lyapunov theorems for the case of systems described by RFDEs (see [5, 6, 10, 19–21, 29] and references therein).
5. It should be emphasized that the result of Theorem 3.4 is novel and interesting since it is a converse Lyapunov theorem for a large class of systems. Many infinite-dimensional systems (e.g., described by partial differential equations) can be handled by Theorem 3.4: Hypothesis (REG3) is satisfied for appropriate (dense) subsets of the state space.

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Chapter 4
External Stability: Notions and Characterizations

4.1 Introduction

This chapter is devoted to the analysis of external global stability notions used in mathematical control and system theories. The presented stability notions are developed in the system-theoretic framework described in Chap. 1 so that one can obtain a wide perspective of the role of stability in various classes of deterministic systems. The results in this chapter are of both theoretic importance and practical relevance since almost all engineering and natural systems are subject to external input signals, which may take diverse forms as reference signals and actuator and sensor disturbances.

Another feature that distinguishes the "external" stability notions developed in this chapter from "internal" stability notions introduced in Chap. 2 is that the notions are *not uniform* with respect to the effect of external inputs. Therefore, the notions are not direct extensions of the corresponding stability notions for dynamical systems with no external inputs, often referred to as the disturbance-free case. More specifically, the notions presented in this chapter are variations of the notion of Input-to-State Stability (ISS), introduced by E.D. Sontag in his seminal work [22]. The ISS property has been proved to be very useful for the study of nonlinear systems, since it captures two main stability notions: Lyapunov stability (describing the behavior of zero-input response with respect to nonzero initial conditions) and bounded-input bounded-state stability (describing the behavior of zero-state response with respect to nonzero inputs); see [24] for further details.

A large part of this chapter is also devoted to the presentation of methods of proving external stability properties. In this chapter, we focus mainly on Lyapunov methods, although transformation methods and analytical solutions can be exploited as in Chap. 2. However, Lyapunov methods have been proved to be much more useful for the derivation of useful inequalities that show specific external stability properties. Small-Gain methods will be the subject of the following chapter.

In what follows, $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ will be a control system with the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Moreover, $u_0 \in M_U$ will be the identically zero input, i.e., $u_0(t) = 0 \in U$

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for all $t \geq 0$. For the output map $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$, we assume that either $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ is continuous or that there exists a partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\Re^+$ with diameter $r > 0$ such that $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ satisfies Hypothesis (L2) in Sect. 1.7 of Chap. 1.

4.2 Definitions

To begin with, consider a RGAOS system Σ. In other words, according to Theorem 2.1, there exist functions $\sigma \in KL$ and $\beta, \mu \in K^+$ such that the following estimate holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$:

$$\begin{aligned} &\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big) \big\|_{\mathcal{Y}} + \mu(t) \big\| \phi(t, t_0, x_0, u_0, d) \big\|_{\mathcal{X}} \\ &\quad \leq \sigma\big(\beta(t_0) \|x_0\|_{\mathcal{X}}, t - t_0\big) \end{aligned}$$

As control engineers, a natural question to ask is whether or not this RGAOS property is robust in the face of nonzero external input $u \in M_U$, i.e., $u(t) \not\equiv 0$. While "robustness" is mathematically interpreted as "structural stability," the essence of the question is to understand the impact of external inputs on the behavior of system Σ, or more precisely the output variables of system Σ.

The answer to the above question is critically important for many practical problems, but the answer to the question in our context of general complex systems is far from obvious. For example, there is this well-known phenomenon of "finite escape" (i.e., solutions blow up in finite time) when Σ is a highly nonlinear system. An elementary example of this kind is the scalar system $\dot{x} = -x + x^2 u$ with external input u. Clearly, the zero-input system $\dot{x} = -x$ is globally exponentially stable, and thus RGAOS. However, for any arbitrarily small constant input $u = u^* \neq 0$, some solutions blow up in finite time. Indeed, for any initial condition $x(0) = x_0$ such that $u^* x_0 > 1$ and for $t > 0$, the associated solution $x(t) = \frac{x_0}{e^t(1-u^* x_0)+u^* x_0}$ goes to $+\infty$ as $t \to \ln(\frac{u^* x_0}{u^* x_0 - 1}) < +\infty$.

Even in the absence of the finite escape phenomenon for every admissible external input $u \in M_U$, there is still no straightforward answer to the question we ask. This is because, even when the solutions are defined for each positive time, it is still possible that certain inputs produce unbounded output responses, i.e., $\limsup_{t \to +\infty} \|H(t, \phi(t, t_0, x_0, u, d), u(t))\|_{\mathcal{Y}} = +\infty$. An interesting question to ask is how to characterize the class of external inputs that do not produce unbounded output responses. An even stronger requirement is the characterization of the class of inputs for which the output response converges, i.e., $\lim_{t \to +\infty} \|H(t, \phi(t, t_0, x_0, u, d), u(t))\|_{\mathcal{Y}} = 0$.

It is reasonable to expect that the magnitude of the external input $\|u(t)\|_{\mathcal{U}}$ will play a significant role. However, it is worth pointing out that the smallness of the size of external inputs (except the trivial case where $\|u(t)\|_{\mathcal{U}} \equiv 0$) does not make the problem go away, as shown in the above-mentioned scalar example.

The notion of (Uniform) (Weighted) Input-to-Output Stability property allows us to study the effect of the magnitude of the external input to the output response. This is the major reason that this stability notion has proved to be very useful in Mathematical Control Theory. Of course, it should be emphasized that the (Uniform) (Weighted) Input-to-Output Stability property has a large number of theoretical and practical applications.

Next, we present the (Uniform) (Weighted) Input-to-Output Stability property.

Definition 4.1 Suppose that Σ is RFC from the input $u \in M_U$.

- If there exist functions $\sigma \in KL$, $\beta, \delta \in K^+$, and $\gamma \in \mathcal{N}$ such that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$m and $t \geq t_0$:

$$\begin{aligned}&\big\|H\big(t, \phi(t, t_0, x_0, u, d), u(t)\big)\big\|_{\mathcal{Y}}\\&\quad\leq \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big) + \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big)\end{aligned} \tag{4.1}$$

  then, we say that Σ satisfies the Weighted Input-to-Output Stability (WIOS) property from the input $u \in M_U$ with gain $\gamma \in \mathcal{N}$ and weight $\delta \in K^+$. Moreover, if $\beta(t) \equiv 1$, then we say that Σ satisfies the Uniform Weighted Input-to-Output Stability (UWIOS) property from the input $u \in M_U$ with gain $\gamma \in \mathcal{N}$ and weight $\delta \in K^+$.
- If there exist functions $\sigma \in KL$, $\beta \in K^+$, and $\gamma \in \mathcal{N}$ such that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, and $t \geq t_0$:

$$\begin{aligned}&\big\|H\big(t, \phi(t, t_0, x_0, u, d), u(t)\big)\big\|_{\mathcal{Y}}\\&\quad\leq \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big) + \sup_{t_0 \leq \tau \leq t} \gamma\big(\big\|u(\tau)\big\|_{\mathcal{U}}\big)\end{aligned} \tag{4.2}$$

  then, we say that Σ satisfies the Input-to-Output Stability (IOS) property from the input $u \in M_U$ with gain $\gamma \in \mathcal{N}$. Moreover, if $\beta(t) \equiv 1$, then we say that Σ satisfies the Uniform Input-to-Output Stability (UIOS) property from the input $u \in M_U$ with gain $\gamma \in \mathcal{N}$.
- Finally, for the special case of the identity output mapping, i.e., $H(t, x, u) := x$, the (Uniform) (Weighted) Input-to-Output Stability property from the input $u \in M_U$ is called (Uniform) (Weighted) Input-to-State Stability ((U)(W)ISS) property from the input $u \in M_U$.

Remark 4.1 Using the inequalities $\max\{a, b\} \leq a + b \leq \max\{a + \rho(a), b + \rho^{-1}(b)\}$ (which hold for all $\rho \in K_\infty$ and $a, b \geq 0$), it should be clear that the WIOS property for $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ can be defined by using an estimate based on "max,"

$$\begin{aligned}&\big\|H\big(t, \phi(t, t_0, x_0, u, d), u(t)\big)\big\|_{\mathcal{Y}}\\&\quad\leq \max\Big\{\sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big), \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big)\Big\}\end{aligned} \tag{4.3}$$

instead of (4.1). Notice that the functions σ and γ involved in (4.1) and (4.3) are not necessarily the same. Similarly, the IOS property for $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ can be defined by using an estimate of the form:

$$\begin{aligned} &\| H(t, \phi(t, t_0, x_0, u, d), u(t)) \|_{\mathcal{Y}} \\ &\quad \le \max\left\{ \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big), \sup_{t_0 \le \tau \le t} \gamma\big(\|u(\tau)\|_{\mathcal{U}}\big) \right\} \end{aligned} \tag{4.4}$$

instead of (4.2).

We call estimate (4.1) "a Sontag-like estimate," because E.D. Sontag invented the notion of ISS in [22] for finite-dimensional continuous-time systems, which is expressed using an estimate of the form

$$\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} \le \max\left\{ \sigma\big(\|x_0\|_{\mathcal{X}}, t - t_0\big), \sup_{t_0 \le \tau \le t} \gamma\big(\|u(\tau)\|_{\mathcal{U}}\big) \right\}$$

Moreover, Sontag and Wang formulated IOS in [27, 28] (see also [8]) for continuous-time finite-dimensional systems using an estimate of the form (4.1) with $\beta(t) \equiv \delta(t) \equiv 1$.

Exactly as in the case of internal stability properties, for external stability notions, we have found the following methods of proving the (U)(W)IOS property in the literature:

(1) Analytical Solutions
For this line of research, basic estimates for the solutions of the system are extracted by actually solving the differential (or difference) equations (or inequalities).

(2) Transformation Methods
In this case, basic estimates for the solutions of the system are derived by transforming the system into a different system with special properties.

(3) The method of Lyapunov functions and functionals
Basic estimates for the solutions are derived by means of a (or many) Lyapunov functional(s) and comparison lemmas.

(4) Small-Gain Methods
Basic estimates for the solutions of the system are derived by means of small-gain arguments.

(5) Qualitative Methods
Certain qualitative properties of the solutions guarantee that the (U)(W)IOS property holds. Working in this way, usually we cannot have an explicit expression for the gain functions or the weight functions.

All the above methods, with the exception of Small-Gain methods, will be explained in the present chapter. The Small-Gain methods will be explained in detail in the following chapter. Again it should be emphasized that the methods of proving external stability properties can be (and usually are) combined.

The following lemmas provide ε–δ characterizations of the WIOS and UWIOS properties.

Lemma 4.1 *Suppose that Σ is RFC from the input $u \in M_U$. Furthermore, suppose that there exist functions $V : \Re^+ \times \mathcal{X} \times U \to \Re^+$ with $V(t, 0, 0) = 0$ for all $t \ge 0$, $\gamma \in \mathcal{N}$, and $\delta \in K^+$ such that the following properties hold:*

P1. *For all* $s \geq 0$ *and* $T \geq 0$, *it holds that*

$$\sup\Big\{V\big(t, \phi(t, t_0, x_0, u, d), u(t)\big) - \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\big);$$
$$t \geq t_0, \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T], d \in M_D, u \in M_U\Big\} < +\infty.$$

P2. *For all* $\varepsilon > 0$ *and* $T \geq 0$, *there exists a* $\delta := \delta(\varepsilon, T) > 0$ *such that*

$$\sup\Big\{V\big(t, \phi(t, t_0, x_0, u, d), u(t)\big) - \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\big);$$
$$t \geq t_0, \|x_0\|_{\mathcal{X}} \leq \delta, t_0 \in [0, T], d \in M_D, u \in M_U\Big\} \leq \varepsilon.$$

P3. *For all* $\varepsilon > 0$, $T \geq 0$, *and* $R \geq 0$, *there exists* $\tau := \tau(\varepsilon, T, R) \geq 0$ *such that*

$$\sup\Big\{V\big(t, \phi(t, t_0, x_0, u, d), u(t)\big) - \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\big);$$
$$t \geq t_0 + \tau, \|x_0\|_{\mathcal{X}} \leq R, t_0 \in [0, T], d \in M_D, u \in M_U\Big\} \leq \varepsilon.$$

Then, there exist functions $\sigma \in KL$ *and* $\beta \in K^+$ *such that the following estimate holds for all* $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, *and* $t \geq t_0$:

$$V\big(t, \phi(t, t_0, x_0, u, d), u(t)\big)$$
$$\leq \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big) + \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\big) \tag{4.5}$$

Moreover, if there exists $a \in \mathcal{N}$ *such that* $\|H(t, x, u)\|_{\mathcal{Y}} \leq a(V(t, x, u))$ *for all* $(t, x, u) \in \Re^+ \times \mathcal{X} \times U$, *then for every* $\rho \in K_\infty$, Σ *satisfies the WIOS property from the input* $u \in M_U$ *with gain* $\tilde{\gamma} \in \mathcal{N}$ *and weight* $\delta \in K^+$, *where* $\tilde{\gamma}(s) := a(\gamma(s) + \rho(\gamma(s)))$.

Lemma 4.2 *Suppose that* Σ *is RFC from the input* $u \in M_U$. *Furthermore, suppose that there exist functions* $V : \Re^+ \times \mathcal{X} \times U \to \Re^+$ *with* $V(t, 0, 0) = 0$ *for all* $t \geq 0$, $\gamma \in \mathcal{N}$, *and* $\delta \in K^+$ *such that the following properties hold*:

P1. *For every* $s \geq 0$, *it holds that*

$$\sup\Big\{V\big(t, \phi(t, t_0, x_0, u, d), u(t)\big) - \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\big);$$
$$t \geq t_0, \|x_0\|_{\mathcal{X}} \leq s, t_0 \geq 0, d \in M_D, u \in M_U\Big\} < +\infty.$$

P2. *For every* $\varepsilon > 0$, *there exists* $\delta := \delta(\varepsilon) > 0$ *such that*

$$\sup\Big\{V\big(t, \phi(t, t_0, x_0, u, d), u(t)\big) - \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\big);$$
$$t \geq t_0, \|x_0\|_{\mathcal{X}} \leq \delta, t_0 \geq 0, d \in M_D, u \in M_U\Big\} \leq \varepsilon.$$

P3. *For all* $\varepsilon > 0$ *and* $R \geq 0$, *there exists* $\tau := \tau(\varepsilon, R) \geq 0$ *such that*

$$\sup\Big\{V\big(t, \phi(t, t_0, x_0, u, d), u(t)\big) - \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\big);$$
$$t \geq t_0 + \tau, \|x_0\|_{\mathcal{X}} \leq R, t_0 \geq 0, d \in M_D, u \in M_U\Big\} \leq \varepsilon.$$

Then, there exists a function $\sigma \in KL$ such that estimate (4.5) holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, and $t \geq t_0$ with $\beta(t) \equiv 1$. Moreover, if there exists $a \in \mathcal{N}$ such that $\|H(t,x,u)\|_{\mathcal{Y}} \leq a(V(t,x,u))$ for all $(t,x,u) \in \Re^+ \times \mathcal{X} \times U$, then for every $\rho \in K_\infty$, Σ satisfies the UWIOS property from the input $u \in M_U$ with gain $\tilde{\gamma} \in \mathcal{N}$ and weight $\delta \in K^+$, where $\tilde{\gamma}(s) := a(\gamma(s) + \rho(\gamma(s)))$.

Remark 4.2 Notice that Lemmas 4.1 and 4.2 can be very useful for the demonstration of the (U)WIOS property, because in practice we show Properties (P1–3) for some Lyapunov function V and not necessarily for the norm of the output map. Moreover, notice that V is not required to be continuous. If $V : \Re^+ \times \mathcal{X} \times U \to \Re^+$ is a continuous functional that maps bounded sets of $\Re^+ \times \mathcal{X} \times U$ into bounded sets of $\Re^+$, then Lemmas 4.1 and 4.2 guarantee that Σ satisfies the WIOS and the UWIOS properties with V as the output, respectively, from the input $u \in M_U$ with gain $\gamma \in \mathcal{N}$ and weight $\delta \in K^+$.

Proof of Lemma 4.1 Let $T,\ h \geq 0$, $s \geq 0$, and define

$$a(T,s) := \sup\Big\{ V\big(t, \phi(t,t_0,x_0,u,d), u(t)\big) - \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\big);\ \|x_0\|_{\mathcal{X}} \leq s, t \geq t_0 \in [0,T], d \in M_D, u \in M_U \Big\} \tag{4.6}$$

$$M(h,T,s) := \sup\Big\{ V\big(t_0+h, \phi(t_0+h,t_0,x_0,u,d), u(t)\big) - \sup_{t_0 \leq \tau \leq t_0+h} \gamma\big(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\big);\ \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0,T], d \in M_D, u \in M_U \Big\} \tag{4.7}$$

First, notice that, by virtue of Property P1, it holds that $a(T,s) < +\infty$ for all $T \geq 0$ and $s \geq 0$. Moreover, notice that since $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ and $V(t,0,0) = 0$ for all $t \geq 0$, we have $a(T,s) \geq 0$ for all $T \geq 0$, $s \geq 0$. Furthermore, notice that M is well defined, since by definitions (4.6), (4.7) the following inequality is satisfied for all $T, h \geq 0$ and $s \geq 0$:

$$0 \leq M(h,T,s) \leq a(T,s) \tag{4.8}$$

Clearly, definition (4.6) implies that, for each fixed $s \geq 0$, $a(\cdot,s)$ is nondecreasing and, for each fixed $T \geq 0$, $a(T,\cdot)$ is nondecreasing. Furthermore, Property P2 asserts that for every $T \geq 0$, $\lim_{s\to 0^+} a(T,s) = 0$. Hence, the inequality $a(T,0) \geq 0$ for all $T \geq 0$, in conjunction with $\lim_{s\to 0^+} a(T,s) = 0$ and the fact that $a(T,\cdot)$ is nondecreasing, implies $a(\cdot,0) = 0$. It turns out from Lemma 2.3 that there exist functions $\zeta \in K_\infty$ and $q \in K^+$ such that

$$a(T,s) \leq \zeta\big(q(T)s\big) \quad \forall (T,s) \in \big(\Re^+\big)^2 \tag{4.9}$$

Without loss of generality, we may assume that $q \in K^+$ is nondecreasing. Moreover, Property P3 guarantees that for all $\varepsilon > 0$, $T \geq 0$, and $R \geq 0$, there exists $\tau = \tau(\varepsilon,T,R) \geq 0$ such that

$$M(h,T,s) \leq \varepsilon \quad \text{for all } h \geq \tau(\varepsilon,T,R) \text{ and } 0 \leq s \leq R \tag{4.10}$$

Let

$$g(s) := \sqrt{s} + s^2 \tag{4.11}$$

and let p be a nondecreasing function of class K^+ with $p(0) = 1$ and

$$\lim_{t \to +\infty} p(t) = +\infty \tag{4.12}$$

Define

$$\mu(h) := \sup\left\{ \frac{M(h,T,s)}{p(T)g(\zeta(q(T)s))}; T \geq 0, s > 0 \right\} \tag{4.13}$$

Obviously, by virtue of (4.8), (4.9), and (4.11), the function $\mu : \Re^+ \to \Re^+$ is well defined and satisfies $\mu(\cdot) \leq 1$. We show that $\lim_{h\to+\infty} \mu(h) = 0$; equivalently, we establish that for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \geq 0$ such that

$$\mu(h) \leq \varepsilon \quad \text{for } h \geq \delta(\varepsilon) \tag{4.14}$$

Notice first that, for any given $\varepsilon > 0$, there exist constants $a := a(\varepsilon)$ and $b := b(\varepsilon)$ with $0 < a < b$ such that

$$x \notin (a,b) \quad \Rightarrow \quad \frac{x}{\sqrt{x} + x^2} \leq \varepsilon \tag{4.15}$$

We next recall (4.12), which asserts that, for the above ε for which (4.15) holds, there exists $c := c(\varepsilon) \geq 0$ such that $p(T) \geq \frac{1}{\varepsilon}$ for all $T \geq c$. By virtue of (4.8), (4.9), (4.11), and (4.15), this yields

$$\frac{M(h,T,s)}{p(T)g(\zeta(q(T)s))} \leq \varepsilon \quad \forall h \geq 0, \text{ when } T \geq c \text{ or } \zeta\big(q(T)s\big) \notin (a,b) \tag{4.16}$$

Hence, in order to establish (4.14), it remains to consider the case

$$a \leq \zeta\big(q(T)s\big) \leq b \quad \text{and} \quad 0 \leq T \leq c \tag{4.17}$$

Since, for each fixed $(h,s) \in (\Re^+)^2$, the mappings $M(h,\cdot,s)$, $M(h,s,\cdot)$, $q(\cdot)$, and $p(\cdot)$ are nondecreasing, we have

$$\frac{M(h,T,s)}{p(T)g(\zeta(q(T)s))} \leq \frac{M\big(h,c,\frac{\zeta^{-1}(b)}{q(0)}\big)}{g(a)} \tag{4.18}$$

where (4.17) was used. By using (4.10) and (4.18) with

$$\varepsilon := \varepsilon g(a) \qquad T := c \qquad R := \frac{\zeta^{-1}(b)}{q(0)}$$

it follows that

$$M\left(h, c, \frac{\zeta^{-1}(b)}{q(0)}\right) \leq \varepsilon g(a) \quad \text{for } h \geq \delta(\varepsilon) := \tau\left(\varepsilon g(a), c, \frac{\zeta^{-1}(b)}{q(0)}\right) \tag{4.19}$$

By taking into account (4.16), (4.17), (4.18), (4.19) and definition (4.13) of $\mu(\cdot)$ it follows that (4.14) holds with $\delta = \delta(\varepsilon)$ as selected in (4.19). Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{h\to+\infty} \mu(h) = 0$. Consequently, there exists a continuous strictly decreasing function $\bar{\mu} : \Re^+ \to (0, +\infty)$ such that $\bar{\mu}(h) \geq \mu(h)$ for all $h \geq 0$ and $\lim_{h\to+\infty} \bar{\mu}(h) = 0$. Thus, by recalling definition (4.13), we obtain

$$M(h, T, s) \leq \bar{\mu}(h)\theta(T, s) \quad \forall (T, s) \in \left(\Re^+\right)^2, \forall h \geq 0 \tag{4.20}$$

where $\theta(T, s) := p(T)g(\zeta(q(T)s))$. Clearly, θ satisfies all hypotheses of Lemma 2.3, and therefore there exist $\zeta_2 \in K_\infty$ and $\beta \in K^+$ such that

$$\theta(T, s) \leq \zeta_2\bigl(\beta(T)s\bigr) \quad \forall (T, s) \in \left(\Re^+\right)^2 \tag{4.21}$$

Thus, definition (4.7) implies that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, and $t \geq t_0$:

$$\begin{aligned} &V\bigl(t, \phi(t, t_0, x_0, u, d), u(t)\bigr) \\ &\quad \leq \bar{\mu}(t - t_0)\zeta_2\bigl(\beta(t_0)\|x_0\|_{\mathcal{X}}\bigr) + \sup_{t_0\leq\tau\leq t} \gamma\bigl(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\bigr) \end{aligned} \tag{4.22}$$

Estimate (4.22) implies (4.5) with $\sigma(s, t) := \bar{\mu}(t)\zeta_2(s)$. □

Proof of Lemma 4.2 As in the proof of Lemma 4.1, let $h \geq 0$, $s \geq 0$, and define

$$\begin{aligned} a(s) := \sup\Bigl\{ &V\bigl(t, \phi(t, t_0, x_0, u, d), u(t)\bigr) - \sup_{t_0\leq\tau\leq t} \gamma\bigl(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\bigr); \\ &\|x_0\|_{\mathcal{X}} \leq s, t \geq t_0 \geq 0, d \in M_D, u \in M_U \Bigr\} \end{aligned} \tag{4.23}$$

$$\begin{aligned} M(h, s) := \sup\Bigl\{ &V\bigl(t_0 + h, \phi(t_0 + h, t_0, x_0, u, d), u(t)\bigr) \\ &- \sup_{t_0\leq\tau\leq t_0+h} \gamma\bigl(\delta(\tau)\|u(\tau)\|_{\mathcal{U}}\bigr); \\ &\|x_0\|_{\mathcal{X}} \leq s, t_0 \geq 0, d \in M_D, u \in M_U \Bigr\} \end{aligned} \tag{4.24}$$

First notice that by Property P1 it holds that $a(s) < +\infty$ for all $s \geq 0$. Moreover, notice that since $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ and $V(t, 0, 0) = 0$ for all $t \geq 0$, we have $a(s) \geq 0$ for all $s \geq 0$. Furthermore, notice that M is well defined, since by definitions (4.23), (4.24) the following inequality is satisfied for all $h \geq 0$ and $s \geq 0$:

$$0 \leq M(h, s) \leq a(s) \tag{4.25}$$

Clearly, (4.23) implies that $a(\cdot)$ is nondecreasing. Furthermore, Property P2 asserts that $\lim_{s\to0^+} a(s) = 0$. Hence, the inequality $a(0) \geq 0$, in conjunction with $\lim_{s\to0^+} a(s) = 0$ and the fact that $a(\cdot)$ is nondecreasing, implies $a(0) = 0$. It turns out that a can be bounded from above by the K_∞ function $\tilde{a}$ defined by $\tilde{a}(s) := s + \frac{1}{s}\int_s^{2s} a(w)\,dw$ for $s > 0$ and $\tilde{a}(0) = 0$. Define

$$\mu(h) := \sup\left\{ \frac{M(h, s)}{g(\tilde{a}(s))}; s > 0 \right\} \tag{4.26}$$

where g is defined by (4.11). Working exactly as in the proof of Lemma 4.1, we can show that the function $\mu : \Re^+ \to \Re^+$ is well defined and satisfies

$\mu(\cdot) \le 1$ and $\lim_{h\to+\infty} \mu(h) = 0$. Consequently, there exists a continuous strictly decreasing function $\bar{\mu} : \Re^+ \to (0, +\infty)$ such that $\bar{\mu}(h) \ge \mu(h)$ for all $h \ge 0$ and $\lim_{h\to+\infty} \bar{\mu}(h) = 0$. Thus, by recalling definition (4.26) we obtain

$$M(h, s) \le \bar{\mu}(h) g\big(\tilde{a}(s)\big) \quad \forall h, s \ge 0 \tag{4.27}$$

Hence definition (4.24) implies that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, and $t \ge t_0$:

$$\begin{aligned} &V\big(t, \phi(t, t_0, x_0, u, d), u(t)\big) \\ &\quad \le \bar{\mu}(t - t_0) g\big(\tilde{a}\big(\|x_0\|_{\mathcal{X}}\big)\big) + \sup_{t_0 \le \tau \le t} \gamma\big(\delta(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big) \end{aligned} \tag{4.28}$$

Estimate (4.28) implies (4.5) with $\beta(t) \equiv 1$ and $\sigma(s, t) := \bar{\mu}(t) g(\tilde{a}(s))$. □

Next, some useful observations for *T-periodic* control systems are provided. It turns out that periodicity guarantees uniformity with respect to the initial time instants.

Lemma 4.3 *Suppose that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic. If Σ satisfies the WIOS property from the input $u \in M_U$, then Σ satisfies the UWIOS property from the input $u \in M_U$.*

Lemma 4.4 *Suppose that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic. If Σ satisfies the IOS property from the input $u \in M_U$, then Σ satisfies the UIOS property from the input $u \in M_U$.*

Proof of Lemmas 4.3 and 4.4 The proof is based on the following observation: if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic, then for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, we have $\phi(t, t_0, x_0, u, d) = \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d)$ and $H(t, \phi(t, t_0, x_0, u, d), u(t)) = H(t - kT, \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d), (P_{kT}u)(t - kT))$, with $k := [t_0/T]$ denoting the integer part of t_0/T and the inputs $P_{kT}u \in M_U$ and $P_{kT}d \in M_D$ as defined in Definition 1.2.

Since $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfies the WIOS property from the input $u \in M_U$, there exist functions $\sigma \in KL$, $\beta, \delta \in K^+$, and $\gamma \in \mathcal{N}$ such that (4.1) holds for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ and $t \ge t_0$. Consequently, it follows that the following estimate holds for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ and $t \ge t_0$:

$$\begin{aligned} &\big\|H\big(t, \phi(t, t_0, x_0, u, d), u(t)\big)\big\|_{\mathcal{Y}} \\ &\quad \le \sigma\big(\beta(t_0 - kT)\|x_0\|_{\mathcal{X}}, t - t_0\big) + \sup_{\tau \in [t_0 - kT, t - kT]} \gamma\big(\delta(\tau)\big\|(P_{kT}u)(\tau)\big\|_{\mathcal{U}}\big) \end{aligned}$$

Setting $\tau = s - kT$ and since $0 \le t_0 - [\frac{t_0}{T}]T < T$ for all $t_0 \ge 0$, we obtain

$$\begin{aligned} &\big\|H\big(t, \phi(t, t_0, x_0, u, d), u(t)\big)\big\|_{\mathcal{Y}} \\ &\quad \le \tilde{\sigma}\big(\|x_0\|_{\mathcal{X}}, t - t_0\big) + \sup_{s \in [t_0, t]} \gamma\big(\delta(s - kT)\big\|(P_{kT}u)(s - kT)\big\|_{\mathcal{U}}\big) \end{aligned} \tag{4.29}$$

where $\tilde{\sigma}(s,t) := \sigma(rs,t)$ and $r := \max\{\beta(t); 0 \le t \le T\}$. Estimate (4.29) and the identity $(P_{kT}u)(s-kT) = u(s)$ for all $s \ge 0$ implies that the following estimate holds for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ and $t \ge t_0$:

$$\begin{aligned} &\big\| H\big(t, \phi(t, t_0, x_0, u, d), u(t)\big)\big\|_{\mathcal{Y}} \\ &\quad \le \tilde{\sigma}\big(\|x_0\|_{\mathcal{X}}, t-t_0\big) + \sup_{s\in[t_0,t]} \gamma\big(\tilde{\delta}(s)\big\|u(s)\big\|_{\mathcal{U}}\big) \end{aligned} \tag{4.30}$$

where $\tilde{\delta}(t) := \max\{\delta(s); s \in [0,t]\}$.

When $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfies the IOS property from the input $u \in M_U$, all arguments above may be repeated with $\delta(t) \equiv 1$. Thus we conclude that (4.30) holds for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ and $t \ge t_0$ with $\tilde{\delta}(t) \equiv 1$. The proof is complete. □

Finally, we end this section by giving a simple example to show how analytical expressions for the solutions can be exploited for the IOS (or ISS) property.

Example 4.2.1 Consider the scalar nonlinear system

$$\dot{x} = -x\beta(x) + u \quad x \in \Re, u \in \Re \tag{4.31}$$

where $\beta : \Re \to \Re$ is a locally Lipschitz mapping that satisfies

$$\inf_{x\in\Re} \beta(x) > 0 \tag{4.32}$$

Although analytical expressions for the solution of (4.31) are not (in general) available, we know that the solution must satisfy the following integral equation (variations of constants) for all $t \ge t_0$ for which the solution exists:

$$x(t) = x(t_0)\exp\left(-\int_{t_0}^{t} \beta\big(x(\tau)\big)\,d\tau\right) + \int_{t_0}^{t} u(\tau)\exp\left(-\int_{\tau}^{t} \beta\big(x(s)\big)\,ds\right)d\tau \tag{4.33}$$

Define $L := \inf_{x\in\Re} \beta(x)$. Using (4.33) and (4.32), we obtain successively:

$$\begin{aligned} \big|x(t)\big| &\le \big|x(t_0)\big|\exp\left(-\int_{t_0}^{t} \beta\big(x(\tau)\big)\,d\tau\right) + \int_{t_0}^{t} \big|u(\tau)\big|\exp\left(-\int_{\tau}^{t} \beta\big(x(s)\big)\,ds\right)d\tau \\ &\le \big|x(t_0)\big|\exp\big(-L(t-t_0)\big) + \sup_{t_0\le\tau\le t}\big|u(\tau)\big|\int_{t_0}^{t} \exp\left(-\int_{\tau}^{t} \beta\big(x(s)\big)\,ds\right)d\tau \\ &\le \big|x(t_0)\big|\exp\big(-L(t-t_0)\big) + \sup_{t_0\le\tau\le t}\big|u(\tau)\big|\int_{t_0}^{t} \exp\big(-L(t-\tau)\big)\,d\tau \\ &= \big|x(t_0)\big|\exp\big(-L(t-t_0)\big) + \sup_{t_0\le\tau\le t}\big|u(\tau)\big|\frac{1-\exp(-L(t-t_0))}{L} \\ &\le \big|x(t_0)\big|\exp\big(-L(t-t_0)\big) + \frac{1}{L}\sup_{t_0\le\tau\le t}\big|u(\tau)\big| \end{aligned}$$

Consequently, it follows that system (4.31) satisfies the UISS property with gain function $\gamma(s) := L^{-1}s$.

4.3 Consequences of the WIOS Property

The consequences of the WIOS property are numerous and various. In this section we focus on certain consequences which will be used in the following sections. More specifically, for a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ which satisfies the WIOS property, we will show that:

1. Σ satisfies the 0-GAOS property,
2. Σ satisfies the Bounded-Weighted-Input-Bounded-Output (BWIBO) property and the Converging-Weighted-Input-Converging-Output (CWICO) property,
3. Σ satisfies the RFC property from the input $u \in M_U$,
4. there is a class of static maps for which the feedback interconnection with Σ produces a RGAOS system.

(a) The 0-GAOS property.

It is clear from Definition 4.1 that if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfies the WIOS property, then the system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M'_U, M_D, \phi, \pi, H)$ with $M'_U = \{u_0\}$ is RGAOS. Notice that $\Sigma' := (\mathcal{X}, \mathcal{Y}, M'_U, M_D, \phi, \pi, H)$ is the original system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ for which the only allowable input is the zero input u_0. Moreover, if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfies the UWIOS property, then the system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M'_U, M_D, \phi, \pi, H)$ with $M'_U = \{u_0\}$ is URGAOS. However, the following example shows that there are systems which satisfy the 0-GAOS property and do not satisfy the WIOS property.

Example 4.3.1 Consider the following scalar system:

$$\dot{x} = -x + x^2 u \qquad Y = x$$
$$x \in \Re, u \in \Re \tag{4.34}$$

It is clear that the 0-GAS property holds for system (4.34), since for $u \equiv 0$, we have $\dot{x} = -x$; the fact that $\dot{x} = -x$ is URGAS can be shown by using the analytical expression of the solution. On the other hand, the constant input $u \equiv 1$ gives the scalar system $\dot{x} = -x + x^2$ which is not RFC; indeed, using the analytical expression for the solution of $\dot{x} = -x + x^2$, it can be shown that for initial conditions greater than 1, the solution cannot be defined for all times. It should be noticed that local asymptotic stability for autonomous systems described by ODEs does imply a local version of the Input-to-State Stability property (see [17], pp. 217–219).

(b) Bounded-Weighted-Input-Bounded-Output (BWIBO) and Converging-Weighted-Input-Converging-Output (CWICO) Properties.

The BWIBO and CWICO properties are direct consequences of the following proposition.

Proposition 4.1 *Consider a system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ and suppose that Σ satisfies the WIOS property from the input $u \in M_U$ with gain $\gamma \in \mathcal{N}$ and weight $\delta \in K^+$. Then the output trajectory $t \to H(t, \phi(t, t_0, x_0, u, d), u(t))$ is bounded* *for all inputs* $u \in M_U$ *with* $\sup_{t\geq 0}\delta(t)\|u(t)\|_{\mathcal{U}} < +\infty$ (*Bounded Weighted Input Bounded Output property*) *and* $\lim_{t\to+\infty} H(t,\phi(t,t_0,x_0,u,d),u(t)) = 0$ *for all inputs* $u \in M_U$ *with* $\lim_{t\to+\infty}\delta(t)\|u(t)\|_{\mathcal{U}} = 0$ (*Converging Weighted Input Converging Output property*).

Proof The fact that the output function $t \to H(t,\phi(t,t_0,x_0,u,d),u(t))$ is bounded for all inputs $u \in M_U$ with $\sup_{t\geq 0}\delta(t)\|u(t)\|_{\mathcal{U}} < +\infty$ is an immediate consequence of estimate (4.1). Next, we show that $\lim_{t\to+\infty} H(t,\phi(t,t_0,x_0,u,d),u(t)) = 0$ for all inputs $u \in M_U$ with $\lim_{t\to+\infty}\delta(t)\|u(t)\|_{\mathcal{U}} = 0$ (Converging Weighted Input Converging Output property). Consider arbitrary $\varepsilon > 0$, $(t_0,x_0,d) \in \Re^+ \times \mathcal{X} \times M_D$, and input $u \in M_U$ with $\lim_{t\to+\infty}\delta(t)\|u(t)\|_{\mathcal{U}} = 0$. Clearly, there exists $T > t_0$ such that $\gamma(\delta(t)\|u(t)\|_{\mathcal{U}}) \leq \frac{\varepsilon}{2}$ for all $t \geq T$. By virtue of the weak semigroup property for Σ and since $(T+r,t_0,x_0,u,d) \in A_\phi$, where $r > 0$ is the constant involved in the weak semigroup property for Σ, there exists $\tilde{T} \in \pi(t_0,x_0,u,d) \cap [T,T+r] \neq \emptyset$ with $\phi(t,\tilde{T},\phi(\tilde{T},t_0,x_0,u,d),u,d) = \phi(t,t_0,x_0,u,d)$ for all $t \geq \tilde{T}$. Inequality (4.1), in conjunction with the fact that $\gamma(\delta(t)\|u(t)\|_{\mathcal{U}}) \leq \frac{\varepsilon}{2}$ for all $t \geq \tilde{T}$, implies, for all $t \geq \tilde{T}$,

$$\begin{aligned}
&\big\|H\big(t,\phi(t,t_0,x_0,u,d),u(t)\big)\big\|_{\mathcal{Y}} \\
&\quad= \big\|H\big(t,\phi\big(t,\tilde{T},\phi\big(\tilde{T},t_0,x_0,u,d\big),u,d\big),u(t)\big)\big\|_{\mathcal{Y}} \\
&\quad\leq \sigma\big(\beta\big(\tilde{T}\big)\big\|\phi\big(\tilde{T},t_0,x_0,u,d\big)\big\|_{\mathcal{X}},t-\tilde{T}\big) + \sup_{\tilde{T}\leq\tau\leq t}\gamma\big(\delta(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big) \\
&\quad\leq \sigma\big(\beta\big(\tilde{T}\big)\big\|\phi\big(\tilde{T},t_0,x_0,u,d\big)\big\|_{\mathcal{X}},t-\tilde{T}\big) + \frac{\varepsilon}{2}
\end{aligned}$$

The above implies that $\|H(t,\phi(t,t_0,x_0,u,d),u(t))\|_{\mathcal{Y}} \leq \varepsilon$ for sufficiently large $t \geq \tilde{T}$. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{t\to+\infty} H\big(t,\phi(t,t_0,x_0,u,d),u(t)\big) = 0. \qquad \square$$

Each of the BWIBO and CWICO properties fails to be sufficient for the WIOS property to hold. The following example illustrates this point.

Example 4.3.2 Consider the scalar system described by ODE

$$\begin{gathered}
\dot{x} = -ux \qquad Y = x \\
x \in \Re,\ u \in [0,+\infty) \subset \Re
\end{gathered} \tag{4.35}$$

The reader should notice that for every measurable and locally essentially bounded input $u : \Re^+ \to [0,+\infty)$, the solution of (4.35) satisfies $|x(t)| \leq |x(t_0)|$ for every $t \geq t_0$. Therefore, the Bounded-Input-Bounded-State property holds. However, system (4.35) does not satisfy the ISS property because the 0-GAS property does not hold. For $u \equiv 0$, we have $\dot{x} = 0$.

Next, consider the scalar system described by ODE

$$\begin{gathered}
\dot{x} = -x + dux \\
x \in \Re,\ u \in \Re,\ d \in [-1,1]
\end{gathered} \tag{4.36}$$

It is clear that the 0-GAS property holds for system (4.36), since for $u \equiv 0$, we have $\dot{x} = -x$; the fact that $\dot{x} = -x$ is URGAS can be shown by using the analytical expression of the solution. Moreover, the analytical expression of the solution of (4.36) $x(t) = x(t_0)\exp(-(t-t_0) + \int_{t_0}^{t} d(\tau)u(\tau)\,d\tau)$ shows that the Converging-Input-Converging-State property holds.

On the other hand, the constant inputs $u \equiv 2$, $d \equiv 1$ give the scalar system $\dot{x} = x$ for which the solutions are not bounded (unless $x(t_0) = 0$). Consequently, the Bounded-Input-Bounded-State property does not hold for system (4.36), and therefore system (4.36) does not satisfy the ISS property.

However, the reader should notice that the UWISS property holds for system (4.36). Indeed, for every $p > 0$, the expression for the solution of (4.36) gives

$$\begin{aligned}
|x(t)| &= |x(t_0)| \exp\left(-(t-t_0) + \int_{t_0}^{t} d(\tau)u(\tau)\,d\tau\right) \\
&\le |x(t_0)| \exp\left(-(t-t_0) + \sup_{t_0 \le \tau \le t}\left(\exp(p\tau)|u(\tau)|\right) \int_{t_0}^{t} \exp(-p\tau)\,d\tau\right) \\
&\le |x(t_0)| \exp\left(-(t-t_0) + p^{-1} \sup_{t_0 \le \tau \le t}\left(\exp(p\tau)|u(\tau)|\right)\right) \\
&\le |x(t_0)| \exp\left(-(t-t_0)\right) + |x(t_0)| \exp\left(-(t-t_0)\right) \\
&\quad \times \left[\exp\left(p^{-1} \sup_{t_0 \le \tau \le t}\left(\exp(p\tau)|u(\tau)|\right)\right) - 1\right]
\end{aligned}$$

Using the inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, we obtain

$$\begin{aligned}
|x(t)| &\le |x(t_0)| \exp\left(-(t-t_0)\right) + \frac{1}{2}|x(t_0)|^2 \exp\left(-2(t-t_0)\right) \\
&\quad + \frac{1}{2}\left[\exp\left(p^{-1} \sup_{t_0 \le \tau \le t}\left(\exp(p\tau)|u(\tau)|\right)\right) - 1\right]^2 \\
&\le |x(t_0)| \exp\left(-(t-t_0)\right) + \frac{1}{2}|x(t_0)|^2 \exp\left(-2(t-t_0)\right) \\
&\quad + \frac{1}{2} \sup_{t_0 \le \tau \le t}\left[\exp\left(p^{-1}\exp(p\tau)|u(\tau)|\right) - 1\right]^2
\end{aligned}$$

The above inequality shows that the UWISS property holds with gain $\gamma(s) := \frac{1}{2}[\exp(p^{-1}s) - 1]^2$ and weight $\delta(t) = \exp(pt)$.

(c) The RFC property from an input.

It is clear that the RFC property from the input $u \in M_U$ is a necessary condition of the WIOS property (a direct consequence of Definition 4.1. Here, we consider control systems which satisfy the additional hypothesis:

(G) U is a cone, i.e., for all $u \in U$ and $\lambda \ge 0$, it follows that $(\lambda u) \in U$. Furthermore, for all $u \in M_U$ and $\lambda \ge 0$, it follows that $(\lambda u) \in M_U$, where $(\lambda u)(t) = \lambda u(t)$ for all $t \ge 0$.

We next consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. If Hypothesis (G) holds, then we are ready to consider the control system $\Sigma' := (\mathcal{X}', \mathrm{Y}, M_{U'}, M_{D'}, \phi', \pi', H)$, where $\mathcal{X}' = \mathcal{X} \times \Re$, $U' = \{0\}$, $D' = D \times \{u \in U : \|u\|_{\mathcal{U}} \le 1\}$, $M_{D'} = \{(d, v) \in M_D \times M_U : \sup_{t \ge 0} \|v(t)\|_{\mathcal{U}} \le 1\}$, and the following equalities hold for all $(t_0, x_0, z_0, d, v) \in \Re^+ \times \mathcal{X} \times \Re \times M_D \times \{v \in M_U : \sup_{t \ge 0} \|v(t)\|_{\mathcal{U}} \le 1\}$ and all $t \in [t_0, t_{\max})$:

$$\phi'\big(t, t_0, x_0, z_0, u_0, (d, v)\big) := \big(\phi\big(t, t_0, x_0, |z_0|v, d\big), z_0\big) \in \mathcal{X} \times \Re \tag{4.37}$$

$$\pi'\big(t_0, x_0, z_0, u_0, (d, v)\big) := \pi\left(t_0, x_0, |z_0|v, d\right) \tag{4.38}$$

where $(|z_0|v)(t) = |z_0|v(t)$ for all $t \ge 0$. The reader can verify that the above definitions guarantee that $\Sigma' := (\mathcal{X}', \mathrm{Y}, M_{U'}, M_{D'}, \phi', \pi', H)$ is a control system with the BIC property and for which $0 \in \mathcal{X}'$ is a robust equilibrium point from the input $u \in M_{U'}$ (notice that $U' = \{0\}$). Moreover, if Σ is RFC from the input $u \in M_U$, then it follows that $\Sigma' := (\mathcal{X}', \mathrm{Y}, M_{U'}, M_{D'}, \phi', \pi', H)$ is RFC from the input $u \in M_{U'}$. Consequently, Lemma 2.7 implies that there exist functions $\mu, \beta \in K^+$ and $\sigma \in KL$ such that, for every $(t_0, x_0, z_0, d, v) \in \Re^+ \times \mathcal{X} \times \Re \times M_D \times \{v \in M_U : \sup_{t \ge 0} \|v(t)\|_{\mathcal{U}} \le 1\}$ and all $t \ge t_0$, we have

$$\mu(t)\big\|\phi'\big(t, t_0, x_0, z_0, u_0, (d, v)\big)\big\|_{\mathcal{X}'} \le \sigma\big(\beta(t_0)\big(\|x_0\|_{\mathcal{X}} + |z_0|\big), t - t_0\big) \tag{4.39}$$

The following observation is crucial: if Σ is RFC from the input $u \in M_U$, then for all $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$ and $t \ge t_0$, there exists $v \in M_U$ with $\sup_{t \ge 0} \|v(t)\|_{\mathcal{U}} \le 1$ such that $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, |z_0|v, d)$, where $z_0 = \sup_{t_0 \le \tau \le t} \|u(\tau)\|_{\mathcal{U}}$. Definition (4.37) and inequality (4.39) give, for all $t \ge t_0$,

$$\mu(t)\big\|\phi(t, t_0, x_0, u, d)\big\|_{\mathcal{X}} \le \sigma\Big(\beta(t_0)\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0 \le \tau \le t}\big\|u(\tau)\big\|_{\mathcal{U}}\Big), t - t_0\Big) \tag{4.40}$$

Using Lemma 3.2, Theorem 3.1, and inequality (4.40), we conclude that for every pair of functions $q, \delta \in K^+$, there exist functions $\sigma \in KL$, $p \in K_\infty$, and $c \in K^+$ such that, for all $t \ge t_0$,

$$\begin{aligned} & c(t)\big\|\phi(t, t_0, x_0, u, d)\big\|_{\mathcal{X}} \\ & \quad \le \sigma\big(q(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big) + \sup_{t_0 \le \tau \le t} p\big(\delta(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big) \end{aligned} \tag{4.41}$$

On the other hand, if inequality (4.41) holds for all $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$, then Σ is RFC from the input $u \in M_U$. Consequently, we obtain the following:

Proposition 4.2 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfying Hypothesis* (G) *with the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Σ is RFC from the input u if and only if for every pair of functions $q, \delta \in K^+$, there exist functions $c \in K^+$, $p \in K_\infty$, and $\sigma \in KL$ such that for every $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$, inequality* (4.41) *holds.*

It is clear that Proposition 4.2 shows that a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfying Hypothesis (G) with the BIC property and for which $0 \in \mathcal{X}$

is a robust equilibrium point from the input $u \in M_U$ is RFC from the input u if and only if there exists $c \in K^+$ such that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfies the (U)(W)IOS property from the input $u \in M_U$ for the output $H(t,x) := c(t)x$. Moreover, the following corollary holds:

Corollary 4.1 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfying Hypothesis* (G) *and the (U)WIOS property with weight $\delta \in K^+$. Then there exists a function $\mu \in K^+$ such that Σ with output $Y = \mu(t)\|x\|_{\mathcal{X}} + \|H(t,x,u)\|_{\mathcal{Y}}$ satisfies the (U)WIOS property with same weight $\delta \in K^+$, i.e., there exist functions $\tilde{\sigma} \in KL$ and $\tilde{\gamma} \in \mathcal{N}$ such that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, and $t \geq t_0$:*

$$
\begin{aligned}
&\left\| H\big(t, \phi(t,t_0,x_0,u,d), u(t)\big)\right\|_{\mathcal{Y}} + \mu(t)\left\|\phi(t,t_0,x_0,u,d)\right\|_{\mathcal{X}} \\
&\quad \leq \tilde{\sigma}\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t-t_0\big) + \sup_{t_0 \leq \tau \leq t} \tilde{\gamma}\big(\delta(\tau)\left\|u(\tau)\right\|_{\mathcal{U}}\big)
\end{aligned}
\tag{4.42}
$$

where $\beta \in K^+$ is the function involved in (4.1).

(d) A class of static maps for which the feedback interconnection with the system produces an RGAOS system.

Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ which satisfies the (U)WIOS property and the following hypothesis:

(G1) There exist functions $P, \mu \geq 0$ and $a \in K_\infty$ such that

$$
\mu(t)\|x\|_{\mathcal{X}} \leq a\big(\left\|H(t,x,u)\right\|_{\mathcal{Y}}\big) + P(t) \quad \forall (t,x,u) \in \Re^+ \times \mathcal{X} \times U.
$$

Without loss of generality, we may assume that the gain function $\gamma \in \mathcal{N}$ is of class K_∞. Next, consider an interconnected system comprised of $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ and a static map $\tilde{H} : \Re^+ \times \mathcal{X} \to U$ that satisfies

$$
\left\|\tilde{H}(t,x)\right\|_{\mathcal{U}} \leq \frac{1}{\delta(t)}\gamma^{-1}\big(\lambda\left\|H\big(t,x,\tilde{H}(t,x)\big)\right\|_{\mathcal{Y}}\big) \quad \forall (t,x) \in \Re^+ \times \mathcal{X} \tag{4.43}
$$

for some constant $\lambda \in (0,1)$. We denote by $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$, where $\tilde{U} = \{0\}$, the control system which results from the feedback interconnection of the system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the static map $\tilde{H} : \Re^+ \times \mathcal{X} \to U$. Notice that here it is implied that the static map $\tilde{H} : \Re^+ \times \mathcal{X} \to U$ satisfies (4.43) and additional requirements so that the feedback interconnection is well defined (in the sense explained in Chap. 1).

Using (4.1), (4.43), and the fact that $u(t) = \tilde{H}(t, x(t))$, we get the following estimate for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \in [t_0, t_{\max})$:

$$
\left\|Y(t)\right\|_{\mathcal{Y}} \leq \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t-t_0\big) + \lambda \sup_{t_0 \leq \tau \leq t}\left\|Y(\tau)\right\|_{\mathcal{Y}} \tag{4.44}
$$

where $x(t) := \tilde{\phi}(t, t_0, x_0, u_0, d)$, $Y(t) := H(t, x(t), \tilde{H}(t, x(t)))$, and $t_{\max}$ is the maximal existence time. Inequality (4.43) implies that the following inequality holds for all $t \in [t_0, t_{\max})$:

$$\sup_{t_0 \le \tau \le t} \|Y(\tau)\|_{\mathcal{Y}} \le \frac{1}{1-\lambda} \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, 0\big) \tag{4.45}$$

The BIC property, in conjunction with (4.45) and Hypothesis (G1), implies that $t_{\max} = +\infty$. Moreover, inequality (4.45) and Hypothesis (G1) imply that $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$ is RFC from the input $u \in M_{\tilde{U}}$ (notice that $\tilde{U} = \{0\}$). It follows from (4.45) that the following inequality holds for all $t \ge t_0$:

$$\|Y(t)\|_{\mathcal{Y}} \le \frac{1}{1-\lambda} \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, 0\big) \tag{4.46}$$

Inequality (4.46) shows that $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$ is Robustly Lagrange Output Stable and Robustly Lyapunov Output Stable. Next, we show that $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$ satisfies the Robust Output Attractivity Property. Let $\varepsilon > 0$, $T, R \ge 0$, and define, for $h \ge 0$,

$$a(h, T, R) := \sup\big\{\|Y(t_0 + h)\|_{\mathcal{Y}} : t_0 \in [0, T], \|x_0\|_{\mathcal{X}} \le R, d \in M_D\big\} \tag{4.47}$$

Inequality (4.46) implies that $a(h, T, R) \le \frac{1}{1-\lambda}\sigma(\max_{0 \le t_0 \le T} \beta(t_0)R, 0)$ for all $h \ge 0$. We define $l(T, R) := \limsup_{h \to +\infty} a(h, T, R)$. Clearly, there exists $\tau(\varepsilon, T, R) \ge 0$ such that $a(h, T, R) \le l(T, R) + \varepsilon$ for all $h \ge \tau(\varepsilon, T, R)$. By virtue of the weak semigroup property, for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \le R$, there exists $\xi \in \tilde{\pi}(t_0, x_0, d)$ such that $\xi \in [t_0 + \tau(\varepsilon, T, R), r + t_0 + \tau(\varepsilon, T, R)]$. It follows from (4.44) and the weak semigroup property that the following inequality holds for all $t \ge \xi$:

$$\|Y(t)\|_{\mathcal{Y}} \le \sigma\big(\beta(\xi)\|x(\xi)\|_{\mathcal{X}}, t - \xi\big) + \lambda \sup_{\xi \le \tau \le t} \|Y(\tau)\|_{\mathcal{Y}} \tag{4.48}$$

Using Hypothesis (G1), (4.48), and the fact that $a(h, T, R) \le l(T, R) + \varepsilon$ for all $h \ge \tau(\varepsilon, T, R)$, we obtain for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$, $\|x_0\|_{\mathcal{X}} \le R$, and $h \ge r + \tau(\varepsilon, T, R)$:

$$\begin{aligned}
&\|Y(t_0 + h)\|_{\mathcal{Y}} \le Q(\varepsilon, T, R, h) \quad \text{with} \\
&Q(\varepsilon, T, R, h) := \sigma\Bigg[\max_{\xi \in [0, r+T+\tau(\varepsilon, T, R)]} \frac{\beta(\xi)}{\mu(\xi)}\Bigg(P(\xi) \\
&\qquad + a\bigg(\frac{1}{1-\lambda}\sigma\Big(\max_{0 \le t_0 \le T} \beta(t_0)R, 0\Big)\bigg)\Bigg), h - r - \tau(\varepsilon, T, R)\Bigg] \\
&\qquad + \lambda l(T, R) + \lambda\varepsilon
\end{aligned} \tag{4.49}$$

It follows from definition (4.47) and inequality (4.49) that $a(h, T, R) \le Q(\varepsilon, T, R, h)$ for all $h \ge r + \tau(\varepsilon, T, R)$. Therefore we must have $l(T, R) = \limsup_{h \to +\infty} a(h, T, R) \le \limsup_{h \to +\infty} Q(\varepsilon, T, R, h) = \lambda l(T, R) + \lambda\varepsilon$, which implies $l(T, R) \le \frac{\lambda}{1-\lambda}\varepsilon$. This implies that $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$ satisfies the Robust Output Attractivity Property. Therefore, we have established the following:

Proposition 4.3 *Consider a control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *satisfying Hypothesis* (G1) *and the WIOS property from the input* $u \in M_U$ *with weight* $\delta \in K^+$ *and gain* $\gamma \in K_\infty$. *Consider* $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$, *where* $\tilde{U} = \{0\}$, *the control system which results from the feedback interconnection of the system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *with a static map* $\tilde{H} : \Re^+ \times \mathcal{X} \to U$ *that satisfies* (4.43) *for certain constant* $\lambda \in (0, 1)$. *Then* $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \pi, H)$ *is RGAOS.*

Next, assume that $\beta \in K^+$ is bounded, i.e., the system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfies the UWIOS property. Without loss of generality we may assume that $\beta(t) \equiv 1$. Moreover, assume that the following hypothesis holds:

(G2) There exists a constant $P \geq 0$ and a function $a \in K_\infty$ such that the inequality $\|x\|_{\mathcal{X}} \leq a(\|H(t, x, u)\|_{\mathcal{Y}}) + P$ holds for all $(t, x, u) \in \Re^+ \times \mathcal{X} \times U$.

Then, the previous analysis in conjunction with (G2) implies that, for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$, the following inequality holds:

$$\|x(t)\|_{\mathcal{X}} \leq a\left(\frac{1}{1-\lambda}\sigma\left(\|x_0\|_{\mathcal{X}}, 0\right)\right) + P \tag{4.50}$$

Inequality (4.46) with $\beta(t) \equiv 1$ shows that $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$ is Uniformly Robustly Lagrange Output Stable and Uniformly Robustly Lyapunov Output Stable.

Next, we show that $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$ satisfies the Uniform Robust Output Attractivity Property. Let $\varepsilon > 0$, $R \geq 0$, and define, for $h \geq 0$,

$$a(h, R) := \sup\left\{\|Y(t_0 + h)\|_{\mathcal{Y}} : t_0 \geq 0, \|x_0\|_{\mathcal{X}} \leq R, d \in M_D\right\} \tag{4.51}$$

Inequality (4.46) with $\beta(t) \equiv 1$ implies that $a(h, R) \leq \frac{1}{1-\lambda}\sigma(R, 0)$ for all $h \geq 0$. We define $l(R) := \limsup_{h \to +\infty} a(h, R)$. Clearly, there exists $\tau(\varepsilon, R) \geq 0$ such that $a(h, R) \leq l(R) + \varepsilon$ for all $h \geq \tau(\varepsilon, R)$. By virtue of the weak semigroup property, for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $\|x_0\|_{\mathcal{X}} \leq R$, there exists $\xi \in \tilde{\pi}(t_0, x_0, d)$ such that $\xi \in [t_0 + \tau(\varepsilon, R), r + t_0 + \tau(\varepsilon, R)]$. It follows from (4.44) and the weak semigroup property that the following inequality holds for all $t \geq \xi$:

$$\|Y(t)\|_{\mathcal{Y}} \leq \sigma\left(\|x(\xi)\|_{\mathcal{X}}, t - \xi\right) + \lambda \sup_{\xi \leq \tau \leq t} \|Y(\tau)\|_{\mathcal{Y}} \tag{4.52}$$

Using (4.50), (4.52), and the fact that $a(h, R) \leq l(R) + \varepsilon$ for all $h \geq \tau(\varepsilon, R)$, we obtain, for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$, $\|x_0\|_{\mathcal{X}} \leq R$, and $h \geq r + \tau(\varepsilon, T, R)$,

$$\begin{aligned} &\|Y(t_0 + h)\|_{\mathcal{Y}} \leq Q(\varepsilon, R, h), \\ &\quad Q(\varepsilon, R, h) := \tilde{\sigma}\left(a\left(\frac{1}{1-\lambda}\sigma\left(\|x_0\|_{\mathcal{X}}, 0\right)\right) + P, h - r - \tau(\varepsilon, R)\right) + \lambda l(R) + \lambda\varepsilon \end{aligned} \tag{4.53}$$

It follows from (4.51) and (4.53) that $a(h, R) \leq Q(\varepsilon, R, h)$ for all $h \geq r + \tau(\varepsilon, R)$. Therefore we must have $l(R) = \limsup_{h \to +\infty} a(h, R) \leq \limsup_{h \to +\infty} Q(\varepsilon, R, h) =$

$\lambda l(R) + \lambda\varepsilon$, which implies $l(R) \leq \frac{\lambda}{1-\lambda}\varepsilon$. Thus $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$ satisfies the Uniform Robust Output Attractivity Property. Therefore, we have shown the following:

Proposition 4.4 *Consider a control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *satisfying Hypothesis* (G2) *and the UWIOS property from the input* $u \in M_U$ *with weight* $\delta \in K^+$ *and gain* $\gamma \in K_\infty$. *Consider* $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \tilde{\pi}, H)$, *where* $\tilde{U} = \{0\}$, *the control system which results from the feedback interconnection of the system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *with a static map* $\tilde{H} : \Re^+ \times \mathcal{X} \to U$ *that satisfies* (4.43) *for certain constant* $\lambda \in (0, 1)$. *Then* $\tilde{\Sigma} := (\mathcal{X}, \mathcal{Y}, M_{\tilde{U}}, M_D, \tilde{\phi}, \pi, H)$ *is URGAOS.*

Propositions 4.3 and 4.4 are important for two reasons:

(a) they allow us to obtain Lyapunov characterizations of external stability properties, and
(b) they allow us to prove (U)RGAOS by invoking the (U)WIOS property.

The following example illustrates how we can use (U)WIOS in order to prove RGAOS.

Example 4.3.3 Consider the scalar system described by (4.36). In Example 4.3.2 we showed that system (4.36) satisfies the UWISS property with gain $\gamma(s) := \frac{1}{2}[\exp(p^{-1}s) - 1]^2$ and weight $\delta(t) = \exp(pt)$ for every $p > 0$. Next, consider the system

$$\begin{aligned} \dot{x} &= -x + d\frac{p}{2}x \exp(-pt) \ln\bigl(1 + 2\lambda|x|\bigr) \\ x &\in \Re, d \in [-1, 1] \end{aligned} \tag{4.54}$$

where $p > 0$ and $\lambda \in (0, 1)$ are constants. The above system can be considered as the feedback interconnection of system (4.36) with the static map

$$u = \frac{p}{2}\exp(-pt) \ln\bigl(1 + 2\lambda|x|\bigr) \tag{4.55}$$

Notice that inequality (4.43) holds. Moreover, Hypothesis (G2) holds. Therefore, we can conclude that system (4.54) is URGAS.

4.4 External Stability Properties for Discrete-Time Systems

In this section we will restrict our attention to discrete-time systems for which the output map is independent of the values of the input, i.e., $H(t, x, u) = H(t, x)$. Furthermore, we consider external stability properties for discrete-time systems (1.110) under Hypotheses (L1–3) for which the partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ is the set of nonnegative integers Z^+. This is a simplifying assumption, which allows an easier derivation of the results of this section. Consequently, a discrete-time system with $\pi = Z^+$ will be written as follows:

$$\begin{aligned} x(t+1) &= f\big(t, d(t), x(t), u(t)\big) \\ Y(t) &= H\big(t, x(t)\big) \end{aligned} \tag{4.56}$$

where $(t, x, d, u) \in Z^+ \times \mathcal{X} \times D \times U$. We denote by $x(t)$ the solution of (4.56) with initial condition $x(t_0) = x_0$ corresponding to inputs $(d, u) \in M_D \times M_U$.

The following proposition provides various characterizations of the WIOS property for time-varying system (4.56), either under Hypotheses (L1–3) or under the pair of Hypothesis (L1–3) and the following hypothesis:

(L5) For every pair of bounded sets $S \subset \mathcal{X} \times U$, $I \subset Z^+$ and for every $\varepsilon > 0$, the set $f(I \times D \times S)$ is bounded, and there exists $\delta > 0$ such that $\sup\{\|f(t,d,x,u) - f(t,d,y,v)\|_{\mathcal{X}}; d \in D\} < \varepsilon$ for all $t \in I$ and $(x,u), (y,v) \in S$ with $\|x - y\|_{\mathcal{X}} + \|u - v\|_{\mathcal{U}} < \delta$.

Remark about Hypothesis (L5) Hypothesis (L5) is "stronger" than Hypothesis (L4), in the sense that the implication (L5) $\Rightarrow$ (L4) holds. Hypothesis (L5) also implies Hypothesis (L1). The proof of the implication (L5) $\Rightarrow$ (L1) is made by defining the following function:

$$a(T,s) := \sup\big\{\big\|f(t,d,x,u)\big\|_{\mathcal{X}}; t \in Z^+, t \le T, \|x\|_{\mathcal{X}} \le s, d \in D, \|u\|_{\mathcal{U}} \le s\big\}$$

which is well defined for all $T, s \ge 0$. Moreover, for all $T, s \ge 0$, the functions $a(\cdot, s)$ and $a(T, \cdot)$ are nondecreasing, and since $f(t, d, 0, 0) = 0 \in \mathcal{X}$ for all $(t, d) \in Z^+ \times D$, we also obtain $a(T, 0) = 0$ for all $T \ge 0$. Finally, let $\varepsilon > 0$ and $T \ge 0$. It can be shown that Hypothesis (L5) guarantees the existence of $\delta := \delta(\varepsilon, T) > 0$ such that $a(T, \delta(\varepsilon, T)) < \varepsilon$, and consequently we have $\lim_{s \to 0^+} a(T, s) = 0$ for all $T \ge 0$. It turns out from Lemma 2.3 that there exist functions $\zeta \in K_\infty$ and $\beta \in K^+$ such that $a(T, s) \le \zeta(\beta(T)s)$ for all $T, s \ge 0$. Consequently, we obtain $\|f(t,d,x,u)\|_{\mathcal{X}} \le \zeta(\beta(t)\max\{\|x\|_{\mathcal{X}}, \|u\|_{\mathcal{U}}\})$ for all $(t, x, d, u) \in Z^+ \times \mathcal{X} \times D \times U$, which directly implies Hypothesis (L1).

Let $\mathcal{X}, \mathcal{Y}$ be a pair of normed linear spaces. We denote by $CU(Z^+ \times A; W)$, where $A \subseteq \mathcal{X}$, the set of all continuous mappings $H : Z^+ \times A \to W \subseteq \mathcal{Y}$ with the following property: "for every pair of bounded sets $I \subset Z^+$, $S \subseteq A$ and for every $\varepsilon > 0$, the set $H(I \times S)$ is bounded, and there exists $\delta > 0$ such that $\|H(t,x) - H(t,x_0)\|_{\mathcal{Y}} < \varepsilon$ for all $t \in I$ and $x, x_0 \in S$ with $\|x - x_0\|_{\mathcal{X}} < \delta$." For example, if Hypotheses (L1–4) hold for system (4.56) and system (4.56) is RGAOS, then Proposition 3.1 implies that $V \in CU(Z^+ \times \mathcal{X}; \Re^+)$.

Proposition 4.5 *Consider system* (4.56) *under Hypotheses* (L1–3) *and* (G). *Then the following statements are equivalent*:

(i) *There exist functions $\sigma \in KL$, $\beta, \delta \in K^+$, and $\rho \in K_\infty$ such that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in Z^+ \times \mathcal{X} \times M_D$, and $t \ge t_0$*:

$$\left\| H\left(t, x(t)\right) \right\|_{\mathcal{Y}} \leq \max\left\{ \sigma\left(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\right), \sup_{\tau \in [t_0, t]} \sigma\left(\rho\left(\delta(\tau)\left\|u(\tau)\right\|_{\mathcal{U}}\right), t - \tau\right) \right\}. \quad (4.57)$$

(ii) *System* (4.56) *satisfies the WIOS property.*

(iii) *There exist functions* $\theta \in K_\infty$ *and* $p \in K^+$ *such that the following system is RGAOS*:

$$x(t+1) = f\left(t, d(t), x(t), p(t)\theta\left(\left\|x(t)\right\|_{\mathcal{X}}\right)d'(t)\right)$$
$$Y(t) = H\left(t, x(t)\right)$$
$$x(t) \in \mathcal{X}, Y(t) \in \mathcal{Y}, \left(d(t), d'(t)\right) \in D \times B_U[0, 1], t \in Z^+ \quad (4.58)$$

where $B_U[0,1] := \{u \in U; \|u\|_{\mathcal{U}} \leq 1\}$.

(iv) *There exist functions* $V : Z^+ \times \mathcal{X} \to \Re^+$, $a_1, a_2, a_3 \in K_\infty$, $\beta, \phi, \mu \in K^+$ *and a constant* $\lambda \in (0, 1)$ *such that, for all* $(t, x, d, u) \in Z^+ \times \mathcal{X} \times D \times U$,

$$a_1\left(\left\|H(t, x)\right\|_{\mathcal{Y}} + \mu(t)\|x\|_{\mathcal{X}}\right) \leq V(t, x) \leq a_2\left(\beta(t)\|x\|_{\mathcal{X}}\right) \quad (4.59)$$
$$V\left(t+1, f(t, d, x, u)\right) \leq \lambda V(t, x) + a_3\left(\phi(t)\|u\|_{\mathcal{U}}\right). \quad (4.60)$$

Moreover, if Hypothesis (L5) *holds, then* $V \in CU(Z^+ \times \mathcal{X}; \Re^+)$.

Proof (i) $\Rightarrow$ (ii) By setting $\gamma(s) := \sigma(\rho(s), 0)$, the desired (4.1) is a consequence of (4.57) and the previous inequality.

(ii) $\Rightarrow$ (iii) By Corollary 4.1, there exist functions $\tilde{\sigma} \in KL$, $\beta, \delta \in K^+$, and $\tilde{\gamma} \in \mathcal{N}$ such that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, and $t \geq t_0$:

$$\left\|Y(t)\right\|_{\mathcal{Y}} + \mu(t)\left\|x(t)\right\|_{\mathcal{X}} \leq \tilde{\sigma}\left(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\right) + \sup_{t_0 \leq \tau \leq t} \tilde{\gamma}\left(\delta(\tau)\left\|u(\tau)\right\|_{\mathcal{U}}\right) \quad (4.61)$$

By Lemma 3.2 there exists $a \in K_\infty$ such that $\tilde{\gamma}(rs) \leq a(r)a(s)$ for all $r, s \geq 0$. Let $\lambda \in (0, 1)$ and define

$$\theta(s) := a^{-1}(s) \quad \text{and} \quad p(t) := \frac{a^{-1}(\lambda\mu(t))}{\delta(t)} \quad (4.62)$$

Notice that (4.62) guarantees that $p(t)\theta(\|x\|_{\mathcal{X}}) \leq \frac{1}{\delta(t)}\tilde{\gamma}^{-1}(\lambda\mu(t)\|x\|_{\mathcal{X}})$. Moreover, system (4.58) is the interconnection of the static map $v(t) = p(t)\theta(\|x(t)\|_{\mathcal{X}})$ with the dynamic system

$$x(t+1) = f\left(t, d(t), x(t), v(t)d'(t)\right)$$
$$\tilde{Y}(t) = \left(H\left(t, x(t)\right), \mu(t)x\right)$$

with $x(t) \in \mathcal{X}, \tilde{Y}(t) \in \tilde{\mathcal{Y}} := \mathcal{Y} \times \mathcal{X}, v(t) \in \Re^+, (d(t), d'(t)) \in D \times B_U[0, 1]$, $t \in Z^+$. Estimate (4.61) guarantees that the above system satisfies the WIOS property from the input $v \in M_{\Re^+}$ with gain $\tilde{\gamma} \in \mathcal{N}$ and weight $\delta \in K^+$. Finally, notice that (4.43) and Hypothesis (G1) hold for the output $\tilde{Y}(t) = (H(t, x(t)), \mu(t)x)$. Proposition 4.3 implies that system (4.58) is RGAOS.

(iii) ⇒ (iv) Notice that since (4.56) satisfies Hypotheses (L1–3) (or (L1–3) and (L5)), it follows that system (4.58) satisfies Hypotheses (L1–3) (or (L1–4)). Let $\lambda \in (0,1)$ and define $c := -\ln(\lambda)$. Theorems 2.1 and 3.1 imply the existence of functions a_1, a_2 of class K_∞ and β, μ of class K^+ such that, for all $t \geq t_0$,

$$a_1\big(\|Y(t)\|_{\mathcal{Y}} + \mu(t)\|x(t)\|_{\mathcal{X}}\big) \leq \exp\big(-2c(t-t_0)\big)a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) \quad (4.63)$$

where $x(t)$ denotes the solution of (4.58) with initial condition $x(t_0) = x_0$ corresponding to inputs $(d, d') \in M_D \times M_{B_U[0,1]}$. In case where (4.56) satisfies Hypotheses (L1–3), we define

$$\begin{aligned} V(t_0, x_0) := \sup\big\{&\exp\big(c(t-t_0)\big)a_1\big(\|Y(t)\|_{\mathcal{Y}} + \mu(t)\|x(t)\|_{\mathcal{X}}\big) : \\ &t \geq t_0, (d, d') \in M_D \times M_{B_U[0,1]}\big\} \end{aligned} \quad (4.64)$$

Definition (4.64), in conjunction with estimate (4.63), guarantees that inequality (4.59) holds. Moreover, exploiting the semigroup property, we obtain

$$\begin{aligned} &V\big(t+1, f(t,d,x,u)\big) \leq \lambda V(t,x) \\ &\quad \forall (t,x,d,u) \in Z^+ \times \mathcal{X} \times D \times U \text{ with } \|u\|_{\mathcal{U}} \leq p(t)\theta\big(\|x\|_{\mathcal{X}}\big) \end{aligned} \quad (4.65)$$

In case where (4.56) satisfies Hypothesis (L5), Proposition 3.1 guarantees the existence of $V \in CU(Z^+ \times \mathcal{X}; \Re^+)$ satisfying (4.59) and (4.65). Define, for all $(t,x,u) \in Z^+ \times \mathcal{X} \times U$,

$$\psi(t,x,u) := \sup\big\{V\big(t+1, f(t,d,x,u)\big); d \in D\big\} \quad (4.66)$$

Clearly, Hypothesis (L1) and inequality (4.59), in conjunction with Lemma 2.3, imply the existence of functions $\omega \in K_\infty$ and $q \in K^+$ such that $\psi(t,x,u) \leq \omega(q(t)\|x\|_{\mathcal{X}}) + \omega(q(t)\|u\|_{\mathcal{U}})$. Moreover, Lemma 2.3 guarantees the existence of functions $a_3 \in K_\infty$ and $\phi \in K^+$ such that $\omega(q(t)\theta^{-1}(\frac{s}{p(t)})) + \omega(q(t)s) \leq a_3(\phi(t)s)$ for all $t, s \geq 0$. Combining the previous inequalities and definition (4.66), we obtain

$$\begin{aligned} &\sup\left\{V\big(t+1, f(t,d,x,u)\big); d \in D, \|u\|_{\mathcal{U}} \leq s, \|x\|_{\mathcal{X}} \leq \theta^{-1}\left(\frac{s}{p(t)}\right)\right\} \\ &\quad \leq a_3\big(\phi(t)s\big) \quad \text{for all } t, s \geq 0 \end{aligned} \quad (4.67)$$

We next establish inequality (4.60), with a_3 as previously, by considering the following two cases:

- $\|u\|_{\mathcal{U}} \leq p(t)\theta(\|x\|_{\mathcal{X}})$. In this case, inequality (4.60) is a direct consequence of (4.65).
- $\|u\|_{\mathcal{U}} \geq p(t)\theta(\|x\|_{\mathcal{X}})$. In this case, inequality (4.60) is a direct consequence of (4.67).

(iv) ⇒ (i) Consider the trajectory $x(t)$ of (4.56) that corresponds to input $(d,u) \in M_D \times M_U$ with initial condition $x(t_0) = x_0 \in \mathcal{X}$ and define $c := -\log(\lambda) > 0$, $V(t) = V(t, x(t))$, and $b(t) := \exp(2ct)a_3(\phi(t)\|u(t)\|_{\mathcal{U}})$ for all $t \geq t_0$. Inequality (4.60) implies that $V(t+1) \leq \exp(-c)V(t) + \exp(-2ct)b(t)$ for all $t \geq t_0$, which, by an induction argument, yields

$$V(t) \le \exp\bigl(-c(t-t_0)\bigr)V(t_0) + \frac{\exp(2c)}{\exp(c)-1}\exp\bigl(-c(t-t_0)\bigr) \times \sup_{t_0 \le \tau \le t}\bigl(\exp(2c\tau)a_3\bigl(\phi(\tau)\|u(\tau)\|_{\mathcal{U}}\bigr)\bigr) \quad \forall t \ge t_0 \tag{4.68}$$

By Lemma 2.3, there exist functions $\rho \in K_\infty$ and $\delta \in K^+$ such that

$$a_2^{-1}\left(\frac{\exp(2c)}{\exp(c)-1}\exp(2ct)a_3\bigl(\phi(t)s\bigr)\right) \le \rho\bigl(\delta(t)s\bigr),$$

where $a_2 \in K_\infty$ is the function involved in (4.59). The previous inequality, together with (4.59), (4.68), and the definition $\sigma(s,t) := 2a_1^{-1}(2\exp(-ct)a_2(s)) \in KL$, implies (4.57). The proof is complete. □

The following proposition provides a sharper characterization of the IOS property for the time-varying case (4.56), which holds only for discrete-time systems with continuous dynamics. For continuous-time systems, the situation is more involved since the finite escape time phenomenon can occur. Further research is required for the case of discrete-time systems with discontinuous dynamics.

Proposition 4.6 *System* (4.56) *under Hypotheses* (L1–3), (L5), *and* (G) *satisfies the WIOS property if and only if the "unforced" system*

$$\begin{aligned} x(t+1) &= f\bigl(t, d(t), x(t), 0\bigr) \\ Y(t) &= H\bigl(t, x(t)\bigr) \end{aligned} \tag{4.69}$$

is RGAOS.

Proof Clearly, if system (4.56) satisfies the WIOS property, then the "unforced" system (4.69) is RGAOS. Therefore it suffices to prove the converse statement.

Notice that by Hypothesis (L5) the "unforced" system (4.69) satisfies Hypothesis (L4). Since the "unforced" system (4.69) is RGAOS, it follows by Proposition 3.1 that there exist functions $V \in CU(Z^+ \times \mathcal{X}; \Re^+)$, a_1, a_2 of class K_∞, and $\tilde{\beta}$ of class K^+ and constant $c > 0$ such that

$$a_1\bigl(\|H(t,x)\|_{\mathcal{Y}}\bigr) \le V(t,x) \le a_2\bigl(\tilde{\beta}(t)\|x\|_{\mathcal{X}}\bigr) \quad \forall (t,x) \in Z^+ \times \mathcal{X} \tag{4.70}$$

$$V\bigl(t+1, f(t,d,x,0)\bigr) \le \exp(-c)V(t,x) \quad \forall (t,x,d) \in Z^+ \times \mathcal{X} \times D \tag{4.71}$$

Define the function

$$\gamma(r,s) := \sup\bigl\{\bigl|V\bigl(t+1, f(t,d,x,u)\bigr) - V\bigl(t+1, f(t,d,x,0)\bigr)\bigr|; \; 0 \le t \le r, d \in D, \|x\|_{\mathcal{X}} \le r, \|u\|_{\mathcal{U}} \le s\bigr\} \tag{4.72}$$

By virtue of the right-hand side of inequality (4.70) and Hypothesis (L1), it follows that $\gamma(r,s) < +\infty$ for all $r, s \ge 0$. Moreover, definition (4.72) guarantees that for all $r, s \ge 0$, the mappings $\gamma(r,\cdot)$ and $\gamma(\cdot,s)$ are nondecreasing with $\gamma(r,0) = 0$. Finally, Hypothesis (L5), in conjunction with the fact that $V \in CU(Z^+ \times \mathcal{X}; \Re^+)$, guarantees that $\lim_{s\to 0^+}\gamma(r,s) = 0$ for all $r \ge 0$. Consequently, Lemma 2.3 guarantees the existence of functions $a_3 \in K_\infty$ and $\phi \in K^+$ such that $\gamma(r,s) \le a_3(\phi(r)s)$

for all $r, s \geq 0$. It follows by definition (4.72) that the following inequality holds for all $(t, x, u) \in Z^+ \times \mathcal{X} \times U$:

$$\left| \sup_{d\in D} V\big(t+1, f(t,d,x,u)\big) - \sup_{d\in D} V\big(t+1, f(t,d,x,0)\big) \right| \\ \leq a_3\big(\phi(t)\|u\|_{\mathcal{U}}\big) + a_3\big(\phi\big(\|x\|_{\mathcal{X}}\big)\|u\|_{\mathcal{U}}\big) \tag{4.73}$$

By Proposition 4.2 there exist functions $\mu \in K^+$ and $a \in K_\infty$ such that for every $(t_0, x_0, d, u) \in Z^+ \times \mathcal{X} \times M_D \times M_U$, the corresponding solution $x(t)$ of (4.56) with $x(t_0) = x_0$ satisfies

$$\frac{\|x(t)\|_{\mathcal{X}}}{\mu(t)} \leq \max\Big\{a\big(\|x_0\|_{\mathcal{X}}\big), \sup_{\tau\in[t_0,t]} a\big(\big\|u(\tau)\big\|_{\mathcal{U}}\big)\Big\} \quad \forall t \geq t_0 \tag{4.74}$$

Using (4.71), (4.73), Lemmas 2.3, and 3.2, we obtain functions $a_5, a_6 \in K_\infty$ and $q \in K^+$ such that, for all $(t, x, d, u) \in Z^+ \times \mathcal{X} \times D \times U$, it holds that

$$\sup_{d\in D} V\big(t+1, f(t,d,x,u)\big) \leq \exp(-c)V(t,x) + \exp(-2ct)a_5\big(q(t)\|u\|_{\mathcal{U}}\big) \\ + \exp(-2ct)a_6\left(\frac{\|x\|_{\mathcal{X}}}{\mu(t)}\right)a_5\big(q(t)\|u\|_{\mathcal{U}}\big) \tag{4.75}$$

Let $(t_0, x_0, d, u) \in Z^+ \times \mathcal{X} \times M_D \times M_U$ and consider the corresponding solution $x(t)$ of (4.56) with $x(t_0) = x_0$. Let $V(t) := V(t, x(t))$. By virtue of (4.74) and (4.75), we obtain

$$V(t+1) \leq \exp(-c)V(t) + \exp(-2ct) \sup_{\tau\in[t_0,t]} a_5\big(q(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big) \\ + \exp(-2ct) \sup_{\tau\in[t_0,t]} \big(a_5\big(q(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big)\big)^2 \\ + \frac{1}{2}\exp(-2ct) \sup_{\tau\in[t_0,t]} \big(a_6\big(a\big(\big\|u(\tau)\big\|_{\mathcal{U}}\big)\big)\big)^2 \\ + \frac{1}{2}\exp(-2ct)\big(a_6\big(a\big(\|x_0\|_{\mathcal{X}}\big)\big)\big)^2$$

or

$$V(t+1) \leq \exp(-c)V(t) + \exp(-2ct) \sup_{\tau\in[t_0,t]} \rho_1\big(r(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big) \\ + \exp(-2ct)\rho_2\big(\|x_0\|_{\mathcal{X}}\big) \tag{4.76}$$

where $\rho_1(s) := a_5(s) + (a_5(s))^2 + \frac{1}{2}(a_6(a(s)))^2$, $\rho_2(s) := \frac{1}{2}(a_6(a(s)))^2$, and $r(t) := q(t) + 1$. Inequality (4.76), in conjunction with (4.70), directly implies, for all $t \geq t_0$,

$$a_1\big(\big\|H\big(t, x(t)\big)\big\|_{\mathcal{Y}}\big) \leq \exp\big(-c(t-t_0)\big)a_2\big(\tilde{\beta}(t_0)\|x_0\|_{\mathcal{X}}\big) \\ + \exp\big(-c(t-t_0)\big) \sup_{\tau\in[t_0,t]} \frac{\exp(2c)}{\exp(c)-1}\rho_1\big(r(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big) \\ + \exp\big(-c(t-t_0)\big)\frac{\exp(2c)}{\exp(c)-1}\rho_2\big(\|x_0\|_{\mathcal{X}}\big) \tag{4.77}$$

Finally, (4.77) implies inequality (4.1) with $\sigma(s,t) := 2a_1^{-1}(2\exp(-ct)a_2(s) + 2\exp(-ct)\frac{\exp(2c)}{\exp(c)-1}\rho_2(s))$, $\beta(t) := \tilde{\beta}(t) + 1$, $\delta(t) := r(t)$, and $\gamma(s) := 2\frac{\exp(2c)}{\exp(c)-1}\rho_1(s)$. The proof is complete. □

Example 4.4.1 Consider the following nonlinear, finite-dimensional, discrete-time, time-varying system:

$$\begin{aligned} x_1(t+1) &= d(t)x_1(t) \\ x_2(t+1) &= 2^{-t}d(t)\left|x_1(t)\right|^{\frac{1}{2}} + u(t) \\ Y(t) &= H\big(t, x(t)\big) := x_2(t) \end{aligned} \tag{4.78}$$

where $x(t) := (x_1(t), x_2(t)) \in \Re^2$, $t \in Z^+$, $d(t) \in [-2,2]$, and $u(t) \in \Re$.

We next show that system (4.78) satisfies the WIOS property by showing that system (4.78) with $u \equiv 0$ is RGAOS. Indeed, noticing that Hypotheses (L1–3), (L5), and (G) are satisfied for system (4.78), the WIOS property is guaranteed by Proposition 4.6.

Consider the trajectory $x(t)$ of (4.78) with $u \equiv 0$ that corresponds to input $d \in M_D$ with initial condition $x(t_0) = x_0 \in \Re^2$. Since $|d(t)| \le 2$, it follows from (4.78) that, for all $t \ge t_0$,

$$\left|x_1(t+1)\right| \le 2\left|x_1(t)\right| \tag{4.79}$$

By induction and with (4.79), we obtain the following estimate for all $t \ge t_0$:

$$\left|x_1(t)\right| \le 2^{t-t_0}\left|x_1(t_0)\right| \tag{4.80}$$

Using (4.78) with $u \equiv 0$, (4.80) and the fact that $|d(t)| \le 2$, we obtain, for all $t \ge t_0 + 1$,

$$\left|x_2(t)\right| \le M\exp\big(-c(t-t_0)\big)\left|x_1(t_0)\right|^{1/2}$$

where $c := \frac{\ln(2)}{2}$ and $M := 2\sqrt{2}$. Consequently, the following inequality holds for all $t \ge t_0$:

$$\left|x_2(t)\right| \le \exp\big(-c(t-t_0)\big)\rho\big(\left|x(t_0)\right|\big) \tag{4.81}$$

where $\rho(s) := \max\{s, M\sqrt{s}\}$. It follows from (4.81) that system (4.78) with $u \equiv 0$ is URGAOS.

It should be noted that although system (4.78) is guaranteed to satisfy the WIOS property, we cannot determine the gain function or the weight function. This happens because we exploited the qualitative characterization of Proposition 4.6.

In order to be able to determine the gain and weight functions for the WIOS property, one has to use comparison lemmas based on difference inequalities. The following lemma illustrates this point.

Lemma 4.5 *Suppose that there exist functions $V : Z^+ \times \mathcal{X} \to \Re^+$, $W : Z^+ \times \mathcal{X} \times U \to \Re$, $\beta, \delta \in K^+$, $\gamma \in \mathcal{N}$, and a positive definite function $\rho \in C^0(\Re^+;\Re^+)$ such that the following implications hold for all $(k,x,d,u) \in Z^+ \times \mathcal{X} \times D \times U$:*

$$V(k,x) \geq W(k,x,u) \quad \Rightarrow \quad V\big(k+1, f(k,d,x,u)\big) \leq V(k,x) - \rho\big(V(k,x)\big) \tag{4.82}$$

$$V(k,x) \leq W(k,x,u) \quad \Rightarrow \quad V\big(k+1, f(k,d,x,u)\big) \leq a\big(\delta(k)\|u\|_{\mathcal{U}}\big) \tag{4.83}$$

Then there exists $\sigma \in KL$ such that, for all $t \geq t_0$,

$$V\big(t,x(t)\big) \leq \max\Big\{\sigma\big(V\big(t_0,x(t_0)\big), t-t_0\big), \max_{t_0 \leq k \leq t} \sigma\big(a\big(\delta(k)\big\|u(k)\big\|_{\mathcal{U}}\big), t-k\big)\Big\}$$

Moreover, if there exist functions $\beta \in K^+$ and $a_1, a_2 \in K_\infty$ such that

$$a_1\big(\big\|H(t,x)\big\|_{\mathcal{Y}}\big) \leq V(t,x) \leq a_2\big(\beta(t)\|x\|_{\mathcal{X}}\big) \quad \forall (t,x) \in Z^+ \times \mathcal{X} \tag{4.84}$$

then, *system* (4.56) *under Hypotheses* (L1–3) *satisfies the WIOS property with gain $\gamma(s) := a_1^{-1}(a(s))$ and weight $\delta \in K^+$. Finally, if $\beta(t) \equiv 1$, then system* (4.56) *under Hypotheses* (L1–3) *satisfies the UWIOS property with gain $\gamma(s) := a_1^{-1}(a(s))$ and weight $\delta \in K^+$.*

Proof Define, for all $(k,s) \in Z^+ \times \mathfrak{R}^+$ and $t \in [k,k+1)$,

$$g(s) := \max_{0 \leq y \leq s} \big(y - \rho(y)\big) \tag{4.85}$$

$$\tilde{\sigma}(s,k) := g^{(k)}(s) := \underbrace{g \circ \cdots \circ g}_{k \text{ times}}(s) \quad \text{and} \quad \tilde{\sigma}(s,0) := s \tag{4.86}$$

$$\tilde{\sigma}(s,t) := (t-k)\tilde{\sigma}(s,k+1) + (k+1-t)\tilde{\sigma}(s,k) \tag{4.87}$$

Definition (4.85) implies that the function $g \in \mathcal{N}$ satisfies $g(s) < s$ for all $s > 0$. A simple contradiction argument shows that $\lim_{k \to +\infty} g^{(k)}(s) = 0$ for all $s \geq 0$. Therefore, definitions (4.86) and (4.87) imply that $\tilde{\sigma} \in KL$.

Consider the trajectory $x(t)$ of (4.56) that corresponds to input $(d,u) \in M_D \times M_U$ with initial condition $x(t_0) = x_0 \in \mathcal{X}$. First suppose that $t \geq t_0 + 1$. We distinguish the following cases:

Case 1: Assume that $V(k,x(k)) \geq W(k,x(k),u(k))$ for all $k = t_0, \ldots, t-1$. Then using implication (4.82) and definitions (4.85), (4.86), we obtain, by using induction,

$$V\big(t,x(t)\big) \leq \tilde{\sigma}\big(V\big(t_0,x(t_0)\big), t-t_0\big). \tag{4.88}$$

Case 2: Assume that there exists $k \in \{t_0, \ldots, t-1\}$ with $V(k,x(k)) < W(k,x(k),u(k))$. Let $l := \max\{k \in \{t_0, \ldots, t-1\} : V(k,x(k)) < W(k,x(k),u(k))\}$. If $l < t-1$, then it holds that $V(k,x(k)) \geq W(k,x(k),u(k))$ for all $k = l+1, \ldots, t-1$. Therefore, by virtue of (4.88) we have

$$V\big(t,x(t)\big) \leq \tilde{\sigma}\big(V\big(l+1,x(l+1)\big), t-l-1\big). \tag{4.89}$$

On the other hand, (4.83) implies that $V(l+1,x(l+1)) \leq a(\delta(l)\|u(l)\|_{\mathcal{U}})$. Therefore, from (4.89) we obtain

$$V\big(t,x(t)\big) \leq \tilde{\sigma}\big(a\big(\delta(l)\big\|u(l)\big\|_{\mathcal{U}}\big), t-l-1\big) \tag{4.90}$$

Notice that inequality (4.90) holds for the case $l = t - 1$. Combining (4.89) and (4.90), we obtain that, for all $t \geq t_0 + 1$,

$$V\big(t, x(t)\big) \leq \max\Big\{\sigma\big(V\big(t_0, x(t_0)\big), t - t_0\big), \max_{t_0 \leq k \leq t} \sigma\big(a\big(\delta(k)\|u(k)\|_{\mathcal{U}}\big), t - k\big)\Big\}$$

where $\sigma(s, t) := s$ for all $(s, t) \in \Re^+ \times [0, 1)$ and $\sigma(s, t) := \tilde{\sigma}(s, t-1)$ for all $(s, t) \in \Re^+ \times [1, +\infty)$. Finally, notice that the previous estimate holds for the case $t = t_0$ as well.

The rest of the proof is a direct consequence of the above estimate. □

Lemma 4.6 *Suppose that there exist functions $V : Z^+ \times \mathcal{X} \to \Re^+$, $W : Z^+ \times \mathcal{X} \to \Re^+$, $\beta, \delta \in K^+$, and $\gamma \in \mathcal{N}$ and a constant $\lambda \in (0, 1)$ such that the solution $x(t)$ of (4.56) that corresponds to input $(d, u) \in M_D \times M_U$ with initial condition $x(t_0) = x_0 \in \mathcal{X}$ satisfies the following inequality for all $t \geq t_0$:*

$$V\big(t+1, x(t+1)\big) \leq \lambda V\big(t, x(t)\big) + \lambda^{t-t_0} W(t_0, x_0) + a\big(\delta(t)\|u(t)\|_{\mathcal{U}}\big) \quad (4.91)$$

Moreover, suppose that there exist functions $\beta \in K^+$ and $a_1, a_2 \in K_\infty$ such that

$$\left.\begin{aligned} &a_1(\|H(t, x)\|_{\mathcal{Y}}) \leq V(t, x) \\ &V(t, x) + \lambda^{-1} W(t, x) \leq a_2(\beta(t)\|x\|_{\mathcal{X}}) \end{aligned}\right\} \quad \forall (t, x) \in Z^+ \times \mathcal{X} \quad (4.92)$$

Then system (4.56) under Hypotheses (L1–3) satisfies the WIOS property with gain $\gamma(s) := a_1^{-1}(\frac{2}{1-\lambda} a(s))$ and weight $\beta, \delta \in K^+$. Finally, if $\beta(t) \equiv 1$, then system (4.56) under Hypotheses (L1–3) satisfies the UWIOS property with gain $\gamma(s) := a_1^{-1}(a(s))$ and weight $\beta, \delta \in K^+$.

Proof Consider the trajectory $x(t)$ of (4.56) that corresponds to input $(d, u) \in M_D \times M_U$ with initial condition $x(t_0) = x_0 \in \mathcal{X}$. Using induction and (4.91), we obtain, for all nonnegative integers $k \geq 0$,

$$\begin{aligned} &V\big(t_0 + k, x(t_0 + k)\big) \\ &\quad \leq \lambda^k V\big(t_0, x(t_0)\big) + \sum_{i=0}^{k-1} \lambda^{k-1-i}\big(\lambda^i W(t_0, x_0) + a\big(\delta(t_0 + i)\|u(t_0 + i)\|_{\mathcal{U}}\big)\big) \end{aligned} \quad (4.93)$$

Using (4.93), we obtain, for all nonnegative integers $k \geq 0$,

$$\begin{aligned} &V\big(t_0 + k, x(t_0 + k)\big) \\ &\quad \leq \lambda^k\big(V\big(t_0, x(t_0)\big) + \lambda^{-1} W(t_0, x_0)\big) + \frac{1}{1-\lambda} \max_{t_0 \leq i \leq t_0+k-1} a\big(\delta(i)\|u(i)\|_{\mathcal{U}}\big) \end{aligned} \quad (4.94)$$

The rest of the proof is a direct consequence of the above estimate. □

Example 4.4.2 Again consider the nonlinear finite-dimensional discrete-time time-varying system (4.78). In order to determine the gain and weight functions, we have to consider the continuous function $V(t, x) := \exp(-t)|x_1| + |x_2|$, which clearly satisfies the following inequality for all $(t, x, d, u) \in Z^+ \times \Re^2 \times [-2, 2] \times \Re$:

$$\begin{aligned}
&V\big(t+1, dx_1, 2^{-t}d|x_1|^{\frac{1}{2}}+u\big)\\
&\quad\le \exp(-t-1)|d||x_1| + 2^{-t}|d||x_1|^{\frac{1}{2}}+|u|\\
&\quad\le 2e^{-1}\exp(-t)|x_1| + 2^{-t+1}|x_1|^{\frac{1}{2}}+|u| \qquad (4.95)
\end{aligned}$$

Let $x(t)$ denote the solution of (4.78) initiated from $x_0 \in \Re^2$ at time $t_0 \in Z^+$ and corresponding to $(d,u) \in M_{[-2,2]} \times M_{\Re}$. Notice that inequality (4.95), in conjunction with (4.80), gives

$$V(t+1) \le 2e^{-1}V(t) + 2^{-\frac{t-t_0}{2}}2|x_0|^{\frac{1}{2}} + \big|u(t)\big| \quad \forall t \ge t_0 \qquad (4.96)$$

Clearly, inequality (4.91) holds for $\lambda = \frac{2}{e}$, $W(t,x) := 2|x|^{1/2}$, $a(s) := s$, and $\delta(t) \equiv 1$. Moreover, inequality (4.92) holds with $a_1(s) := s$, $\beta(t) \equiv 1$, and $a_2(s) := 2s + \frac{4}{e}s^{1/2}$. We conclude from Lemma 4.6 that system (4.78) satisfies the UIOS property with gain $\gamma(s) := \frac{2e}{e-2}s$.

Finally, we end this section by providing characterizations of the UIOS property for system (4.56).

Proposition 4.7 *Consider system* (4.56) *under Hypotheses* (L1–3), (L5), (G), *and* (G2). *Suppose that* (4.56) *is T-periodic. Then the following statements are equivalent*:

(i) *There exist functions $\sigma \in KL$ and $\rho \in \mathcal{N}$ such that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in Z^+ \times \mathcal{X} \times M_D$, and $t \ge t_0$*:

$$\big\|H\big(t,x(t)\big)\big\|_{\mathcal{Y}} \le \max\Big\{\sigma\big(\|x_0\|_{\mathcal{X}}, t-t_0\big), \sup_{\tau\in[t_0,t]} \sigma\big(\rho\big(\big\|u(\tau)\big\|_{\mathcal{U}}\big), t-\tau\big)\Big\}. \qquad (4.97)$$

(ii) *System* (4.56) *satisfies the UIOS property.*

(iii) *There exists a function $\theta \in K_\infty$ such that the following system is URGAOS*:

$$\begin{aligned}
&x(t+1) = f\big(t, d(t), x(t), \theta\big(\big\|H\big(t,x(t)\big)\big\|_{\mathcal{Y}}\big)d'(t)\big)\\
&Y(t) = H\big(t, x(t)\big)\\
&\quad x(t) \in \mathcal{X}, Y(t) \in \mathcal{Y}, \big(d(t), d'(t)\big) \in D \times B_U[0,1], t \in Z^+ \qquad (4.98)
\end{aligned}$$

where $B_U[0,1] := \{u \in U; \|u\|_{\mathcal{U}} \le 1\}$.

(iv) *There exist functions $V \in CU(Z^+ \times \mathcal{X}; \Re^+)$, which is T-periodic, and $a_1, a_2, a_3 \in K_\infty$ and a constant $\lambda \in (0,1)$ such that, for all $(t,x,d,u) \in Z^+ \times \mathcal{X} \times D \times U$,*

$$a_1\big(\big\|H(t,x)\big\|_{\mathcal{Y}}\big) \le V(t,x) \le a_2\big(\|x\|_{\mathcal{X}}\big) \qquad (4.99)$$

$$V\big(t+1, f(t,d,x,u)\big) \le \lambda V(t,x) + a_3\big(\|u\|_{\mathcal{U}}\big). \qquad (4.100)$$

Proof (i) $\Rightarrow$ (ii) The desired (4.1) is a consequence of (4.97) with $\gamma(s) := \sigma(\rho(s), 0)$.

(ii) ⇒ (iii) Without loss of generality we may assume that the gain function γ is of class K_∞. Let $\lambda \in (0,1)$ and define $\theta(s) := \gamma^{-1}(\lambda s)$. System (4.98) is the feedback interconnection of the system

$$\begin{aligned} &x(t+1) = f\big(t, d(t), x(t), v(t)d'(t)\big) \\ &Y(t) = H\big(t, x(t)\big) \\ &\quad x(t) \in \mathcal{X}, Y(t) \in \mathcal{Y}, v(t) \in \Re^+, \big(d(t), d'(t)\big) \in D \times B_U[0,1], t \in Z^+ \end{aligned}$$

with the static map $v(t) = \theta(\|H(t, x(t))\|_{\mathcal{Y}})$. Clearly, the above system satisfies the UIOS property from the input $v \in M_{\Re^+}$. Finally, notice that (4.43) and Hypothesis (G2) hold. Proposition 4.4 implies that system (4.98) is URGAOS.

(iii) ⇒ (iv) Since (4.56) satisfies Hypotheses (L1–3) and (L5), it follows that system (4.98) satisfies Hypotheses (L1–4). Proposition 3.1 implies the existence of mappings $V \in CU(Z^+ \times \mathcal{X}; \Re^+)$,which is T-periodic, and $a_1, a_2 \in K_\infty$ and a constant $\lambda \in (0,1)$ such that inequality (4.99) holds and

$$\begin{aligned} &V\big(t+1, f(t,d,x,u)\big) \le \lambda V(t,x) \quad \forall (t,x,d,u) \in Z^+ \times \mathcal{X} \times D \times U \\ &\quad \text{with } \|u\|_{\mathcal{U}} \le \theta\big(\big\|H(t,x)\big\|_{\mathcal{Y}}\big) \end{aligned} \tag{4.101}$$

Since the mappings $f : Z^+ \times D \times \mathcal{X} \times U \to \mathcal{X}$ and $H : Z^+ \times \mathcal{X} \to \mathcal{Y}$ are T-periodic, from Hypotheses (L1), (L2), (L3), Fact I in Sect. 2.2, and Lemma 2.4, there follows the existence of $a \in K_\infty$ such that

$$\begin{aligned} &\big\|f(t,d,x,u)\big\|_{\mathcal{X}} + \big\|H(t,x)\big\|_{\mathcal{Y}} \le a\big(\|x\|_{\mathcal{X}}\big) + a\big(\|u\|_{\mathcal{U}}\big) \\ &\quad \forall (t,x,d,u) \in Z^+ \times \mathcal{X} \times D \times U \end{aligned} \tag{4.102}$$

Define, for all $s \ge 0$,

$$\begin{aligned} \psi(s) := \sup\big\{&V\big(t+1, f(t,d,x,u)\big) - V\big(t+1, f(t,d,x,0)\big); \\ &d \in D, t \ge 0, \|u\|_{\mathcal{U}} \le s, \big\|H(t,x)\big\|_{\mathcal{Y}} \le \theta^{-1}(s)\big\} \end{aligned} \tag{4.103}$$

Clearly, Hypothesis (G2) and inequalities (4.99), (4.102) imply that the mapping $\psi : \Re^+ \to \Re^+$ is well defined, locally bounded, and nondecreasing. Moreover, Hypotheses (L5), (G2) and the fact that $V \in CU(Z^+ \times \mathcal{X}; \Re^+)$ guarantees that $\lim_{s\to 0^+} \psi(s) = \psi(0) = 0$. Consequently, Lemma 2.4 implies the existence of $a_3 \in K_\infty$ such that $\psi(s) \le a_3(s)$ for all $s \ge 0$.

We next establish inequality (4.100), with a_3 as previously, by considering the following two cases:

- $\|u\|_{\mathcal{U}} \le \theta(\|H(t,x)\|_{\mathcal{Y}})$. In this case inequality (4.100) is a direct consequence of (4.101).
- $\|u\|_{\mathcal{U}} \ge \theta(\|H(t,x)\|_{\mathcal{Y}})$. In this case inequality (4.100) is a direct consequence of (4.101) and (4.103).

(iv) ⇒ (i) Notice that inequalities (4.99), (4.100) guarantee that the assumptions of Lemma 4.5 are satisfied with $W(t,x,u) := \frac{2}{1-\lambda} a_3(\|u\|_{\mathcal{U}})$, $a(s) := \frac{1+\lambda}{1-\lambda} a_3(s)$, $\rho(s) := \frac{1-\lambda}{2} s$, and $\delta(t) \equiv 1$. The proof is complete. □

4.5 Transformations Preserving WIOS

The method of verifying external stability properties using transformations is used frequently in Mathematical Control Theory and Stability Theory. Roughly speaking, we want to verify the WIOS property for a system Σ which is the transformation of another system Σ' (in the sense described in Sect. 1.6). Suppose that we can establish that Σ' satisfies the WIOS property (using another method, e.g., by solving the differential (or difference) equations (or inequalities)). Then Σ satisfies the WIOS property.

Proposition 4.8 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ which satisfies the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Suppose that Σ satisfies the WIOS property. Let $\Phi : \Re^+ \times \mathcal{X} \to \mathcal{X}$ be a change of coordinates and $q : \Re^+ \times V \to U$ be a transformation of V onto U, $V \subseteq \mathcal{U}$. Then the system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ defined by* (1.106), (1.107), (1.108) *satisfies the WIOS property. Moreover, if Σ satisfies the UWIOS property and there exists $a \in K_\infty$ such that* (2.65) *holds, then the system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ defined by* (1.106), (1.107), (1.108) *satisfies the UWIOS property. Finally, if Σ satisfies the IOS property and there exists $\zeta \in K_\infty$ such that*

$$\left\|q(t,v)\right\|_{\mathcal{U}} \leq \zeta\left(\|v\|_{\mathcal{U}}\right) \quad \forall (t,v) \in \Re^+ \times V \tag{4.104}$$

then, the system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ defined by (1.106), (1.107), (1.108) *satisfies the IOS property.*

Proof By Lemma 1.2, $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ is a deterministic control system which satisfies the BIC property. Moreover, $0 \in \mathcal{X}$ is a robust equilibrium point from the input $v \in M_V$ for Σ'. By Lemma 2.8, the system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ is RFC from the input $v \in M_V$.

Finally, notice that by Definition 4.1, there exist functions $\sigma \in KL$, $\beta, \delta \in K^+$, and $\gamma \in \mathcal{N}$ such that estimate (4.1) holds for all $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$ and $t \geq t_0$. Definitions (1.106) and (1.108), in conjunction with (4.1), imply that the following estimate holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, $v \in M_V$ and $t \geq t_0$:

$$\begin{aligned}
&\left\|H'\big(t, \phi'(t, t_0, x_0, v, d), v(t)\big)\right\|_{\mathcal{Y}} \\
&\quad = \left\|H\big(t, \phi\big(t, t_0, \Phi^{-1}(t_0, x_0), Qv, d\big), q\big(t, v(t)\big)\big)\right\|_{\mathcal{Y}} \\
&\quad \leq \sigma\big(\beta(t_0)\left\|\Phi^{-1}(t_0, x_0)\right\|_{\mathcal{X}}, t - t_0\big) + \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\left\|q\big(\tau, v(\tau)\big)\right\|_{\mathcal{U}}\big)
\end{aligned} \tag{4.105}$$

Since $\Phi : \Re^+ \times \mathcal{X} \to \mathcal{X}$ is a change of coordinates, it follows from Lemma 3.2 that there exist $a \in K_\infty$ and $p \in K^+$ such that $\|\Phi^{-1}(t, x)\|_{\mathcal{X}} \leq p(t)a(\|x\|_{\mathcal{X}})$ for all $(t, x) \in \Re^+ \times \mathcal{X}$. Since $q : \Re^+ \times V \to U$ is a transformation of V onto U, it follows from Lemma 2.3 that there exist functions $\zeta \in K_\infty$ and $\mu \in K^+$ such that $\delta(t)\|q(t, v)\|_{\mathcal{U}} \leq \zeta(\mu(t)\|v\|_{\mathcal{U}})$ for all $(t, v) \in \Re^+ \times V$. Consequently, from (4.105) we obtain

$$\begin{aligned}&\left\| H'\big(t, \phi'(t, t_0, x_0, v, d), v(t)\big)\right\|_{\mathcal{Y}}\\&\quad \le \sigma\big(\beta(t_0)p(t_0)a\big(\|x_0\|_{\mathcal{X}}\big), t-t_0\big) + \sup_{t_0\le\tau\le t} \gamma\big(\zeta\big(\mu(\tau)\|v(\tau)\|_{\mathcal{U}}\big)\big)\end{aligned} \tag{4.106}$$

The above inequality shows that the system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ defined by (1.106), (1.107), (1.108) satisfies the WIOS property.

All other cases are treated similarly. The proof is complete. □

Example 4.5.1 All linear time-varying systems described by ODEs

$$\dot{x} = A(t)x + B(t)u \quad x \in \Re^n, u \in U \subseteq \Re^m \tag{4.107}$$

where the matrices $A(t) \in \Re^{n\times n}$ and $B(t) \in \Re^{n\times m}$ have continuous components, satisfy the UWISS property from the input $u \in M_U$ if and only if the system $\dot{x} = A(t)x, x \in \Re^n$, is URGAS. To see this, notice that if the system $\dot{x} = A(t)x, x \in \Re^n$, is URGAS then by Theorem 3.9, p. 143, in [17] there exist constants $M, \sigma > 0$ such that the fundamental solution matrix of the system satisfies

$$\big|\Phi(t, t_0)\big| \le M \exp\big(-\sigma(t-t_0)\big) \quad \text{for all } t \ge t_0 \ge 0 \tag{4.108}$$

The variations of constants formula for the solution of (4.107) implies that

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\, d\tau \tag{4.109}$$

Let $\delta \in K^+$ be a function that satisfies $|B(t)| \le \delta(t)$. Then by combining (4.109) with (4.108) we obtain, for the solution of (4.107),

$$\big|x(t)\big| \le M \exp\big(-\sigma(t-t_0)\big)\big|x(t_0)\big| + M\sigma^{-1} \sup_{t_0\le\tau\le t} \big(\delta(\tau)\big|u(\tau)\big|\big) \tag{4.110}$$

Consequently, system (4.107) satisfies the UWISS property with gain function $\gamma(s) := M\sigma^{-1}s$ and weight $\delta \in K^+$.

By Proposition 4.8, it follows that the UWISS property holds for all nonlinear systems of the form

$$\dot{x} = f(t, x, u) \quad x \in \Re^n, u \in U \subseteq \Re^m \tag{4.111}$$

which satisfy Hypotheses (H1), (H3), (H4) in Sect. 1.2 and for which there exist a continuously differentiable change of coordinates $\Phi : \Re^+ \times \Re^n \to \Re^n$ satisfying

$$\big|\Phi(t, x)\big| \le a\big(|x|\big) \quad \text{and} \quad \big|\Phi^{-1}(t, x)\big| \le a\big(|x|\big) \tag{4.112}$$

for all $(t, x) \in \Re^+ \times \Re^n$, for certain $a \in K_\infty$ and a continuous mapping $p : \Re^+ \times U \to V$ with $V \subseteq \Re^m$ and $p(t, 0) = 0$ for all $t \ge 0$, for which the following system of partial differential equations holds for all $(t, x, u) \in \Re^+ \times \Re^n \times U$:

$$\frac{\partial \Phi}{\partial t}(t, x) + \frac{\partial \Phi}{\partial x}(t, x) f(t, x, u) = A(t)\Phi(t, x) + B(t)p(t, u) \tag{4.113}$$

where the matrices $A(t) \in \Re^{n\times n}$ and $B(t) \in \Re^{n\times m}$ have continuous components, and the system $\dot{x} = A(t)x, x \in \Re^n$, is URGAS.

4.6 Qualitative Characterizations of WIOS

Consider a system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with outputs $H : \Re^+ \times \mathcal{X} \to \mathcal{Y}$, which is Robustly Forward Complete from the input $u \in M_U$ and satisfies the following hypotheses:

(CON1) There exists a partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$with finite diameter such that

1. for each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, it holds that $\pi \cap (t_0, +\infty) \subset \pi(t_0, x_0, u, d)$, where $\pi(t_0, x_0, u, d)$ is the set involved in Property 4 of Definition 1.1 (weak semigroup property),
2. for each bounded set $S \subset \mathcal{X}$ and for every $(T, R, \varepsilon) \in \Re^+ \times \Re^+ \times (0, +\infty)$, there exists $\delta > 0$ such that $\sup\{\|\phi(\tau, t_0, x, u, d) - \phi(\tau, t_0, x_0, v, d)\|_{\mathcal{X}}; d \in M_D, \tau \in [t_0, q_\pi(t_0)]\} < \varepsilon$ for all $t_0 \in [0, T] \cap \pi$, $x, x_0 \in S$, and $u, v \in \mathcal{M}(B_U[0, R]) \cap M_U$ with $\|x - x_0\|_{\mathcal{X}} + \sup_{t_0 \le \tau \le t} \|u(\tau) - v(\tau)\|_{\mathcal{U}} < \delta$,
3. for any sequences $\{d_i \in M_D\}_{i=0}^{\infty}$ and $\{u_i \in M_U\}_{i=0}^{\infty}$, the mappings $Pd : \Re^+ \to D$ and $Pu : \Re^+ \to U$ with $Pd(t) := d_i(t)$, $Pu(t) := u_i(t)$ for all $T_i \le t < T_{i+1}$, $i = 0, 1, 2, \ldots$, belong to M_D and M_U, respectively.

(CON2) *Complete Continuity of the Output Map*: For every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|H(t, x) - H(t, x_0)\|_{\mathcal{Y}} < \varepsilon$ for all $t \in I$ and $x, x_0 \in S$ with $\|x - x_0\|_{\mathcal{X}} < \delta$.

Hypothesis (CON2) is a standard continuity assumption, which can be verified easily for a very wide class of systems. Hypothesis (CON1) is a technical hypothesis that guarantees the following two properties:

(a) the continuity of the transition map with respect to the external input and the initial state,
(b) the set of times that satisfies the semigroup property, i.e., the set $\pi(t_0, x_0, u, d)$, contains all members of the partition $\pi = \{T_i\}_{i=0}^{\infty}$ greater than t_0. Thus the partition $\pi = \{T_i\}_{i=0}^{\infty}$ can be used in order to discretize time for all initial conditions and inputs. Notice that this requirement is automatically satisfied if $\pi(t_0, x_0, u, d) = [t_0, +\infty)$ (i.e., the case where the system is RFC from the input $u \in M_U$ and satisfies the classical semigroup property).

Let us here remark one case where Hypothesis (CON1) is (generally) not satisfied, the case of control systems with variable sampling partition (1.57).

For systems satisfying Hypotheses (CON1), (CON2), the following result provides equivalent characterizations for the WIOS property.

Theorem 4.1 (Necessary and sufficient conditions for the WIOS property) *Consider a system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ which is RFC from the input $u \in M_U$ and satisfies Hypotheses* (CON1), (CON2), (G). *Moreover, suppose that* $0 \in \mathcal{X}$ *is a robust equilibrium point from the input $u \in M_U$ for Σ. The following statements are equivalent*:

(i) *there exist functions $\sigma \in KL$, $\beta, \gamma \in K^+$, and $\rho \in K_\infty$ such that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, and $t \ge t_0$*:

$$\begin{aligned}&\left\| H\big(t, \phi(t, t_0, x_0, u, d), u(t)\big)\right\|_{\mathcal{Y}}\\&\quad \leq \max\left\{\sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big), \sup_{\tau\in[t_0,t]} \sigma\big(\rho\big(\gamma(\tau)\left\|u(\tau)\right\|_{\mathcal{U}}\big), t - \tau\big)\right\}.\end{aligned} \tag{4.114}$$

(ii) *Σ satisfies the WIOS property from the input $u \in M_U$.*
(iii) *Σ satisfies the* 0*-GAOS property.*

Estimates of the form (4.114) ("fading memory estimates") were first used by Praly and Wang [21] for the formulation of exp-ISS and by Grüne [3, 4] for the formulation of Input-to-State Dynamical Stability (ISDS) with $H(t, x) = x$ and $\beta(t) \equiv \gamma(t) \equiv 1$, which was proved to be qualitatively equivalent with (4.2) for finite-dimensional continuous-time systems. It is clear that Theorem 4.1 shows that for the WIOS property, the "fading memory" estimate (4.114) is qualitatively equivalent to the "Sontag-like" estimate (4.1).

Proof of Theorem 4.1 Notice that the implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are immediate. Thus we are left with the proof of the implication (iii) $\Rightarrow$ (i).

By virtue of Proposition 4.2 and since (i) Hypothesis (G) holds, i.e., $\tilde{u} \in M_U$ for all $(u, \lambda) \in M_U \times \Re^+$, where $\tilde{u}$ is the input that satisfies $\tilde{u}(t) = \lambda u(t)$ for all $t \geq 0$, (ii) Σ is RFC from the input $u \in M_U$, and (iii) $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ for Σ, there exist functions $\mu \in K^+$ and $a \in K_\infty$ such that the following estimate holds for all $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$:

$$\left\|\phi(t, t_0, x_0, u, d)\right\|_{\mathcal{X}} \leq \mu(t) a\Big(\|x_0\|_{\mathcal{X}} + \sup_{\tau\in[t_0,t]} \left\|u(\tau)\right\|_{\mathcal{U}}\Big) \quad \forall t \geq t_0 \tag{4.115}$$

By virtue of Theorem 2.1 and since $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ satisfies the 0-GAOS property, we guarantee the existence of functions $\sigma \in KL$ and $\beta \in K^+$ such that the following estimate holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$:

$$\left\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\right\|_{\mathcal{Y}} \leq \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big) \tag{4.116}$$

Let $\pi = \{T_i\}_{i=0}^{\infty}$ the partition of $\Re^+$ for which Hypothesis (CON1) is satisfied, and let $r > 0$ be its diameter, i.e., $r := \sup\{T_{i+1} - T_i; i = 0, 1, 2, \ldots\} < +\infty$.

Let $MU \subseteq M_U$ be the cone of inputs $u \in M_U$ with $\sup_{t\geq 0} \|u(t)\|_{\mathcal{U}} < +\infty$, which is a subset of the normed linear space $M\mathcal{U}$ of bounded functions $u : \Re^+ \to \mathcal{U}$ with norm $\|u\|_{M\mathcal{U}} := \sup_{t\geq 0} \|u(t)\|_{\mathcal{U}}$.

The proof is divided into two parts:

Part 1: We apply an abstract discretization technique, which provides an infinite-dimensional discrete-time system satisfying the WIOS property.

Part 2: The solution of the discrete-time system obtained from the 1st part is related to the solution of Σ, and we show that estimate (4.114) holds for all bounded inputs $u \in MU$.

Since estimate (4.114) holds for all bounded inputs $u \in MU$, we conclude that Σ satisfies the WIOS property.

Part 1: Abstract Discretization

Define the normed linear space FX of bounded functions $x : [0, r] \to \mathcal{X}$ with norm $\|x\|_{\text{FX}} := \sup_{\theta \in [0,r]} \|x(\theta)\|_{\mathcal{X}}$. Let CM_D and CMU denote the set of sequences with values in M_D and MU, respectively. For $(i, x, d, u) \in Z^+ \times \text{FX} \times M_D \times M_U$, where $x(\theta) \in \mathcal{X}$ for $\theta \in [0, r]$, define

$$f(i, d, x, u) := \begin{cases} \phi(\theta - r + T_{i+1}, T_i, x(r), u, d) & \theta \in [r - T_{i+1} + T_i, r] \\ 0 & \theta \in [0, r - T_{i+1} + T_i] \end{cases} \tag{4.117}$$

a map, which is well defined and satisfies (see (4.115))

$$\|f(i, d, x, u)\|_{\text{FX}} \le \tilde{\mu}(i) a\Big(\|x\|_{\text{FX}} + \sup_{T_i \le s \le T_{i+1}} \|u(s)\|_{\mathcal{U}}\Big)$$
$$\forall (i, x, d, u) \in Z^+ \times \text{FX} \times M_D \times M_U \tag{4.118}$$

where $\tilde{\mu}(i) := \max_{T_i \le t \le T_{i+1}} \mu(t)$. By Hypothesis (CON1) and inequality (4.118) it follows that f is completely continuous, i.e., satisfies the following hypothesis:

(CON3) For all bounded sets $S \subset \text{FX} \times MU$ and $I \subset Z^+$ and for every $\varepsilon > 0$, the set $f(I \times M_D \times S)$ is bounded, and there exists $\delta > 0$ such that

$$\sup\Big\{\big\|f(i, d, x, u) - f(i, d, x_0, u_0)\big\|_{\mathcal{X}}; d \in M_D\Big\} < \varepsilon$$

for all $i \in I$, $(x, u) \in S$, and $(x_0, u_0) \in S$ with $\|x - x_0\|_{\text{FX}} + \|u - u_0\|_{M\mathcal{U}} < \delta$. Moreover, $f(i, d, 0, 0) = 0$ for all $(i, d) \in Z^+ \times M_D$.

Define the normed linear space FY of bounded functions $Y : [0, r] \to \mathcal{Y}$ with norm $\|Y\|_{\text{FY}} := \sup_{\theta \in [0,r]} \|Y(\theta)\|_{\mathcal{Y}}$. Define the output map for $(i, x) \in Z^+ \times \text{FX}$:

$$\tilde{H}(i, x) := H\big(\max\{0, T_i - r + \theta\}, x(\theta)\big) \quad \theta \in [0, r] \tag{4.119}$$

By virtue of Hypothesis (CON2) and the fact that the continuous map $H : \Re^+ \times \mathcal{X} \to \mathcal{Y}$ maps bounded sets of $\Re^+ \times \mathcal{X}$ into bounded sets of $\mathcal{Y}$, it follows that $\tilde{H}$ is completely continuous, i.e., satisfies the following hypothesis:

(CON4) For every pair of bounded sets $I \subset Z^+$, $S \subset \text{FX}$ and for every $\varepsilon > 0$, the set $\tilde{H}(I \times S)$ is bounded, and there exists $\delta > 0$ such that $\|\tilde{H}(i, x) - \tilde{H}(i, x_0)\|_{\mathcal{Y}} < \varepsilon$ for all $i \in I$ and $x, x_0 \in S$ with $\|x - x_0\|_{\mathcal{X}} < \delta$. Moreover, $\tilde{H}(i, 0) = 0$ for all $t \in Z^+$.

Next consider the discrete-time system

$$x_{i+1} = f(i, d_i, x_i, u_i)$$
$$Y_i = \tilde{H}(i, x_i)$$
$$(i, x_i, Y_i, d_i, u_i) \in Z^+ \times \text{FX} \times \text{FY} \times M_D \times MU \tag{4.120}$$

By virtue of all the above, the discrete-time system (4.120) satisfies Hypotheses (L1–3) in Sect. 1.7. Using induction and the weak semigroup property for Σ, it can be easily shown that, for every sequence $\{d_i\}_{i=0}^{\infty} \in CM_D$, the solution of (4.120) with initial condition $x_{i_0} = x_0 \in \mathrm{FX}$ and corresponding to input $\{d_i\}_{i=0}^{\infty} \in CM_D$ and $u_i \equiv 0$ for all $i \geq i_0$ satisfies, for all $i \geq i_0 + 1$,

$$x_i = \begin{cases} \phi(\theta - r + T_i, T_{i_0}, x_0(r), u_0, Pd) & \theta \in [r - T_i + T_{i-1}, r] \\ 0 & \theta \in [0, r - T_i + T_{i-1}] \end{cases} \tag{4.121}$$

where $Pd : \Re^+ \to D$ with $Pd(t) := d_i(t)$ for all $T_i \leq t < T_{i+1}$, $i = 0, 1, 2, \ldots$, which by Hypothesis (CON1) belongs to M_D. Moreover, using (4.119) in conjunction with (4.121), we obtain, for all $i \geq i_0 + 1$,

$$\|Y_i\|_{\mathrm{FY}} \leq \sigma\left(\beta(T_{i_0})\|x_{i_0}\|_{\mathrm{FX}}, \max\{0; T_i - T_{i_0} - r\}\right) \tag{4.122}$$

Estimate (4.122) implies that system (4.120) satisfies the Robust Output Attractivity Property (P3) of Definition 2.2. Thus by Lemmas 1.3, 1.4, and 2.1, system (4.120) is RGAOS. Consequently, by virtue of Proposition 4.6 and Properties (CON3), (CON4) above (which imply Hypotheses (L1–5) for (4.120)), we conclude that the discrete-time system (4.120) satisfies the WIOS property.

Since the discrete-time system (4.120) satisfies the WIOS property, by Proposition 4.5, there exist functions $\tilde{\sigma} \in KL$, $\tilde{\beta}, \tilde{\gamma} \in K^+$, and $\tilde{\rho} \in K_\infty$ such that for all $\{u_i \in MU\}_{i=0}^{\infty}$ and $(i_0, x_0, \{d_i\}_{i=0}^{\infty}) \in Z^+ \times \mathrm{FX} \times CM_D$, the following estimate holds for all $i \geq i_0$ for the solution x_i of (4.120) with initial condition $x_{i_0} = x_0$ and corresponding to inputs $\{u_i\}_{i=0}^{\infty} \in CMU$ and $\{d_i\}_{i=0}^{\infty} \in CM_D$:

$$\begin{aligned}\left\|\tilde{H}(i, x_i)\right\|_{\mathrm{FY}} \leq \max\Big\{&\tilde{\sigma}\left(\tilde{\beta}(i_0)\|x_0\|_{\mathrm{FX}}, i - i_0\right), \\ &\sup_{i_0 \leq j \leq i} \tilde{\sigma}\left(\tilde{\rho}\left(\tilde{\gamma}(j) \sup_{\tau \geq 0}\left\|u_j(\tau)\right\|_{\mathcal{U}}\right), i - j\right)\Big\}\end{aligned} \tag{4.123}$$

Part 2: Proof of estimate (4.114)

Notice that definition (4.117) of the evolution map f of the discrete-time system shows that the solution x_i of (4.120) with arbitrary initial condition $x_{i_0} = x_0$ and corresponding to arbitrary inputs $\{u_i\}_{i=0}^{\infty} \in CMU$ and $\{d_i\}_{i=0}^{\infty} \in CM_D$ coincides, for all $i \geq i_0 + 1$, with the solution with arbitrary initial condition x_{i_0} that satisfies $x_{i_0}(r) = x_0(r)$ corresponding to inputs $\{d_i\}_{i=0}^{\infty} \in CM_D$ and $\{\tilde{u}_i\}_{i=0}^{\infty} \in CMU$ with $\tilde{u}_j(t) = u_j(t)$ for all $t \in [T_j, T_{j+1}]$ and $j = i_0, i_0 + 1, \ldots, i - 1$. Thus by (4.123) it follows that for all $\{u_i \in MU\}_{i=0}^{\infty}$ and$(i_0, x_0, \{d_i\}_{i=0}^{\infty}) \in Z^+ \times \mathrm{FX} \times CM_D$, the following estimate holds for all $i \geq i_0 + 1$ for the solution x_i of (4.120) with initial condition $x_{i_0} = x_0$ and corresponding to inputs $\{u_i\}_{i=0}^{\infty} \in CMU$ and $\{d_i\}_{i=0}^{\infty} \in CM_D$:

$$\begin{aligned}\left\|\tilde{H}(i, x_i)\right\|_{\mathrm{FY}} \leq \max\Big\{&\tilde{\sigma}\left(\tilde{\beta}(i_0)\left\|x_0(r)\right\|_{\mathcal{X}}, i - i_0\right), \\ &\sup_{i_0 \leq j \leq i-1} \tilde{\sigma}\left(\tilde{\rho}\left(\tilde{\gamma}(j) \sup_{T_j \leq s \leq T_{j+1}}\left\|u_j(s)\right\|_{\mathcal{U}}\right), i - j\right)\Big\}\end{aligned} \tag{4.124}$$

Let arbitrary $(d, u) \in M_D \times MU$, $(t_0, x_0) \in \Re^+ \times \mathcal{X}$ and consider the transition map $\phi(t, t_0, x_0, u, d)$ of Σ. The definitions (4.117) and (4.119) of the evolution and output maps $f, \tilde{H}$ of the discrete-time system (4.120) implies that the transition map $\phi(t, t_0, x_0, u, d)$ of Σ satisfies the following identities for all $i \geq i_0 + 1$:

$$x(t) = \phi(t, t_0, x_0, u, d) = x_i(t + r - T_i) \quad \forall t \in [T_{i-1}, T_i]$$
$$Y(t) = H\big(t, \phi(t, t_0, x_0, u, d)\big) = Y_i(t + r - T_i) \quad \forall t \in [T_{i-1}, T_i]$$

where $T_{i_0} = q_\pi(t_0)$, and x_i denotes the solution of (4.120) with arbitrary initial condition $x_{i_0} \in \mathrm{FX}$ that satisfies $x_{i_0}(r) = \phi(T_{i_0}, t_0, x_0, u, d)$ and corresponding to the constant inputs $\{u_i \equiv u\}_{i=0}^{\infty} \in CMU$ and $\{d_i \equiv d\}_{i=0}^{\infty} \in CM_D$. Combining the above identities with (4.125), we obtain, for all $i \geq i_0 + 1$,

$$\sup_{T_{i-1} \leq t \leq T_i} \|Y(t)\|_{\mathcal{Y}} \leq \max\Big\{\tilde{\sigma}\big(\tilde{\beta}(i_0)\|x_{i_0}(r)\|_{\mathcal{X}}, i - i_0\big), \sup_{i_0 \leq j \leq i-1} \tilde{\sigma}\Big(\tilde{\rho}\Big(\tilde{\gamma}(j) \sup_{T_j \leq s \leq T_{j+1}} \|u(s)\|_{\mathcal{U}}\Big), i - j\Big)\Big\} \tag{4.125}$$

Let $\bar{\beta}, \bar{\gamma} \in K^+$ be nondecreasing functions that satisfy $\bar{\beta}(T_{j-1}) \geq \tilde{\beta}(j)$ and $\bar{\gamma}(T_j) \geq \tilde{\gamma}(j)$ for all integers $j \geq 1$. Estimate (4.125), in conjunction with the trivial inequalities $T_{i_0} - r \leq t_0 < T_{i_0}$ and the causality property for Σ (which shows that $Y(t)$ depends only on the values of $u \in MU$ in the interval $[t_0, t]$), implies (notice that without loss of generality we may assume that $\sigma(s, t)$ is of class K_∞ for each $t \geq 0$)

$$\|Y(t)\|_{\mathcal{Y}} \leq \max\left\{\tilde{\sigma}\left(\bar{\beta}(t_0)\|x(T_{i_0})\|_{\mathcal{X}}, \frac{t - t_0 - r}{r}\right), \sup_{t_0 \leq \tau \leq t} \tilde{\sigma}\left(\tilde{\rho}\big(\bar{\gamma}(\tau)\|u(\tau)\|_{\mathcal{U}}\big), \frac{t - \tau}{r}\right)\right\} \quad \forall t \geq t_0 + r \tag{4.126}$$

Without loss of generality, we may assume that the function $\mu \in K^+$ involved in (4.115) is nondecreasing. Inequality (4.115), in conjunction with (4.126), implies the following estimate for all $t \geq t_0 + r$:

$$\|Y(t)\|_{\mathcal{Y}} \leq \max\left\{\tilde{\sigma}\left(\bar{\beta}(t_0)\mu(t_0 + r)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|_{\mathcal{U}}\Big), \frac{t - t_0 - r}{r}\right), \sup_{t_0 \leq \tau \leq t} \tilde{\sigma}\left(\tilde{\rho}\big(\bar{\gamma}(\tau)\|u(\tau)\|_{\mathcal{U}}\big), \frac{t - \tau}{r}\right)\right\} \tag{4.127}$$

It follows from Hypothesis (CON2) in conjunction with the fact that for every bounded set $S \subset \Re^+ \times \mathcal{X}$, the set $H(S)$ is bounded and Lemma 2.3 that there exists a pair of functions $\zeta \in K_\infty$ and $\delta \in K^+$ such that

$$\|H(t, x)\|_{\mathcal{Y}} \leq \zeta\big(\delta(t)\|x\|_{\mathcal{X}}\big) \quad \forall (t, x) \in \Re^+ \times \mathcal{X} \tag{4.128}$$

Without loss of generality we may assume that the function $\delta \in K^+$ involved in (4.128) is nondecreasing. Combining the above inequality with estimate (4.115), it follows that, for all $t \in [t_0, t_0 + r]$,

$$\|Y(t)\|_{\mathcal{Y}} \leq \zeta\Big(\delta(t_0 + r)\mu(t_0 + r)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|_{\mathcal{U}}\Big)\Big) \tag{4.129}$$

Defining

$$\omega(s,t) := \exp(r-t)\zeta(s) + \begin{cases} \tilde{\sigma}\left(s, \frac{t-r}{r}\right) & \text{if } t > r \\ \exp(r-t)\tilde{\sigma}(s,0) & \text{if } 0 \le t \le r \end{cases} \tag{4.130}$$

$$\hat{\beta}(t) := \bar{\beta}(t)\big(1+\mu(t+r)\big) + \delta(t+r)\mu(t+r) \tag{4.131}$$

and combining estimates (4.127) and (4.129), we obtain the following estimate for all $t \ge t_0$:

$$\big\|Y(t)\big\|_{\mathcal{Y}} \le \max\Big\{\omega\Big(\hat{\beta}(t_0)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0\le\tau\le t}\big\|u(\tau)\big\|_{\mathcal{U}}\Big), t-t_0\Big), \sup_{t_0\le\tau\le t}\omega\big(\tilde{\rho}\big(\bar{\gamma}(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big), t-\tau\big)\Big\} \tag{4.132}$$

Lemma 2.3 implies the existence of a pair of nondecreasing functions $\tilde{a} \in K_\infty$ and $q \in K^+$ such that $\hat{\beta}(t)a(2s) \le \tilde{a}(q(t)s)$ for all $t, s \ge 0$. Since, for all $t \ge t_0$,

$$\omega\Big(\hat{\beta}(t_0)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0\le\tau\le t}\big\|u(\tau)\big\|_{\mathcal{U}}\Big), t-t_0\Big) \le \max\Big\{\omega\big(\hat{\beta}(t_0)a\big(2\|x_0\|_{\mathcal{X}}\big), t-t_0\big), \omega\Big(\hat{\beta}(t_0)a\Big(2\sup_{t_0\le\tau\le t}\big\|u(\tau)\big\|_{\mathcal{U}}\Big), t-t_0\Big)\Big\} \tag{4.133}$$

and $\hat{\beta}(t)a(2s) \le \tilde{a}(q(t)s)$ for all $t, s \ge 0$ with $q \in K^+$ being nondecreasing, from (4.133) we obtain

$$\omega\Big(\hat{\beta}(t_0)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0\le\tau\le t}\big\|u(\tau)\big\|_{\mathcal{U}}\Big), t-t_0\Big) \le \max\Big\{\bar{\omega}\big(q(t_0)\|x_0\|_{\mathcal{X}}, t-t_0\big), \sup_{t_0\le\tau\le t}\bar{\omega}\big(q(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}, t-\tau\big)\Big\} \tag{4.134}$$

where $\bar{\omega}(s,t) := \omega(\tilde{a}(s),t)$. Finally, let $\beta(t) := 1 + q(t) + \hat{\beta}(t)$, $\sigma(s,t) := \bar{\omega}(s,t) + \omega(s,t)$, $\rho(s) := s + \tilde{\rho}(s)$, and $\gamma(t) := \bar{\gamma}(t) + q(t)$. The previous definitions, in conjunction with (4.132) and (4.134), imply estimate (4.114).

The proof is complete. □

It should be noticed that for systems satisfying Hypotheses (CON1), (CON2), and (G), Theorem 4.1 gives qualitative characterizations for the WIOS property which cannot be further weakened. On the other hand, Example 4.3.2 shows that the result cannot be extended to the IOS property.

4.7 Lyapunov-Like Necessary and Sufficient Conditions for WIOS

The main purpose of this section is to present necessary and sufficient Lyapunov-like conditions for WIOS and IOS for certain types of systems. We start with the finite-dimensional case.

4.7.1 Control Systems Described by ODEs

In this section, we will restrict our attention to systems for which the output map is independent of the values of the input, i.e., $H(t, x, u) = H(t, x)$.

Theorem 4.2 *Consider system* (1.3) *under Hypotheses* (H1–5) *and suppose that* $H : \Re^+ \times \Re^n \to \Re^k$ *is locally Lipschitz and independent of* $u \in U$. *Moreover, suppose that the following hypotheses hold*:

(H6) *For every bounded* $I \subseteq \Re^+$ *and for every bounded* $S \subset \Re^n \times U$, *there exists a constant* $L \geq 0$ *such that*

$$\left| f(t, x, u, d) - f(t, x, v, d) \right| \leq L|u - v|$$
$$\forall t \in I, \forall (x, u, x, v) \in S \times S, \forall d \in D$$

(H7) U *is a cone, i.e., for all* $u \in U$ *and* $\lambda \geq 0$, *it follows that* $(\lambda u) \in U$. *Then, the following statements are equivalent*:

(a) *System* (1.3) *is robustly forward complete* (*RFC*) *from the input* $u \in M_U$, *and there exist functions* $\sigma \in KL$, $\beta, \delta \in K^+$, *and* $\zeta \in K$ *such that for all* $(d, u) \in M_D \times M_U$ *and* $(t_0, x_0) \in \Re^+ \times \Re^n$, *the solution* $x(t)$ *of* (1.3) *with* $x(t_0) = x_0$ *corresponding to* $(d, u) \in M_D \times M_U$ *satisfies the following estimate for all* $t \geq t_0$:

$$\left| H\big(t, x(t)\big) \right| \leq \max\left\{ \sigma\big(\beta(t_0)|x_0|, t - t_0\big), \sup_{t_0 \leq \tau \leq t} \sigma\Big(\zeta\big(\delta(\tau)\left|u(\tau)\right|\big), t - \tau\Big) \right\}. \quad (4.135)$$

(b) *System* (1.3) *satisfies the WIOS property from the input* $u \in M_U$.

(c) *There exist a locally Lipschitz function* $\theta \in K_\infty$ *and functions* $\phi, \mu \in K^+$ *such that the following system is RGAOS*:

$$\dot{x}(t) = f\left(t, x(t), \frac{\theta(|x(t)|)}{\phi(t)} d'(t), d(t)\right) \qquad Y(t) = \tilde{H}\big(t, x(t)\big) \quad (4.136)$$

where $\Delta := B_U[0, 1] \times D$, $\tilde{H}(t, x) := (H(t, x), \mu(t)x) \in \Re^k \times \Re^n$.

(d) *There exist a locally Lipschitz Lyapunov function* $V : \Re^+ \times \Re^n \to \Re^+$, *functions* a_1, a_2, a_3 *of class* K_∞, *and* β, δ, μ *of class* K^+ *such that*

$$a_1\Big(\left|H(t, x)\right| + \mu(t)|x|\Big) \leq V(t, x) \leq a_2\big(\beta(t)|x|\big)$$
$$\forall (t, x) \in \Re^+ \times \Re^n \quad (4.137)$$
$$V^0\big(t, x; f(t, x, u, d)\big) \leq -V(t, x) + a_3\big(\delta(t)|u|\big)$$
$$\forall (t, x, u, d) \in \Re^+ \times \Re^n \times U \times D \quad (4.138)$$

(e) *System* (1.3) *is RFC from the input* $u \in M_U$, *and there exist a locally Lipschitz Lyapunov function* $V : \Re^+ \times \Re^n \to \Re^+$, *functions* a_1, a_2, ζ *of class* K_∞, β, δ *of class* K^+, *and a locally Lipschitz positive definite function*

$\rho : \Re^+ \to \Re^+$ *such that*

$$a_1\big(\big|H(t,x)\big|\big) \le V(t,x) \le a_2\big(\beta(t)|x|\big) \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{4.139}$$

$$V^0\big(t,x; f(t,x,u,d)\big) \le -\rho\big(V(t,x)\big) \quad \forall (t,x,u,d) \in \Re^+ \times \Re^n \times U \times D \text{ with } \zeta\big(\delta(t)|u|\big) \le V(t,x) \tag{4.140}$$

(f) *System* (1.3) *is RFC from the input $u \in M_U$, and system* (1.3) *with $u \equiv 0$ is RGAOS.*

Moreover, if there exist functions $p \in K_\infty$ and $\mu \in K^+$ and a constant $R \ge 0$ such that $p(\mu(t)|x|) \le V(t,x) + R$ for all $(t,x) \in \Re^+ \times \Re^n$, then the requirement that (1.3) *is RFC from the input $u \in M_U$ is not needed in statement* (e) *above.*

The proof of Theorem 4.2 relies on the following lemma, which shows how Lyapunov functions can be used in order to prove the WIOS property and determine the weight and gain functions.

Lemma 4.7 *Consider system* (1.3) *under Hypotheses* (H1–4) *and suppose that it is RFC from the input $u \in M_U$. Furthermore, suppose that there exist locally Lipschitz functions $V : \Re^+ \times \Re^n \to \Re^+$, a_1, a_2, ζ of class K_∞, β, δ of class K^+, and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that inequalities* (4.139), (4.140) *hold.*

Then, system (1.3) *satisfies the WIOS property with gain $\gamma(s) := a_1^{-1}(\zeta(s))$ and weight δ. Furthermore, there exists $\sigma \in KL$ such that statement* (a) *of Theorem* 4.2 *holds. Moreover, if $\beta(t) \equiv 1$, then* (1.3) *satisfies the UWIOS property with gain $\gamma(s) := a_1^{-1}(\zeta(s))$ and weight δ. Finally, if there exist functions $a \in K_\infty$ and $\mu \in K^+$ and a constant $R \ge 0$ such that $a(\mu(t)|x|) \le V(t,x) + R$ for all $(t,x) \in \Re^+ \times \Re^n$, then, the requirement that* (1.3) *is RFC from the input $u \in M_U$ is not needed.*

Proof Consider a solution $x(t)$ of (1.3) under Hypotheses (H1–4) corresponding to arbitrary $(d,u) \in M_D \times M_U$ with initial condition $x(t_0) = x_0 \in \Re^n$. Clearly, the solution exists for $t \in [t_0, t_{\max})$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Notice that the mapping $t \to V(t, x(t))$ is absolutely continuous on $[t_0, t_{\max})$ and we have

$$\frac{d}{dt} V\big(t, x(t)\big) = V^0\big(t, x(t); f\big(t, x(t), u(t), d(t)\big)\big) \quad \text{a.e.} \quad \text{on } [t_0, t_{\max}) \tag{4.141}$$

where $V^0(t,x;v) := \limsup_{h \to 0^+} \frac{V(t+h, x+hv) - V(t,x)}{h}$. Indeed, notice that, for all $t \in [t_0, t_{\max}) \backslash I$, where $I \subset [t_0, t_{\max})$ is the set of zero Lebesgue measure such that $D^+x(t) = \lim_{h \to 0^+} h^{-1}(x(t+h) - x(t))$, or $\frac{d}{dt} V(t, x(t))$ is not defined on I, or $D^+x(t) \ne f(t, x(t), u(t), d(t))$, we get, for all sufficiently small $h > 0$ and all $t \in [t_0, t_{\max}) \backslash I$,

$$x(t+h) - x(t) - hD^+x(t) = hy_h \tag{4.142}$$

where

$$y_h = \frac{x(t+h) - x(t)}{h} - D^+ x(t)$$

Since $\lim_{h \to 0^+} \frac{x(t+h)-x(t)}{h} = D^+ x(t)$, we obtain that $y_h \to 0$ as $h \to 0^+$. Notice that since $V : \Re^+ \times \Re^n \to \Re^+$ is locally Lipschitz, we have

$$\begin{aligned} V^0\big(t, x(t); D^+ x(t)\big) &= \limsup_{h \to 0^+} \frac{V(t+h, x(t) + hD^+ x(t)) - V(t, x(t))}{h} \\ &= \limsup_{h \to 0^+} \frac{V(t+h, x(t) + hD^+ x(t) + hy_h) - V(t, x(t))}{h} \end{aligned}$$

The above equality, in conjunction with (4.142) and the facts that $\frac{d}{dt} V(t, x(t)) = \limsup_{h \to 0^+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h}$ and $D^+ x(t) = f(t, x(t), u(t), d(t))$ for all $t \in [t_0, t_{\max}) \setminus I$, implies (4.141).

It follows from Lemma 2.14 and inequalities (4.140), (4.141) that there exists $\tilde{\sigma} \in KL$ with $\tilde{\sigma}(s, 0) = s$ for all $s \geq 0$ such that, for all $t \in [t_0, t_{\max})$,

$$V\big(t, x(t)\big) \leq \max\Big\{\tilde{\sigma}\big(V(t_0, x_0), t - t_0\big), \sup_{t_0 \leq \tau \leq t} \tilde{\sigma}\big(\zeta\big(\delta(\tau)\big|u(\tau)\big|\big), t - \tau\big)\Big\} \tag{4.143}$$

Next, we distinguish the following cases:

1. If (1.3) is RFC, then (4.143) holds for all $t \geq t_0$. It follows from (4.143) and (4.139) that (4.1) holds with $\gamma(s) := a_1^{-1}(\zeta(s))$ (recall that $\tilde{\sigma}(s, 0) = s$ for all $s \geq 0$). Moreover, it follows from (4.143) and (4.139) that statement (a) of Theorem 4.2 holds with $\sigma(s, t) := a_1^{-1}(\tilde{\sigma}(a_2(s), t))$.
2. If there exist functions $a \in K_\infty$ and $\mu \in K^+$ and a constant $R \geq 0$ such that $a(\mu(t)|x|) \leq V(t, x) + R$ for all $(t, x) \in \Re^+ \times \Re^n$, then (4.143), in conjunction with (4.139) and the fact that $\tilde{\sigma}(s, 0) = s$ for all $s \geq 0$, implies the following estimate:

$$\big|x(t)\big| \leq \frac{1}{\mu(t)} a^{-1}\Big(R + a_2\big(\beta(t_0)|x_0|\big) + \sup_{t_0 \leq \tau \leq t} \zeta\big(\delta(\tau)\big|u(\tau)\big|\big)\Big) \tag{4.144}$$

for all $t \in [t_0, t_{\max})$. Estimate (4.144), in conjunction with the BIC property, implies that $t_{\max} = +\infty$. Moreover, (4.144) and Definition 1.4 show that system (1.3) is RFC from the input $u \in M_U$, since we have, for all $r, T \geq 0$,

$$\begin{aligned} &\sup\Big\{\big|x(t_0 + s)\big|; s \in [0, T], |x_0| \leq r, t_0 \in [0, T], (d, u) \in M_D \times M_U, \\ &\quad \sup_{0 \leq t} \big|u(\tau)\big| \leq r\Big\} \\ &\leq \frac{1}{\min_{0 \leq t \leq 2T} \mu(t)} a^{-1}\Big(R + a_2\Big(r \max_{0 \leq t \leq T} \beta(t)\Big) + \zeta\Big(r \max_{0 \leq t \leq 2T} \delta(t)\Big)\Big) < +\infty \end{aligned}$$

The proof is complete. □

Proof of Theorem 4.2 We prove implications (a) ⇒ (b), (b) ⇒ (c), (c) ⇒ (d), (d) ⇒ (e), and (e) ⇒ (a). The equivalence between (f) and (b) is a direct consequence of Theorem 4.1.

(a) $\Rightarrow$ (b): Suppose that there exist functions $\sigma \in KL$, $\beta, \delta \in K^+$, $\zeta \in K_\infty$ such that the estimate (4.135) holds for all $(d, u) \in M_D \times M_U$, $(t_0, x_0) \in \Re^+ \times \Re^n$ and $t \geq t_0$. If we set $\gamma(s) := \sigma(\zeta(s), 0)$ (that obviously is of class $\mathcal{N}$), the desired (4.1) is a consequence of (4.135) and the previous definition. Thus statement (b) holds.

(b) $\Rightarrow$ (c): By virtue of Corollary 4.1 and Hypothesis (H7), it follows that the system

$$\dot{x}(t) = f\big(t, x(t), v(t)d'(t), d(t)\big) \qquad Y(t) = \tilde{H}\big(t, x(t)\big) \tag{4.145}$$

where $\Delta := B_U[0, 1] \times D$, $\tilde{H}(t, x) := (H(t, x), \mu(t)x) \in \Re^k \times \Re^n$, and $v \in M_{\Re^+}$ satisfies the WIOS property from the input $v \in M_{\Re^+}$. More specifically, there exist functions $\mu, \beta, \delta \in K^+$, $\tilde{\sigma} \in KL$, and $\tilde{\gamma} \in \mathcal{N}$ such that (4.42) holds for the solutions of (4.145) for all $v \in M_{\Re^+}$, $(t_0, x_0, (d', d)) \in \Re^+ \times \mathcal{X} \times M_\Delta$, and $t \geq t_0$. Without loss of generality we may assume that $\tilde{\gamma} \in K_\infty$. By virtue of Lemma 3.2 we obtain that there exists $\kappa \in K_\infty$ such that $\tilde{\gamma}(rs) \leq \kappa(r)\kappa(s)$ for all $(r, s) \in (\Re^+)^2$. Let $\phi(t) := \delta(t)\kappa^{-1}(\frac{\mu(t)}{2})$, and let $\theta \in K_\infty$ be a locally Lipschitz function that satisfies $\theta(s) \leq \kappa^{-1}(s)$ for all $s \geq 0$. The previous definitions imply that

$$\frac{\theta(|x|)}{\phi(t)} \leq \frac{1}{\delta(t)}\tilde{\gamma}^{-1}\left(\frac{\mu(t)}{2}|x|\right) \quad \forall (t, x) \in \Re^+ \times \Re^n \tag{4.146}$$

Finally, the reader should notice that system (4.136) is the feedback interconnection of system (4.145) with the static map $v(t) := \frac{\theta(|x(t)|)}{\phi(t)}$. The result follows from Proposition 4.3 and inequality (4.146).

(c) $\Rightarrow$ (d): Suppose that (4.136) is RGAOS. Hypotheses (H1–7) for system (1.3) guarantee that Hypotheses (H1–5) hold for system (4.136). Therefore, Theorem 3.5 implies that there exists a locally Lipschitz mapping $(t, x) \in \Re^+ \times \Re^n \to V(t, x) \in \Re^+$ with the following properties:

- There exist functions $a_1, a_2 \in K_\infty$ and $\beta \in K^+$ such that (4.137) holds.
- It holds that

$$\begin{aligned} &V^0\left(t, x; f\left(t, x, \frac{\theta(|x|)}{\phi(t)}d', d\right)\right) \leq -V(t, x) \\ &\forall \big(t, x, d', d\big) \in \Re^+ \times \Re^n \times \Delta. \end{aligned} \tag{4.147}$$

Notice that inequality (4.147) implies the following inequality:

$$\begin{aligned} &V^0\big(t, x; f(t, x, u, d)\big) \leq -V(t, x) \\ &\quad \forall (t, x, u, d) \in \Re^+ \times \Re^n \times U \times D \text{ with } \phi(t)|u| \leq \theta\big(|x|\big) \end{aligned} \tag{4.148}$$

Using definition (2.128) and Hypothesis (H6), we obtain, for all $(t, x, u, d) \in \Re^+ \times \Re^n \times U \times D$,

$$\big|V^0\big(t, x; f(t, x, u, d)\big) - V^0\big(t, x; f(t, x, 0, d)\big)\big| \leq M\big(t + |x| + |u|\big)|u| \tag{4.149}$$

for certain nondecreasing function $M : \Re^+ \to \Re^+$. Define

$$\psi(t, s) := \sup\big\{M\big(t + |x| + |u|\big)|u|;\, |x| \leq \theta^{-1}\big(\phi(t)s\big),\, |u| \leq s\big\} \tag{4.150}$$

Without loss of generality we may assume that the function $\phi \in K^+$ is nondecreasing. Clearly, $\psi : \Re^+ \times \Re^+ \to \Re^+$ is a mapping with $\psi(t, 0) = 0$ for all $t \geq 0$ and such that

(i) for each fixed $t \geq 0$, the mapping $\psi(t, \cdot)$ is nondecreasing;
(ii) for each fixed $s \geq 0$, the mapping $\psi(\cdot, s)$ is nondecreasing; and
(iii) $\lim_{s \to 0^+} \psi(t, s) = 0$ for all $t \geq 0$.

Hence, by employing Lemma 2.3 we obtain functions $a_3 \in K_\infty$ and $\delta \in K^+$ such that $\psi(t, s) \leq a_3(\delta(t)s)$.

We next establish inequality (4.138), with a_3 as previously, by considering the following two cases:

- $\theta^{-1}(\phi(t)|u|) \leq |x|$. In this case inequality (4.138) is a direct consequence of (4.148).
- $\theta^{-1}(\phi(t)|u|) \geq |x|$. In this case, by virtue of inequalities (4.148), (4.149), definition (4.150), and the definition of a_3, we have

$$V^0\big(t, x; f(t, x, u, d)\big) \leq V^0\big(t, x; f(t, x, 0, d)\big) + \psi\big(t, |u|\big)$$
$$\leq -V(t, x) + a_3\big(\delta(t)|u|\big)$$

which implies (4.138).

(d) $\Rightarrow$ (e): Notice that (4.138) implies (4.140) with $\zeta(s) := 2a_3(s)$ and $\rho(s) := \frac{1}{2}s$. The fact that system (1.3) is RFC from the input $u \in M_U$ follows directly from Lemma 4.7.

(e) $\Rightarrow$ (a): Follows directly from Lemma 4.7.

The proof is complete. □

Next, we present necessary and sufficient conditions for the UIOS property.

Theorem 4.3 *Consider system* (1.3) *under Hypotheses* (H1–7) *and suppose that* $H : \Re^+ \times \Re^n \to \Re^k$ *is locally Lipschitz and independent of* $u \in U$. *Moreover, suppose that* (1.3) *is* T*-periodic and that Hypothesis* (G2) *holds. Then the following statements are equivalent*:

(a) *System* (1.3) *is robustly forward complete* (RFC) *from the input* $u \in M_U$, *and there exist functions* $\sigma \in KL$ *and* $\zeta \in K$ *such that for all* $(d, u) \in M_D \times M_U$ *and* $(t_0, x_0) \in \Re^+ \times \Re^n$, *the solution* $x(t)$ *of* (1.3) *with* $x(t_0) = x_0$ *corresponding to* $(d, u) \in M_D \times M_U$ *satisfies estimate* (4.135) *with* $\beta(t) = \delta(t) \equiv 1$ *for all* $t \geq t_0$.
(b) *System* (1.3) *satisfies the UIOS property from the input* $u \in M_U$.
(c) *There exist a locally Lipschitz function* $\theta \in K_\infty$ *and functions* $\phi, \mu \in K^+$ *such that the following system is RGAOS*:

$$\dot{x}(t) = f\big(t, x(t), \theta\big(\big|H(t, x)\big|\big)d'(t), d(t)\big) \qquad Y(t) = H\big(t, x(t)\big) \tag{4.151}$$

where $\Delta := B_U[0, 1] \times D$.
(d) *There exist a* T*-periodic locally Lipschitz Lyapunov function* $V : \Re^+ \times \Re^n \to \Re^+$ *and functions* a_1, a_2, a_3 *of class* K_∞ *such that*

$$a_1\big(\big|H(t, x)\big|\big) \leq V(t, x) \leq a_2\big(|x|\big) \quad \forall (t, x) \in \Re^+ \times \Re^n \tag{4.152}$$

$$V^0\big(t, x; f(t, x, u, d)\big) \leq -V(t, x) + a_3\big(|u|\big)$$
$$\forall (t, x, u, d) \in \Re^+ \times \Re^n \times U \times D. \tag{4.153}$$

(e) *There exist a locally Lipschitz Lyapunov function $V : \Re^+ \times \Re^n \to \Re^+$, functions a_1, a_2, ζ of class K_∞, and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that*

$$a_1\big(\big|H(t,x)\big|\big) \le V(t,x) \le a_2\big(|x|\big) \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{4.154}$$

$$V^0\big(t,x; f(t,x,u,d)\big) \le -\rho\big(V(t,x)\big)$$
$$\forall (t,x,u,d) \in \Re^+ \times \Re^n \times U \times D \text{ with } \zeta\big(|u|\big) \le V(t,x). \tag{4.155}$$

Proof of Theorem 4.3 We prove implications (b) ⇒ (c), (c) ⇒ (d). All other implications are proved in exactly the same way as in the proof of Theorem 4.2.

(b) ⇒ (c): By Hypothesis (H7) it follows that the system

$$\dot{x}(t) = f\big(t, x(t), v(t)d'(t), d(t)\big) \qquad Y(t) = H\big(t, x(t)\big) \tag{4.156}$$

where $\Delta := B_U[0,1] \times D$ and $v \in M_{\Re^+}$, satisfies the UIOS property from the input $v \in M_{\Re^+}$. More specifically, there exist functions $\sigma \in KL$ and $\gamma \in \mathcal{N}$ such that (4.2) holds with $\beta(t) \equiv 1$ for the solutions of (4.156) for all $v \in M_{\Re^+}$, $(t_0, x_0, (d', d)) \in \Re^+ \times \mathcal{X} \times M_\Delta$, and $t \ge t_0$. Without loss of generality we may assume that $\gamma \in K_\infty$. Let $\theta \in K_\infty$ be a locally Lipschitz function that satisfies $\theta(s) \le \gamma^{-1}(\frac{1}{2}s)$ for all $s \ge 0$. The previous definitions imply that

$$\theta\big(\big|H(t,x)\big|\big) \le \gamma^{-1}\left(\frac{1}{2}\big|H(t,x)\big|\right) \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{4.157}$$

Finally, notice that system (4.151) is the feedback interconnection of system (4.156) with the static map $v(t) := \theta(|H(t,x(t))|)$. The result follows from Proposition 4.4 and inequality (4.157).

(c) ⇒ (d): Suppose that (4.151) is URGAOS. Hypotheses (H1–7) for system (1.3) guarantee that Hypotheses (H1–5) hold for system (4.151). Therefore, Theorem 3.5 implies that there exists a T-periodic locally Lipschitz mapping $(t,x) \in \Re^+ \times \Re^n \to V(t,x) \in \Re^+$ with the following properties:

- there exist functions $a_1, a_2 \in K_\infty$ such that (4.152) holds.
- it holds that, for all $(t,x,d',d) \in \Re^+ \times \Re^n \times \Delta$,

$$V^0\big(t,x; f\big(t,x,\theta\big(\big|H(t,x)\big|\big)d', d\big)\big) \le -V(t,x). \tag{4.158}$$

Notice that inequality (4.158) implies the following inequality:

$$V^0\big(t,x; f(t,x,u,d)\big) \le -V(t,x)$$
$$\forall (t,x,u,d) \in \Re^+ \times \Re^n \times U \times D \text{ with } |u| \le \theta\big(\big|H(t,x)\big|\big) \tag{4.159}$$

Using definition (2.128), the facts that (1.3) is T-periodic and that V is T-periodic and Hypothesis (H6), we obtain, for all $(t,x,u,d) \in \Re^+ \times \Re^n \times U \times D$,

$$\big|V^0\big(t,x; f(t,x,u,d)\big) - V^0\big(t,x; f(t,x,0,d)\big)\big| \le M\big(|x| + |u|\big)|u| \tag{4.160}$$

for certain nondecreasing function $M : \Re^+ \to \Re^+$. Define

$$\psi(s) := \sup\big\{M\big(|x| + |u|\big)|u|;\ |x| \le a\big(\theta^{-1}(s)\big) + P,\ |u| \le s\big\} \tag{4.161}$$

where $a \in K_\infty$ and $P \geq 0$ are the function and constant, respectively, involved in Hypothesis (G1). By employing Lemma 2.4, we obtain a function $a_3 \in K_\infty$ such that $\psi(s) \leq a_3(s)$.

We next establish inequality (4.153), with a_3 as previously, by considering the following two cases:

- $|u| \leq \theta(|H(t,x)|)$. In this case inequality (4.153) is a direct consequence of (4.159).
- $|u| \geq \theta(|H(t,x)|)$. In this case, by virtue of inequalities (4.159), (4.160), definition (4.161), Hypothesis (G1), and the definition of a_3, we have $V^0(t,x; f(t,x,u,d)) \leq V^0(t,x; f(t,x,0,d)) + \psi(|u|) \leq -V(t,x) + a_3(|u|)$, which implies (4.153).

The proof is complete. □

Using Proposition 4.2 and Theorem 4.2, we obtain the following characterization for the RFC property from the input $u \in M_U$.

Corollary 4.2 *Consider system* (1.3) *under Hypotheses* (H1–3) *and* (H5–7). *Then the following statements are equivalent*:

(a) *System* (1.3) *is robustly forward complete* (*RFC*) *from the input* $u \in M_U$.
(b) *There exist a locally Lipschitz function* $V : \Re^+ \times \Re^n \to \Re^+$, *functions* a_1, a_2, a_3 *of class* K_∞, *and* β, δ, μ *of class* K^+ *such that*

$$a_1\big(\mu(t)|x|\big) \leq V(t,x) \leq a_2\big(\beta(t)|x|\big) \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{4.162}$$

$$V^0\big(t,x; f(t,x,u,d)\big) \leq -V(t,x) + a_3\big(\delta(t)|u|\big) \quad \forall (t,x,u,d) \in \Re^+ \times \Re^n \times U \times D. \tag{4.163}$$

(c) *There exist a locally Lipschitz function* $V : \Re^+ \times \Re^n \to \Re^+$, *functions* a_1, a_2, a_3 *of class* K_∞, *and* β, δ, μ, p, q *of class* K^+ *such that*

$$a_1\big(\mu(t)|x|\big) - p(t) \leq V(t,x) \leq a_2\big(\beta(t)|x|\big) + p(t), \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{4.164}$$

$$V^0\big(t,x; f(t,x,u,d)\big) \leq \big(q(t) + a_3\big(\delta(t)|u|\big)\big)V(t,x) + a_3\big(\delta(t)|u|\big) \quad \forall (t,x,u,d) \in \Re^+ \times \Re^n \times U \times D. \tag{4.165}$$

Proof The equivalence between (a) and (b) is a direct consequence of Proposition 4.2 and Theorem 4.2. Implication (b) ⇒ (c) is obvious. We prove implication (c) ⇒ (a).

Let $T, R \geq 0$ arbitrary and consider the solution $x(t)$ of (1.3) corresponding to arbitrary $(d,u) \in M_D \times M_U$ with initial condition $x(t_0) = x_0 \in \Re^n$ with $t_0 \in [0,T]$ satisfying $|x_0| \leq R$, $\sup_{t \geq 0} |u(t)| \leq R$. Clearly, the solution exists for $t \in [t_0, t_{\max})$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Exactly as in the proof of Lemma 4.7 and using (4.165), we may establish that, for almost all $t \in [t_0, t_{\max})$,

$$\frac{d}{dt}V\big(t,x(t)\big) \le \big(q(t)+a_3\big(\delta(t)R\big)\big)V\big(t,x(t)\big)+a_3\big(\delta(t)R\big) \tag{4.166}$$

For all $t \in [t_0, t_0+T] \cap [t_0, t_{\max})$, it follows from (4.166) that

$$\frac{d}{dt}V\big(t,x(t)\big) \le A(T,R)V\big(t,x(t)\big)+B(T,R) \quad \text{a.e.} \quad \text{on } [t_0,t_{\max}) \tag{4.167}$$

where

$$A(T,R) := \max_{0\le t\le 2T}\big(q(t)+a_3\big(\delta(t)R\big)\big) \quad \text{and}$$
$$B(T,R) := \max_{0\le t\le 2T} a_3\big(\delta(t)R\big).$$

Using the absolutely continuous function $W(t) := \exp(-(t-t_0)A(T,R))V(t,x(t))$ and (4.167), we establish that, for all $t \in [t_0, t_0+T] \cap [t_0, t_{\max})$, it holds that

$$\frac{d}{dt}W(t) \le \exp\big(-(t-t_0)A(T,R)\big)B(T,R) \quad \text{a.e.} \quad \text{on } [t_0,t_{\max}) \tag{4.168}$$

Using the fact that $W(t) := \exp(-(t-t_0)A(T,R))V(t,x(t))$ is absolutely continuous and (4.168), we obtain

$$V\big(t,x(t)\big) \le \exp\big(TA(T,R)\big)V(t_0,x_0)+\exp\big(TA(T,R)\big)\frac{B(T,R)}{A(T,R)}$$
$$\text{for all } t \in [t_0,t_0+T]\cap[t_0,t_{\max}) \tag{4.169}$$

Finally, using (4.164) and (4.169), we obtain

$$a_1\big(\mu(t)\big|x(t)\big|\big) \le p(t)+\exp\big(TA(T,R)\big)Q(T,R)+\exp\big(TA(T,R)\big)\frac{B(T,R)}{A(T,R)}$$
$$\text{for all } t \in [t_0,t_0+T]\cap[t_0,t_{\max}) \tag{4.170}$$

where $Q(T,R) := \max_{0\le t\le T}(p(t)+a_2(\beta(t)R))$. Using (4.170) and the BIC property, we can establish that $t_{\max} = +\infty$. Moreover, using (4.170) and Definition 1.4, we establish that (1.3) is robustly forward complete (RFC) from the input $u \in M_U$. The proof is complete. □

The following example shows how we can utilize Lemma 4.7 in order to prove the UISS property.

Example 4.7.1 (The chemostat with time-varying dilution rate) Here we consider again the chemostat with one microbial species and one nutrient, already studied in Example 1.2.1 with Monod specific growth rate, i.e.,

$$\mu(s) = \frac{as}{k+s} \tag{4.171}$$

where $a, k > 0$ are positive constants, zero mortality rate (i.e., $b = 0$), constant yield coefficient (i.e., $K(s) \equiv K > 0$), and constant inlet nutrient concentration (i.e., $s_{\text{in}}(t) \equiv s_{\text{in}}^* > 0$). If $0 < D^* < \frac{as_{\text{in}}^*}{k+s_{\text{in}}^*}$, then there exists a unique vector $(X^*, s^*) \in (0,+\infty)\times(0,+\infty)$ satisfying the equilibrium conditions (1.5). Therefore, model (1.7) takes the form

$$\begin{aligned}
\dot{x}_1 &= \frac{a}{p+1}\left[p\frac{\exp(x_2)-1}{p+\exp(x_2)} - v\right] \\
\dot{x}_2 &= \frac{a}{p+1}(1+v)\exp(-x_2)\big(1-\exp(x_2)\big) \\
&\quad + \frac{aM}{p+1}\exp(-x_2)\left[v+1-\frac{(p+1)\exp(x_1+x_2)}{p+\exp(x_2)}\right] \\
& x = (x_1, x_2) \in \Re^2,\ v \in [-1, +\infty)
\end{aligned} \tag{4.172}$$

where $v = \frac{D-D^*}{D^*}$ and $p = \frac{k}{s^*}$. It should be noticed that the "unforced" system (4.172) (i.e., system (4.172) with $v \equiv 0$) is URGAS in this case. The analysis presented in Example 2.7.3 holds (because the Monod specific growth rate defined by (4.171) is a strictly increasing function). More specifically, the Lyapunov function defined by (2.158) takes the simple form

$$\begin{aligned}
V(x) &= \exp(x_1) - x_1 - 1 + \frac{p}{M(p+1)}\big(\exp(x_2) - x_2 - 1\big) \\
&\quad + Q\big(M\exp(x_1) + \exp(x_2) - 1 - M\big)^2
\end{aligned} \tag{4.173}$$

where $Q > 0$ is arbitrary. The derivative $\dot{V}(x)$ of V along the trajectories of (4.172) is given by the following expression:

$$\begin{aligned}
\frac{p+1}{a}\dot{V}(x) &= -\frac{p}{p+1}\left[\frac{1+v}{M} + \frac{p}{p+\exp(x_2)}\right]\exp(-x_2)\big(1-\exp(x_2)\big)^2 \\
&\quad + \frac{p}{p+1}\big(\exp(x_2)-1\big)\exp(-x_2)v - \big(\exp(x_1)-1\big)v \\
&\quad - 2Q(1+v)\big(M\exp(x_1) + \exp(x_2) - 1 - M\big)^2
\end{aligned} \tag{4.174}$$

Here, we will study the robustness properties of system (4.172) with respect to the external input $v \in [-1, +\infty)$. It should be noted that if v is allowed to take values greater than p, then we obtain from (4.172) that $\dot{x}_1 \leq -\frac{a(v-p)}{p+1}$. The previous differential inequality shows that $x_1(t)$ can be unbounded for a constant input, which shows that the UISS property from the input $v \in [-1, +\infty)$ cannot hold for system (4.172).

Therefore, we will try to show that the uniform ISS property holds when v is restricted to take values in an appropriate interval $(-1+A, B)$, where the constants $A \in (0, 1)$ and $B > 0$ satisfy $A(1+\frac{p}{p+1}) \leq 1$ and $B \geq \frac{Ap}{p+1}$. The following inequality is a direct consequence of (4.174) and the fact that $v+1 \geq A$:

$$\begin{aligned}
\frac{p+1}{a}\dot{V}(x) &\leq -\frac{p}{p+1}\left[\frac{A}{M} + \frac{p}{p+\exp(x_2)}\right]\exp(-x_2)\big(1-\exp(x_2)\big)^2 \\
&\quad + \left[\frac{p}{p+1}\exp(-x_2) + \frac{1}{M}\right]\big(\exp(x_2)-1\big)v \\
&\quad - \big(M\exp(x_1) + \exp(x_2) - 1 - M\big)\frac{v}{M} \\
&\quad - 2QA\big(M\exp(x_1) + \exp(x_2) - 1 - M\big)^2
\end{aligned} \tag{4.175}$$

Define

$$c(x_2) := \frac{p+1}{pM}\big(p\exp(-x_2)+1\big)\frac{\left[\frac{Mp}{p+1}+\exp(x_2)\right]^2}{(A+M)p+A\exp(x_2)} \tag{4.176}$$

Using the inequalities

$$\begin{aligned}
&-\big(M\exp(x_1)+\exp(x_2)-1-M\big)\frac{v}{M}\\
&\quad\le QA\big(M\exp(x_1)+\exp(x_2)-1-M\big)^2+\frac{1}{4M^2QA}v^2\\
&\left[\frac{p}{p+1}\exp(-x_2)+\frac{1}{M}\right]\big(\exp(x_2)-1\big)v\\
&\quad\le \frac{c(x_2)}{2}v^2+\frac{1}{2c(x_2)}\left[\frac{p}{p+1}\exp(-x_2)+\frac{1}{M}\right]^2\big(\exp(x_2)-1\big)^2
\end{aligned}$$

and definition (4.176), it follows from (4.175) that the following inequality holds:

$$\begin{aligned}
\frac{p+1}{a}\dot{V}(x) \le{}& -\frac{p}{2(p+1)}\left[\frac{A}{M}+\frac{p}{p+\exp(x_2)}\right]\exp(-x_2)\big(1-\exp(x_2)\big)^2\\
&+\frac{1}{2}\left(\frac{1}{2M^2QA}+c(x_2)\right)v^2\\
&-QA\big(M\exp(x_1)+\exp(x_2)-1-M\big)^2
\end{aligned} \tag{4.177}$$

Define the functions

$$\begin{aligned}
\Phi(x_2) :={}& \frac{p}{p+1}\left(\frac{1}{2M^2QA}+c(x_2)\right)^{-1}\left[\frac{A}{M}+\frac{p}{p+\exp(x_2)}\right]\\
&\times\exp(-x_2)\big(1-\exp(x_2)\big)^2
\end{aligned} \tag{4.178}$$

$$\begin{aligned}
P(x) ={}& \Phi(x_2)+2QA\left(\frac{1}{2M^2QA}+c(x_2)\right)^{-1}\\
&\times\big(M\exp(x_1)+\exp(x_2)-1-M\big)^2
\end{aligned} \tag{4.179}$$

and notice that definitions (4.176), (4.178) imply that $\lim_{x_2\to+\infty}\Phi(x_2)=\frac{A^2p^2}{(p+1)^2}>0$ and $\lim_{x_2\to-\infty}\Phi(x_2)=\frac{(A+M)^2}{M^2}>0$. Moreover, notice that $\Phi(x_2)>0$ for all $x_2\neq 0$ and $P(x)>0$ for all $x\neq 0$.

Therefore, if a sequence $\{\xi_i\in\Re^2\}_{i=0}^{\infty}$ satisfies $P(\xi_i)\to 0$, then we must necessarily have $\xi_i\to 0\in\Re^2$. It follows that

$$\eta(s) := \inf\left\{\sqrt{P(x)} : x\in\Re^2,\ V(x)\ge s\right\} \tag{4.180}$$

is a nondecreasing function which satisfies $\eta(0)=0$, $\eta(s)>0$ for all $s>0$, and $\lim_{s\to+\infty}\eta(s)=L\le\frac{Ap}{p+1}$. The reader should notice that definitions (4.178), (4.180), in conjunction with inequality (4.177), show that the following implication holds for every $\lambda\in(0,1)$:

$$|v| \le \lambda\eta\big(V(x)\big) \quad \text{and} \quad v \in (-1+A, B)$$
$$\Rightarrow \quad \frac{p+1}{a}\dot{V}(x) \le -\frac{(1-\lambda^2)}{2}\left(\frac{1}{2M^2QA} + c(x_2)\right)P(x) \tag{4.181}$$

Since $\lim_{s\to+\infty}\eta(s) = L \le \frac{Ap}{p+1}$, the inclusion $v \in (-1+A, B)$ holds automatically if $|v| \le \lambda\eta(V(x))$, $A(1+\frac{\lambda p}{p+1}) < 1$, and $B > \lambda\frac{Ap}{p+1}$.

Next, we consider the following input transformation:

$$v = \lambda L \frac{\exp(u)-1}{1+\exp(u)} \quad \text{with } u \in \Re \tag{4.182}$$

Implication (4.181) and definition (4.182) imply the following implication:

$$\zeta\big(|u|\big) \le V(x)$$
$$\Rightarrow \quad \frac{p+1}{a}\dot{V}(x) \le -\frac{(1-\lambda^2)}{2}\left(\frac{1}{2M^2QA} + c(x_2)\right)P(x) \tag{4.183}$$

where $\zeta \in K_\infty$ is defined by $\zeta(s) := \tilde{\eta}^{-1}(L\frac{\exp(s)-1}{1+\exp(s)})$, and $\tilde{\eta}(s) := \frac{1}{s+1}\int_0^s \eta(w)\,dw$ is a strictly increasing, continuous function with $\lim_{s\to+\infty}\tilde{\eta}(s) = L$ and $\tilde{\eta}(s) \le \eta(s)$ for all $s \ge 0$.

Proposition 2.2, in conjunction with Lemma 4.7 and implication (4.183), shows that the system

$$\dot{x}_1 = \frac{a}{p+1}\left[p\frac{\exp(x_2)-1}{p+\exp(x_2)} - \lambda L\frac{\exp(u)-1}{1+\exp(u)}\right]$$
$$\dot{x}_2 = \frac{a}{p+1}\left(1+\lambda L\frac{\exp(u)-1}{1+\exp(u)}\right)\exp(-x_2)\big(1-\exp(x_2)\big)$$
$$+ \frac{aM}{p+1}\exp(-x_2)\left[\lambda L\frac{\exp(u)-1}{1+\exp(u)} + 1 - \frac{(p+1)\exp(x_1+x_2)}{p+\exp(x_2)}\right]$$
$$x = (x_1, x_2) \in \Re^2, u \in \Re \tag{4.184}$$

satisfies the UISS property from the input $u \in \Re$.

4.7.2 Control Systems Described by RFDEs

We consider systems described by uncertain time-varying RFDEs of the form (1.10), and we state characterizations for the WIOS property. We assume Hypotheses (S1–5) as well as the following hypotheses:

(S6) The mapping $u \to f(t,x,u,d)$ is Lipschitz on bounded sets, in the sense that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r,0];\Re^n) \times U$, there exists a constant $L_U \ge 0$ such that

$$\big|f(t,x,u,d) - f(t,x,v,d)\big| \le L_U|u-v|$$
$$\forall t \in I, (x,u,x,v) \in S \times S, d \in D$$

Hypothesis (S6) is equivalent to the existence of a continuous function $L_U : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $t \geq 0$, the mappings $L_U(t,\cdot)$ and $L_U(\cdot,t)$ are nondecreasing, with the following property:

$$\begin{aligned} &\bigl|f(t,x,u,d) - f(t,x,v,d)\bigr| \leq L_U\bigl(t, \|x\|_r + |u| + |v|\bigr)|u-v| \\ &\quad \forall (t,x,d,u,v) \in \Re^+ \times C^0\bigl([-r,0];\Re^n\bigr) \times D \times U \times U \end{aligned} \tag{4.185}$$

(S7) U is a positive cone, i.e., for all $u \in U$ and $\lambda \geq 0$, we have $(\lambda u) \in U$.

Theorem 4.4 *The following statements are equivalent for system* (1.10) *under Hypotheses* (S1–7):

(a) *System* (1.10) *is robustly forward complete* (*RFC*) *from the input* $u \in M_U$, *and there exist functions* $\sigma \in KL$, $\beta, \phi \in K^+$, *and* $\rho \in K$ *such that for all* $(d,u) \in M_D \times M_U$ *and* $(t_0, x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$, *the solution* $x(t)$ *of* (1.10) *with* $T_r(t_0)x = x_0$ *corresponding to* $(d,u) \in M_D \times M_U$ *satisfies the following estimate for all* $t \geq t_0$:

$$\begin{aligned} &\bigl\|H\bigl(t, T_r(t)x\bigr)\bigr\|_{\mathcal{Y}} \\ &\quad \leq \max\Bigl\{\sigma\bigl(\beta(t_0)\|x_0\|_r, t-t_0\bigr), \sup_{t_0 \leq \tau \leq t} \sigma\bigl(\rho\bigl(\phi(\tau)\bigl|u(\tau)\bigr|\bigr), t-\tau\bigr)\Bigr\} \end{aligned} \tag{4.186}$$

(b) *System* (1.10) *satisfies the WIOS property from the input* $u \in M_U$.

(c) *There exist a locally Lipschitz function* $\theta \in K_\infty$ *and functions* $\phi, \mu \in K^+$ *such that the following system is RGAOS*:

$$\begin{aligned} &\dot{x}(t) = f\left(t, T_r(t)x, \frac{\theta(\|T_r(t)x\|_r)}{\phi(t)} d'(t), d(t)\right) \\ &Y(t) = \tilde{H}\bigl(t, T_r(t)x\bigr) \end{aligned} \tag{4.187}$$

where $\Delta := B_U[0,1] \times D$, $\tilde{H}(t,x) := (H(t,x), \mu(t)x) \in \mathcal{Y} \times C^0([-r,0];\Re^n)$.

(d) *There exist a Lyapunov functional* $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, *which is almost Lipschitz on bounded sets, functions* a_1, a_2, a_3 *of class* K_∞, *and* β, δ, μ *of class* K^+ *such that*

$$\begin{aligned} &a_1\bigl(\bigl\|H(t,x)\bigr\|_{\mathcal{Y}} + \mu(t)\|x\|_r\bigr) \leq V(t,x) \leq a_2\bigl(\beta(t)\|x\|_r\bigr) \\ &\quad \forall (t,x) \in \Re^+ \times C^0\bigl([-r,0];\Re^n\bigr) \end{aligned} \tag{4.188}$$

$$\begin{aligned} &V^0\bigl(t,x; f(t,x,u,d)\bigr) \leq -V(t,x) + a_3\bigl(\delta(t)|u|\bigr) \\ &\quad \forall (t,x,u,d) \in \Re^+ \times C^0\bigl([-r,0];\Re^n\bigr) \times U \times D \end{aligned} \tag{4.189}$$

(e) *System* (1.10) *is RFC from the input* $u \in M_U$, *and there exist a Lyapunov functional* $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, *which is almost Lipschitz on bounded sets, functions* a_1, a_2, ζ *of class* K_∞, β, δ *of class* K^+, *and a locally Lipschitz positive definite function* $\rho : \Re^+ \to \Re^+$ *such that*

$$\begin{aligned} &a_1\bigl(\bigl\|H(t,x)\bigr\|_{\mathcal{Y}}\bigr) \leq V(t,x) \leq a_2\bigl(\beta(t)\|x\|_r\bigr) \\ &\quad \forall (t,x) \in \Re^+ \times C^0\bigl([-r,0];\Re^n\bigr) \end{aligned} \tag{4.190}$$

$$\begin{aligned} &V^0\bigl(t,x; f(t,x,u,d)\bigr) \leq -\rho\bigl(V(t,x)\bigr) \\ &\quad \forall (t,x,u,d) \in \Re^+ \times C^0\bigl([-r,0];\Re^n\bigr) \times U \times D \text{ with } \zeta\bigl(\delta(t)|u|\bigr) \leq V(t,x) \end{aligned} \tag{4.191}$$

(f) *System* (1.10) *is RFC from the input* $u \in M_U$ *and satisfies the* 0-*GAOS property.*

Moreover,

(i) *If* $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ *is equivalent to the finite-dimensional continuous mapping* $h : [-r,+\infty) \times \Re^n \to \Re^p$, *then inequality* (4.190) *in the above statement* (e) *can be replaced by the following inequality*:

$$a_1\big(\big|h\big(t,x(0)\big)\big|\big) \le V(t,x) \le a_2\big(\beta(t)\|x\|_r\big) \quad \forall (t,x) \in \Re^+ \times C^0\big([-r,0];\Re^n\big). \tag{4.192}$$

(ii) *If there exist functions* $a \in K_\infty$ *and* $\mu, R \in K^+$ *such that* $a(\mu(t)|x(0)|) \le \|H(t,x)\|_{\mathcal{Y}} + R(t)$ *for all* $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, *then the requirement that* (1.10) *is RFC from the input* $u \in M_U$ *is not needed in statement* (a) *above*.

(iii) *If there exist functions* $p \in K_\infty$ *and* $\mu, R \in K^+$ *such that* $p(\mu(t)|x(0)|) \le V(t,x) + R(t)$ *for all* $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, *then the requirement that* (1.10) *is RFC from the input* $u \in M_U$ *is not needed in statement* (e) *above*.

In order to obtain characterizations of the UIOS property, we need an extra hypothesis for system (1.10).

(S8) There exists a constant $R \ge 0$ and a function $a \in K_\infty$ such that the inequality $\|x\|_r \le a(\|H(t,x)\|_{\mathcal{Y}}) + R$ holds for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$.

Hypothesis (S8) holds for the important case of the output map $H(t,x) := d(x(\theta),\Gamma); \theta \in [-r,0]$, where $\Gamma \subset \Re^n$ is a compact set which contains $0 \in \Re^n$, and $d(x,\Gamma)$ denotes the distance of the point $x \in \Re^n$ from the set $\Gamma \subset \Re^n$. Notice that it is not required that $\Gamma \subset \Re^n$ is positively invariant for (1.10) with $u \equiv 0$.

Hypothesis (S8) is essentially Hypothesis (G2) and allows us to provide characterizations for the UIOS property for periodic uncertain systems.

Theorem 4.5 *Suppose that system* (1.10) *under Hypotheses* (S1–8) *is* T*-periodic. The following statements are equivalent*:

(a) *There exist functions* $\sigma \in KL$ *and* $\rho \in K_\infty$ *such that for all* $(d,u) \in M_D \times M_U$ *and* $(t_0,x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$, *the solution* $x(t)$ *of* (1.10) *with* $T_r(t_0)x = x_0$ *corresponding to* $(d,u) \in M_D \times M_U$ *satisfies the following estimate for all* $t \ge t_0$:

$$\big\|H\big(t,T_r(t)x\big)\big\|_{\mathcal{Y}} \le \max\Big\{\sigma\big(\|x_0\|_r, t-t_0\big), \sup_{t_0 \le \tau \le t} \sigma\big(\rho\big(\big|u(\tau)\big|\big), t-\tau\big)\Big\}. \tag{4.193}$$

(b) *System* (1.10) *satisfies the UIOS property.*

(c) *There exists a locally Lipschitz function* $\theta \in K_\infty$ *such that* $0 \in C^0([-r,0];\Re^n)$ *is URGAOS for the system*:

$$\begin{aligned} \dot{x}(t) &= f\big(t,T_r(t)x,\theta\big(\big\|H\big(t,T_r(t)x\big)\big\|_{\mathcal{Y}}\big)d'(t),d(t)\big), \\ Y(t) &= H\big(t,T_r(t)x\big) \end{aligned} \tag{4.194}$$

where $\Delta := B_U[0,1] \times D$.

(d) *There exist a T-periodic Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, and functions a_1, a_2, a_3 of class K_∞ such that, for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$,*

$$a_1\big(\|H(t,x)\|_{\mathcal{Y}}\big) \le V(t,x) \le a_2\big(\|x\|_r\big) \tag{4.195}$$

$$V^0\big(t,x; f(t,x,u,d)\big) \le -V(t,x) + a_3\big(|u|\big). \tag{4.196}$$

(e) *There exist a Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a_1, a_2, ζ of class K_∞, and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that*

$$a_1\big(\|H(t,x)\|_{\mathcal{Y}}\big) \le V(t,x) \le a_2\big(\|x\|_r\big) \quad \forall (t,x) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \tag{4.197}$$

$$V^0\big(t,x; f(t,x,u,d)\big) \le -\rho\big(V(t,x)\big) \quad \forall (t,x,u,d) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \times U \times D \text{ with } \zeta\big(|u|\big) \le V(t,x). \tag{4.198}$$

Finally, if $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional continuous T-periodic mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$, then inequalities (4.195) *and* (4.197) *in the above statements* (d) *and* (e), *respectively, can be replaced by the following inequality*:

$$a_1\big(\big|h\big(t,x(0)\big)\big|\big) \le V(t,x) \le a_2\big(\|x\|_r\big) \quad \forall (t,x) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \tag{4.199}$$

The following result provides sufficient Lyapunov-like conditions for the (U)WIOS property. The proof of implications (e) $\Rightarrow$ (a) of Theorem 4.4 and (e) $\Rightarrow$ (a) of Theorem 4.5 are based on the result of Theorem 4.6, which gives quantitative estimates of the solutions of (1.10) under Hypotheses (S1–7). The gain functions and the weights of the WIOS property can be determined *explicitly* in terms of the functions involved in the assumptions of Theorem 4.6.

Theorem 4.6 *Consider system* (1.10) *under Hypotheses* (S1–7) *and suppose that there exist a Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a, ζ of class K_∞, β, δ of class K^+, and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that*

$$V(t,x) \le a\big(\beta(t)\|x\|_r\big) \quad \forall (t,x) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \tag{4.200}$$

$$V^0\big(t,x; f(t,x,u,d)\big) \le -\rho\big(V(t,x)\big) \quad \forall (t,x,u,d) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \times U \times D \text{ with } \zeta\big(\delta(t)|u|\big) \le V(t,x) \tag{4.201}$$

Moreover, suppose that one of the following holds:

(a) *system* (1.10) *is RFC from the input $u \in M_U$,*
(b) *there exist functions $p \in K_\infty$ and $\mu, R \in K^+$ such that $p(\mu(t)|x(0)|) \le V(t,x) + R(t)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$.*

Then, system (1.10) *is RFC from the input* $u \in M_U$, *and there exists a function* $\sigma \in KL$ *with* $\sigma(s,0) = s$ *for all* $s \geq 0$ *such that for all* $(d,u) \in M_D \times M_U$ *and* $(t_0, x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$, *the solution* $x(t)$ *of* (1.10) *with* $T_r(t_0)x = x_0$ *corresponding to* $(d,u) \in M_D \times M_U$ *satisfies the following estimate for all* $t \geq t_0$:

$$V\big(t, T_r(t)x\big) \leq \max\Big\{\sigma\big(a\big(\beta(t_0)\|x_0\|_r\big), t-t_0\big), \sup_{t_0 \leq \tau \leq t} \sigma\big(\zeta\big(\delta(\tau)\big|u(\tau)\big|\big), t-\tau\big)\Big\} \quad (4.202)$$

Finally,

1. *If there exists a function* a_1 *of class* K_∞ *such that* $a_1(\|H(t,x)\|_{\mathcal{Y}}) \leq V(t,x)$ *for all* $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, *then system* (1.10) *satisfies the WIOS property from the input* $u \in M_U$ *with gain* $\gamma(s) := a_1^{-1}(\zeta(s))$ *and weight* δ. *Moreover, if, in addition,* $\beta(t) \equiv 1$, *then system* (1.10) *satisfies the UWIOS property from the input* $u \in M_U$ *with gain* $\gamma(s) := a_1^{-1}(\zeta(s))$ *and weight* δ.
2. *If* $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ *is equivalent to the finite-dimensional continuous mapping* $h : [-r,+\infty) \times \Re^n \to \Re^p$ *and there exist functions* a_1, a_2 *of class* K_∞ *such that* $a_1(|h(t,x(0))|) \leq V(t,x)$, $\|H(t,x)\|_{\mathcal{Y}} \leq a_2(\sup_{\theta\in[-r,0]} |h(t+\theta, x(\theta))|)$ *for all* $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, *then system* (1.10) *satisfies the WIOS property from the input* $u \in M_U$ *with gain* $\gamma(s) := a_2(a_1^{-1}(\zeta(s)))$ *and weight* δ.

Proof of Theorem 4.4 We prove implications (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), (d) $\Rightarrow$ (e), (e) $\Rightarrow$ (a). The equivalence between (f) and (b) is a direct consequence of Theorem 4.1.

(a) $\Rightarrow$ (b): Suppose that there exist functions $\sigma \in KL$, $\beta, \phi \in K^+$, and $\rho \in K_\infty$ such that the estimate (4.186) holds for all $(d,u) \in M_D \times M_U$, $(t_0, x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$, and $t \geq t_0$. If we set $\gamma(s) := \sigma(\rho(s), 0)$, the desired (4.1) is a consequence of (4.186) and the previous definition. Thus statement (b) holds if (1.10) is RFC from the input $u \in M_U$. If the hypothesis that (1.10) is RFC from the input $u \in M_U$ is not included in statement (a), then there exist functions $a \in K_\infty$ and $\mu, R \in K^+$ such that $a(\mu(t)|x(0)|) \leq \|H(t,x)\|_{\mathcal{Y}} + R(t)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$. It follows from (4.186) and previous definitions that for all $(t_0, x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ and $(d,u) \in M_D \times M_U$, the corresponding solution $x(t)$ of (1.10) with $T_r(t_0)x = x_0$ satisfies the following estimate for all $t \geq t_0$:

$$a\big(\mu(t)\big|x(t)\big|\big) \leq R(t) + \max\Big\{\sigma\big(\beta(t_0)\|x_0\|_r, t-t_0\big), \sup_{t_0 \leq \tau \leq t} \gamma\big(\delta(\tau)\big|u(\tau)\big|\big)\Big\}$$

The above estimate in conjunction with Definition 1.4 implies that (1.10) is RFC from the input $u \in M_U$.

(b) $\Rightarrow$ (c): By virtue of Corollary 4.1 and Hypothesis (S7), it follows that the system

$$\dot{x}(t) = f\big(t, T_r(t)x, v(t)d'(t), d(t)\big) \qquad Y(t) = \tilde{H}\big(t, T_r(t)x\big) \quad (4.203)$$

where $\Delta := B_U[0,1] \times D$, $\tilde{H}(t,x) := (H(t,x), \mu(t)x) \in \mathcal{Y} \times C^0([-r,0];\Re^n)$, and $v \in M_{\Re^+}$ satisfies the WIOS property from the input $v \in M_{\Re^+}$. More specifically, there exist functions $\mu, \beta, \delta \in K^+$, $\tilde{\sigma} \in KL$, and $\tilde{\gamma} \in \mathcal{N}$ such that (4.42) holds for the solutions of (4.203) for all $v \in M_{\Re^+}$, $(t_0, x_0, (d', d)) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_\Delta$, and $t \geq t_0$. Without loss of generality we may assume that $\tilde{\gamma} \in K_\infty$. By virtue of Lemma 3.2 we obtain $\kappa \in K_\infty$ such that $\tilde{\gamma}(rs) \leq \kappa(r)\kappa(s)$ for all $(r,s) \in (\Re^+)^2$. Let $\phi(t) := \delta(t)\kappa^{-1}(\frac{\mu(t)}{2})$, and let $\theta \in K_\infty$ be a locally Lipschitz function that satisfies $\theta(s) \leq \kappa^{-1}(s)$ for all $s \geq 0$. The previous definitions imply that

$$\frac{\theta(\|x\|_r)}{\phi(t)} \leq \frac{1}{\delta(t)}\tilde{\gamma}^{-1}\left(\frac{\mu(t)}{2}\|x\|_r\right) \quad \forall (t,x) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \tag{4.204}$$

Finally, the reader should notice that system (4.187) is the feedback interconnection of system (4.203) with the static map $v(t) := \frac{\theta(\|T_r(t)\|_r)}{\phi(t)}$. The result follows from Proposition 4.3 and inequality (4.204).

(c) ⇒ (d): Suppose that (4.187) is RGAOS. Theorem 3.6 implies that there exists a continuous mapping $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \to V(t,x) \in \Re^+$, which is almost Lipschitz on bounded sets, with the following properties:

– there exist functions $a_1, a_2 \in K_\infty$ and $\beta \in K^+$ such that

$$a_1\big(\big\|H(t,x)\big\|_{\mathcal{Y}} + \mu(t)\|x\|_r\big) \leq V(t,x) \leq a_2\big(\beta(t)\|x\|_r\big)$$
$$\forall (t,x) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \tag{4.205}$$

– it holds that

$$V^0\left(t,x; f\left(t,x,\frac{\theta(\|x\|_r)}{\phi(t)}d', d\right)\right) \leq -V(t,x)$$
$$\forall \big(t,x,d',d\big) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \times \Delta \tag{4.206}$$

Notice that inequality (4.206) implies the following inequality:

$$V^0\big(t,x; f(t,x,u,d)\big) \leq -V(t,x)$$
$$\forall (t,x,u,d) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \times U \times D \text{ with } \phi(t)|u| \leq \theta\big(\|x\|_r\big) \tag{4.207}$$

Using Property (P1) of Definition 2.4 for the continuous mapping $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \to V(t,x) \in \Re^+$, we obtain, for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$,

$$\big|V^0\big(t,x; f(t,x,u,d)\big) - V^0\big(t,x; f(t,x,0,d)\big)\big|$$
$$\leq M\big(t + \|x\|_r + 1\big)\big|f(t,x,u,d) - f(t,x,0,d)\big|$$

The above inequality, together with (4.185), implies that the following inequality holds for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$:

$$\big|V^0\big(t,x; f(t,x,u,d)\big) - V^0\big(t,x; f(t,x,0,d)\big)\big|$$
$$\leq M\big(t + \|x\|_r + 1\big)L_U\big(t, \|x\|_r + |u|\big)|u| \tag{4.208}$$

Define

$$\psi(t,s) := \sup\big\{M\big(t + \|x\|_r + 1\big)L_U\big(t, \|x\|_r + |u|\big)|u|;$$
$$\|x\|_r \leq \theta^{-1}\big(\phi(t)s\big), |u| \leq s\big\} \tag{4.209}$$

Without loss of generality we may assume that the function $\phi \in K^+$ is nondecreasing. Clearly, $\psi : \Re^+ \times \Re^+ \to \Re^+$ is a mapping with $\psi(t,0) = 0$ for all $t \geq 0$, such that

(i) for each fixed $t \geq 0$, the mapping $\psi(t,\cdot)$ is nondecreasing;
(ii) for each fixed $s \geq 0$, the mapping $\psi(\cdot,s)$ is nondecreasing; and
(iii) $\lim_{s\to 0^+} \psi(t,s) = 0$ for all $t \geq 0$.

Hence, by employing Lemma 2.3 we obtain functions $a_3 \in K_\infty$ and $\delta \in K^+$ such that $\psi(t,s) \leq a_3(\delta(t)s)$.

We next establish inequality (4.189), with a_3 as previously, by considering the following two cases:

- $\theta^{-1}(\phi(t)|u|) \leq \|x\|_r$. In this case inequality (4.189) is a direct consequence of (4.207).
- $\theta^{-1}(\phi(t)|u|) \geq \|x\|_r$. In this case, by virtue of inequalities (4.207), (4.208), definition (4.209), and the definition of a_3, we have

$$\begin{aligned} V^0\big(t,x; f(t,x,u,d)\big) &\leq V^0\big(t,x; f(t,x,0,d)\big) + \psi\big(t,|u|\big) \\ &\leq -V(t,x) + a_3\big(\delta(t)|u|\big) \end{aligned}$$

which implies (4.189).

(d) $\Rightarrow$ (e): Notice that (4.189) implies (4.191) with $\zeta(s) := 2a_3(s)$ and $\rho(s) := \frac{1}{2}s$. The fact that system (1.10) is RFC follows directly from Theorem 4.6.

(e) $\Rightarrow$ (a): Theorem 4.6 implies that system (1.10) is RFC from the input $u \in M_U$ and that (4.202) holds. Next, we distinguish the following cases:

1. If (4.190) holds, then (4.186) is a direct consequence of (4.202) and (4.190).
2. If (4.192) holds, then (4.202) implies the following estimate for all $t \geq t_0$:

$$\begin{aligned} \big|h\big(t,x(t)\big)\big| \leq \max\Big\{ & a_1^{-1}\big(\sigma\big(a_2\big(\beta(t_0)\|x_0\|_r\big), t-t_0\big)\big), \\ & \sup_{t_0\leq s\leq t} a_1^{-1}\big(\sigma\big(\zeta\big(\delta(s)|u(s)|\big), t-s\big)\big)\Big\} \end{aligned}$$

Since $h : [-r,+\infty) \times \Re^n \to \Re^p$ is continuous with $h(t,0) = 0$ for all $t \geq -r$, it follows from Lemma 2.3 that there exist functions $p \in K_\infty$ and $\phi \in K^+$ such that

$$\big|h(t-r,x)\big| \leq p\big(\phi(t)|x|\big) \quad \forall (t,x) \in \Re^+ \times \Re^n$$

Combining the two previous inequalities, we obtain, for all $t \geq t_0$,

$$\begin{aligned} & \sup_{\theta\in[-r,0]} \big|h\big(t+\theta, x(t+\theta)\big)\big| \\ & \quad \leq \max\Big\{\omega\big(q(t_0)\|x_0\|_r, t-t_0\big), \sup_{t_0\leq s\leq t} \omega\big(\zeta\big(\delta(s)\big|u(s)\big|\big), t-s\big)\Big\} \end{aligned}$$

where $q(t) := \beta(t) + \max_{t\leq\tau\leq t+r}\phi(\tau)$ $\omega(s,t) := \max\{p(s), a_1^{-1}(\sigma(s+a_2(s),0))\}$ for $t \in [0,r)$, and $\omega(s,t) := \max\{\exp(r-t)p(s), a_1^{-1}(\sigma(s+a_2(s),t-r))\}$ for $t \geq r$. The above estimate, in conjunction with the fact that $H : \Re^+ \times C^0([-r,0];\Re^n) \to$

$\mathcal{Y}$ is equivalent to the finite-dimensional mapping h, shows that (1.10) satisfies inequality (4.186).

The proof is complete. □

Proof of Theorem 4.5 The proofs of implications (a) ⇒ (b), (d) ⇒ (e), and (e) ⇒ (a) follow the same methodology as in the proof of Theorem 4.4. Particularly, in the proof of implication (e) ⇒ (a), we use in addition the fact that since $h : [-r, +\infty) \times \Re^n \to \Re^p$ is continuous and T-periodic with $h(t, 0) = 0$ for all $t \geq -r$, it follows from Lemma 2.4 that there exists a function $p \in K_\infty$ such that

$$\big|h(t - r, x)\big| \leq p\big(|x|\big) \quad \forall (t, x) \in \Re^+ \times \Re^n$$

The proof of implication (c) ⇒ (d) differs from the corresponding proof in Theorem 4.4 in the definition of ψ. Specifically, we first notice that the fact that V is T-periodic implies that $V^0(t, x; f(t, x, u, d))$ is T-periodic. Using Property (P1) of Definition 2.4 for the continuous mapping $(t, x) \in \Re^+ \times C^0([-r, 0]; \Re^n) \to V(t, x) \in \Re^+$, we obtain, for all $(t, x, u, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times D$,

$$\begin{aligned} &\big|V^0\big(t, x; f(t, x, u, d)\big) - V^0\big(t, x; f(t, x, 0, d)\big)\big| \\ &\quad \leq M\big(T + \|x\|_r + 1\big)\big|f(t, x, u, d) - f(t, x, 0, d)\big| \end{aligned}$$

for certain nondecreasing function $M : \Re^+ \to \Re^+$. The above inequality, in conjunction with (4.185) and the fact that f is T-periodic, implies that the following inequality holds for all $(t, x, u, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times D$:

$$\begin{aligned} &\big|V^0\big(t, x; f(t, x, u, d)\big) - V^0\big(t, x; f(t, x, 0, d)\big)\big| \\ &\quad \leq M\big(T + \|x\|_r + 1\big)L_U\big(T, \|x\|_r + |u|\big)|u| \end{aligned}$$

We next define

$$\begin{aligned} \psi(s) := \sup\big\{&M\big(T + \|x\|_r + 1\big)L_U\big(T, \|x\|_r + |u|\big)|u|; \\ &\|x\|_r \leq R + a\big(\theta^{-1}(s)\big), |u| \leq s\big\} \end{aligned}$$

Notice that Hypothesis (S8) implies that $\psi(s) \leq a_3(s)$ for all $s \geq 0$, where $a_3(s) := \tilde{M}(T + R + 1 + a(\theta^{-1}(s)))L_U(T, R + a(\theta^{-1}(s)) + s)s$, $R \geq 0$ is the constant involved in Hypothesis (S8), $a \in K_\infty$ is the function involved in Hypothesis (S8) and $\tilde{M}(s)$ is a continuous positive function which satisfies $\tilde{M}(s) \geq M(s)$ for all $s \geq 0$. From this point the proof of implication (c) ⇒ (d) is exactly the same as in Theorem 4.4 (i.e., by distinguishing the cases $\theta^{-1}(|u|) \leq \|H(t, x)\|_{\mathcal{Y}}$ and $\theta^{-1}(|u|) \geq \|H(t, x)\|_{\mathcal{Y}}$).

Finally, we continue with the proof of implication (b) ⇒ (c). By virtue of Hypothesis (S7), it follows that the system

$$\dot{x}(t) = f\big(t, x(t), v(t)d'(t), d(t)\big) \qquad Y(t) = H\big(t, x(t)\big) \tag{4.210}$$

where $\Delta := B_U[0, 1] \times D$ and $v \in M_{\Re^+}$, satisfies the UIOS property from the input $v \in M_{\Re^+}$. More specifically, there exist functions $\sigma \in KL$ and $\gamma \in \mathcal{N}$ such that (4.2) holds with $\beta(t) \equiv 1$ for the solutions of (4.210) for all $v \in M_{\Re^+}$,

$(t_0, x_0, (d', d)) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times M_\Delta$, and $t \geq t_0$. Without loss of generality we may assume that $\gamma \in K_\infty$. Let $\theta \in K_\infty$ be a locally Lipschitz function that satisfies $\theta(s) \leq \gamma^{-1}(\frac{1}{2}s)$ for all $s \geq 0$. The previous definitions imply that

$$\theta\big(\|H(t,x)\|_{\mathcal{Y}}\big) \leq \gamma^{-1}\left(\frac{1}{2}\|H(t,x)\|_{\mathcal{Y}}\right) \quad \forall (t,x) \in \Re^+ \times C^0\big([-r,0]; \Re^n\big) \tag{4.211}$$

Finally, noticing that system (4.194) is the feedback interconnection of system (4.210) with the static map $v(t) := \theta(\|H(t, T_r(t)x)\|_{\mathcal{Y}})$, the result follows from Proposition 4.4 and inequality (4.211).

The proof is complete. □

Proof of Theorem 4.6 Consider a solution of (1.10) under Hypotheses (S1–7) corresponding to arbitrary $(u, d) \in M_U \times M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \Re^n)$. By Lemma 2.17, for every $T \in (t_0, t_{\max})$, the mapping $[t_0, T] \ni t \to V(t, T_r(t)x)$ is absolutely continuous. It follows from (4.201) and Lemma 2.16 that there exists a set $I \subset [t_0, T]$ of zero Lebesgue measure such that the following implication holds for all $t \in [t_0, T] \backslash I$:

$$V\big(t, T_r(t)x\big) \geq \zeta\big(\delta(t)\big|u(t)\big|\big) \quad \Rightarrow \quad \frac{d}{dt}\big(V\big(t, T_r(t)x\big)\big) \leq -\rho\big(V\big(t, T_r(t)x\big)\big)$$

Lemma 2.14 implies the existence of a continuous function σ of class KL, with $\sigma(s, 0) = s$ for all $s \geq 0$ such that

$$V\big(t, T_r(t)x\big) \leq \max\Big\{\sigma\big(V\big(t_0, T_r(t_0)x\big), t - t_0\big), \sup_{t_0 \leq s \leq t} \sigma\big(\zeta\big(\delta(s)\big|u(s)\big|\big), t - s\big)\Big\} \quad \forall t \in [t_0, T] \tag{4.212}$$

with $T \in (t_0, t_{\max})$. Notice that for the case where (1.10) is RFC from the input $u \in M_U$, $t_{\max} = +\infty$. For the case that there exist functions $p \in K_\infty$ and $\mu, R \in K^+$ such that $p(\mu(t)|x(0)|) \leq V(t, x) + R(t)$ for all $(t, x) \in \Re^+ \times C^0([-r, 0]; \Re^n)$, combining the previous inequality and (4.212), we obtain, for every $T \in (t_0, t_{\max})$,

$$\begin{aligned} &p\big(\mu(t)\big|x(t)\big|\big) \\ &\quad \leq R(t) + \max\Big\{\sigma\big(V\big(t_0, T_r(t_0)x\big), t - t_0\big), \sup_{t_0 \leq s \leq t} \sigma\big(\zeta\big(\delta(s)\big|u(s)\big|\big), t - s\big)\Big\} \\ &\quad \forall t \in [t_0, T] \end{aligned} \tag{4.213}$$

It follows from estimate (4.213) that $t_{\max} = +\infty$. An immediate consequence of Lemma 2.18 is that estimate (4.212) holds for all $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times M_D \times M_U$ and $t \geq t_0$. Moreover, if there exist functions $p \in K_\infty$ and $\mu, R \in K^+$ such that $p(\mu(t)|x(0)|) \leq V(t, x) + R(t)$ for all $(t, x) \in \Re^+ \times C^0([-r, 0]; \Re^n)$, then Lemma 2.18 implies that (4.213) also holds for all $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times M_D \times M_U$ and $t \geq t_0$. In this case the fact that system (1.10) is RFC from the input $u \in M_U$ is a consequence of (4.213) and Definition 1.4. Notice that (4.202) is a consequence of (4.212) and (4.200). Finally, (i) and (ii) are direct consequences of (4.202). The proof is complete. □

The following example presents an autonomous time-delay system which satisfies the UWIOS property and does not satisfy the UIOS property. The analysis is performed with the help of Theorems 4.4 and 4.6.

Example 4.7.2 Consider the following autonomous time-delay system:

$$\begin{aligned}
&\dot{x}_1(t) = d(t)x_1(t)\\
&\dot{x}_2(t) = -x_2(t) + x_1(t-r)u(t)\\
&Y(t) = x_2(t)\\
&\qquad \big(x_1(t), x_2(t)\big)' \in \Re^2, d(t) \in D := [-1, 1], u(t) \in U := \Re, Y(t) \in \Re
\end{aligned} \tag{4.214}$$

Consider the functional:

$$\begin{aligned}
V(t, x_1, x_2) &:= \exp(-8t)x_1^4(0)\\
&\quad + \exp(-4t)x_1^2(0) + \frac{1}{2}x_2^2(0) + \frac{1}{4}\exp(-8t)\int_{-r}^{0} x_1^4(s)\,ds
\end{aligned} \tag{4.215}$$

First notice that the functional $V : \Re^+ \times C^0([-r, 0]; \Re^2) \to \Re^+$ defined by (4.215) is almost Lipschitz on bounded sets. Moreover, inequalities (4.190), (4.200) are satisfied for this functional with $a(s) = a_2(s) := (1 + \frac{r}{4})s^4 + s^2$, $a_1(s) := \frac{1}{2}s^2$, and $\beta(t) \equiv 1$. We next estimate an upper bound for the Dini derivative of the functional $V : \Re^+ \times C^0([-r, 0]; \Re^2) \to \Re^+$ along the solutions of system (4.214). We have, for all $(t, x, u, d) \in \Re^+ \times C^0([-r, 0]; \Re^2) \times \Re \times [-1, 1]$,

$$\begin{aligned}
&V^0\big(t, x_1, x_2; \big(dx_1(0), -x_2(0) + x_1(-r)u\big)\big)\\
&\quad = -8\exp(-8t)x_1^4(0) + 4d\exp(-8t)x_1^4(0) - 4\exp(-4t)x_1^2(0)\\
&\qquad + 2d\exp(-4t)x_1^2(0) - x_2^2(0) + x_2(0)x_1(-r)u\\
&\qquad - 2\exp(-8t)\int_{-r}^{0} x_1^4(s)\,ds + \frac{1}{4}\exp(-8t)x_1^4(0) - \frac{1}{4}\exp(-8t)x_1^4(-r)
\end{aligned}$$

Using the inequalities $|d| \le 1$, $x_2(0)x_1(-r)u \le \frac{1}{2}x_2^2(0) + \frac{1}{2}x_1^2(-r)u^2$, and $\frac{1}{2}x_1^2(-r)u^2 \le \frac{1}{4}\exp(-8t)x_1^4(-r) + \frac{1}{4}\exp(8t)u^4$, we are in a position to estimate, for all $(t, x, u, d) \in \Re^+ \times C^0([-r, 0]; \Re^2) \times \Re \times [-1, 1]$,

$$\begin{aligned}
&V^0\big(t, x_1, x_2; \big(dx_1(0), -x_2(0) + x_1(-r)u\big)\big)\\
&\quad \le -3\exp(-8t)x_1^4(0) - 2\exp(-4t)x_1^2(0) - \frac{1}{2}x_2^2(0)\\
&\qquad + \frac{1}{4}\exp(8t)u^4 - 2\exp(-8t)\int_{-r}^{0} x_1^4(s)\,ds
\end{aligned}$$

Finally, using the above inequality and definition (4.215), we obtain, for all $(t, x, u, d) \in \Re^+ \times C^0([-r, 0]; \Re^2) \times \Re \times [-1, 1]$,

$$\begin{aligned}
&V^0\big(t, x_1, x_2; \big(dx_1(0), -x_2(0) + x_1(-r)u\big)\big)\\
&\quad \le -V(t, x_1, x_2) + \frac{1}{4}\exp(8t)u^4
\end{aligned} \tag{4.216}$$

Inequality (4.216) guarantees that (4.191) and (4.201) hold with $\rho(s) := \frac{1}{2}s$, $\zeta(s) := \frac{1}{2}s^4$, and $\delta(t) := \exp(2t)$. Definition (4.215) guarantees that there exist functions $p \in K_\infty$ and $\mu \in K^+$ and a constant $R \geq 0$ such that $p(\mu(t)|x(0)|) \leq V(t,x) + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^2)$ (e.g., $p(s) := \frac{1}{2}s^2$, $\mu(t) := \exp(-2t)$, and $R := 0$). It follows from Theorem 4.4 (statement (e)) that system (4.214) satisfies the UWIOS property from the input $u \in M_U$. In order to be able to determine the gain and weight functions, we utilize the result of Theorem 4.6. Indeed, it follows from Theorem 4.6 that system (4.214) satisfies the UWIOS property from the input $u \in M_U$ with gain $\gamma(s) := a_1^{-1}(\zeta(s)) = s^2$ and weight $\delta(t) := \exp(2t)$.

It should be emphasized that system (4.214) does not satisfy the UIOS property from the input $u \in M_U$. This can be shown by considering the solution of (4.214) corresponding to inputs $d(t) \equiv 1$ and $u(t) \equiv 1$. It can be shown that, for $x_1(0) \neq 0$, the output of (4.214) is not bounded and satisfies $\lim_{t\to+\infty} |Y(t)| = +\infty$. Consequently, bounded inputs can produce unbounded outputs, which contradicts the requirements of the UIOS property from the input $u \in M_U$.

4.8 Bibliographical and Historical Notes

1. As remarked earlier, the ISS property was introduced for finite-dimensional systems described by ODEs in [22]. It was quickly generalized to ISpS (input-to-state practical stability) and IOpS (input-to-output practical stability) in [8] for finite-dimensional systems described by ODEs. The work [27, 28] presents equivalent characterizations of IOS. The WIOS and WISS properties were introduced in [13, 14] for a wide class of systems and were based on the work centered around "nonuniform in time IOS" and "nonuniform in time ISS" in [10–12, 15, 19]. As it is pointed out in [24], the IOS and ISS properties have been proved to be very useful in Mathematical Control Theory for a very important reason: it combines features of internal stability properties (Lyapunov stability looking at the effect of initial conditions for zero-input response) with features of external stability properties (input-output stability looking at the effect of nonzero inputs for zero-state response) proposed earlier in the literature; see the work of Zames, Willems, and others [24].
2. There is still another variation of the ISS property that was not examined in this chapter: the integral ISS (for short, iISS) property. This property was introduced in [23] for finite-dimensional systems described by ODEs and, roughly speaking, requires that there exist functions $\sigma \in KL$ and $\gamma \in \mathcal{N}$ such that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \Re^n \times M_D$, and $t \geq t_0$:

$$\left|x(t)\right| \leq \sigma\left(|x_0|, t - t_0\right) + \int_{t_0}^{t} \gamma\left(\left|u(\tau)\right|\right) d\tau \tag{4.217}$$

where $x(t)$ denotes the unique solution of (1.3) with initial condition $x(t_0) = x_0$ corresponding to inputs $(d,u) \in M_D \times M_U$. The iISS property cannot be easily generalized to other types of systems (notice that (4.217) implies that $u \in M_U$

is Lebesgue integrable which holds for finite-dimensional systems described by ODEs but not necessarily for other systems). Using Lemma 2.3, we are in a position to find functions $\delta \in K^+$ and $\tilde{\gamma} \in \mathcal{N}$ such that $\exp(t)\gamma(s) \leq \tilde{\gamma}(\delta(t)s)$ for all $t, s \geq 0$. Consequently, the integral on the right-hand side of (4.217) satisfies

$$\int_{t_0}^{t} \gamma\left(\left|u(\tau)\right|\right) d\tau = \int_{t_0}^{t} \exp(-\tau)\exp(\tau)\gamma\left(\left|u(\tau)\right|\right) d\tau$$
$$\leq \int_{t_0}^{t} \exp(-\tau)\tilde{\gamma}\left(\delta(\tau)\left|u(\tau)\right|\right) d\tau \leq \sup_{t_0 \leq \tau \leq t} \tilde{\gamma}\left(\delta(\tau)\left|u(\tau)\right|\right)$$

Therefore, combining (4.217) with the above inequality, we easily obtain that if the iISS property holds, then the WISS property holds as well with weight $\delta \in K^+$. The Lyapunov characterizations in Sect. 4.7.A show that, if the ISS property holds, then the iISS property necessarily holds (see also [2, 23]).

3. Various characterizations for the IOS and ISS properties for finite-dimensional systems described by ODEs were provided in [6, 25–28]. Among them, we should point out one characterization which was not given in the present chapter: the so-called "asymptotic gain" characterization.
4. Another variation for the ISS property which is not studied in this book is the "local ISS property". See [9, 17].
5. For finite-dimensional discrete-time systems the ISS property was studied in [7]. In [7] one can find various characterizations of the ISS property for discrete-time finite-dimensional systems.
6. For systems described by RFDEs, sufficient conditions for the ISS property were provided in [20, 29]. Necessary and sufficient conditions were presented in [16] (the results of Sect. 4.7.B are heavily based on the results in [16]).
7. As remarked earlier, estimates of the form (4.114) ("fading memory estimates") were first used by Praly and Wang [21] for the formulation of exp-ISS and by Grüne [3, 4] for the formulation of Input-to-State Dynamical Stability (ISDS) with $H(t, x) = x$ and $\beta(t) \equiv \gamma(t) \equiv 1$, which was proved to be qualitatively equivalent with (4.2) for finite-dimensional systems described by ODEs. It is clear that Theorem 4.1 shows that for the WIOS property, the "fading memory" estimate (4.114) is qualitatively equivalent to the "Sontag-like" estimate (4.1). Applications of such "fading memory" estimates can be found in robust and adaptive nonlinear control; see Jiang and Praly (1998).
8. Lemmas 4.3 and 4.4 should be compared with Lemma 1.1, p. 131, in [5]. The results of Sects. 4.4 and 4.6 of the present chapter are heavily based on (and slightly generalize) the results in [11, 13]. The invariance of the external stability properties to system transformations was discussed in [24].
9. Lyapunov-like characterizations for forward completeness of finite-dimensional systems described by ODEs were provided in [1]. It should be noted that certain results in [18] imply that forward completeness is RFC, and consequently, all characterizations given in Sect. 4.7.A hold as well.

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Chapter 5
Advanced Stability Methods and Applications

5.1 Introduction

The present chapter is devoted to the description of advanced analysis methods for checking internal and external global stability properties introduced previously for various important classes of nonlinear dynamic systems. We will particularly focus attention on methods based upon

- small-gain techniques and
- vector Lyapunov functionals.

The method of proving stability by means of vector Lyapunov functionals is an extension of the method of Lyapunov functionals described in previous chapters and has a long history. However, proving stability by means of small-gain results is a method radically different from all other methods of proving stability presented previously. The obtained small-gain results permit the derivation of novel vector Lyapunov characterizations of global stability notions. A special feature of small-gain results is the utilization of estimates of the solutions: these estimates must be proved by means of other methods (e.g., by means of Lyapunov methods or comparison principles or analytical solutions). Another important feature of the small-gain methods is that they can easily handle both static and dynamic uncertainties. In Chap. 6, we will see that small-gain techniques play a tremendous role in the design of nonlinear controllers with guaranteed robustness to static, parametric, input, and state dynamic uncertainties.

An additional feature of the methods presented in this chapter is the capability of dealing with large-scale systems. Large-scale systems are encountered frequently in mathematical biology, engineering, and mathematical economics. Moreover, it is known that (generally) it is more difficult to establish global stability notions for large-scale systems than for systems described by a small number of (differential or difference) equations. The stability analysis methods presented in this chapter can exploit the structure of a given large-scale system in order to derive useful estimates for the solutions.

Notice that sufficient conditions for global stability notions are developed in the system-theoretic framework described in Chap. 1.

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In what follows, $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ will be a control system with the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Moreover, $u_0 \in M_U$ will be the identically zero input, i.e., $u_0(t) = 0 \in U$ for all $t \geq 0$. For the output map $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$, we assume that either $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ is continuous or that there exists a partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\Re^+$ with diameter $r > 0$ such that $H : \Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ satisfies Hypothesis (L2) in Sect. 1.7 of Chap. 1.

5.2 The Small-Gain Theorem

The main purpose of this section is to state and prove our main small-gain results. In order to develop the small-gain toolkit, we begin with certain notions of monotone operators. The background materials on the theory of monotone operators are first provided.

5.2.1 Results on Monotone Operators

The following technical definitions were used in [27] and are needed here.

Definition 5.1 Let $x = (x_1, \ldots, x_n)' \in \Re^n$ and $y = (y_1, \ldots, y_n)' \in \Re^n$. We define $z = \text{MAX}\{x, y\}$, where $z = (z_1, \ldots, z_n)' \in \Re^n$ satisfies $z_i = \max\{x_i, y_i\}$ for $i = 1, \ldots, n$. Similarly, for $u_1, \ldots, u_m \in \Re^n$, $z = \text{MAX}\{u_1, \ldots, u_m\}$ is a vector $z = (z_1, \ldots, z_n)' \in \Re^n$ with $z_i = \max\{u_{1i}, \ldots, u_{mi}\}$, $i = 1, \ldots, n$.

Definition 5.2 We say that $\Gamma : \Re^n_+ \to \Re^n_+$ is *MAX-preserving* if $\Gamma : \Re^n_+ \to \Re^n_+$ is nondecreasing and for every $x, y \in \Re^n_+$, the following equality holds:

$$\Gamma\big(\text{MAX}\{x, y\}\big) = \text{MAX}\big\{\Gamma(x), \Gamma(y)\big\} \tag{5.1}$$

The above defined MAX-preserving maps enjoy the following important property (see [27]).

Proposition 5.1 *$\Gamma : \Re^n_+ \to \Re^n_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ is MAX-preserving if and only if there exist nondecreasing functions $\gamma_{i,j} : \Re^+ \to \Re^+$, $i, j = 1, \ldots, n$, with $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$ for all $x \in \Re^n_+$, $i = 1, \ldots, n$.*

Proof Define $\gamma_{i,j}(s) := \Gamma_i(se_j)$ for all $s \geq 0$, where $\{e_i\}_{i=1}^n$ denotes the standard basis of $\Re^n$. Let $x \in \Re^n_+$, i.e., $x = x_1e_1 + \cdots + x_ne_n$ with $x_i \geq 0$, $i = 1, \ldots, n$. Notice that $x = \text{MAX}\{x_1e_1, \ldots, x_ne_n\}$ and consequently $\Gamma(x) = \text{MAX}\{\Gamma(x_1e_1), \ldots, \Gamma(x_ne_n)\}$. Therefore, $\Gamma_i(x) = \max\{\Gamma_i(x_1e_1), \ldots, \Gamma_i(x_ne_n)\} = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$. The converse statement is a direct consequence of the definition $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$. □

The following class of MAX-preserving mappings plays an important role in what follows.

Definition 5.3 Let $\Gamma : \Re^n_+ \to \Re^n_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ be a MAX-preserving mapping for which there exist functions $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, n$, such that, for each $i = 1, 2, \ldots, n$, $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$ for all $x \in \Re^n_+$. We say that $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions if the following inequalities hold:

$$\gamma_{i,i}(s) < s \quad \text{for all } s > 0, i = 1, \ldots, n \tag{5.2}$$

and if $n > 1$, then for each $r = 2, \ldots, n$, it holds that

$$(\gamma_{i_1,i_2} \circ \gamma_{i_2,i_3} \circ \cdots \circ \gamma_{i_r,i_1})(s) < s \quad \text{for all } s > 0 \tag{5.3}$$

for all $i_j \in \{1, \ldots, n\}$, $i_j \neq i_k$ if $j \neq k$.

The following proposition provides useful characterizations for MAX-preserving mappings which satisfy the cyclic small-gain conditions.

Proposition 5.2 *Suppose that $\Gamma : \Re^n_+ \to \Re^n_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ is MAX-preserving and there exist functions $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, n$, with $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$, $i = 1, \ldots, n$. The following statements are equivalent*:

(i) $\lim_{k\to\infty} \Gamma^{(k)}(x) = 0$ *for all* $x \in \Re^n_+$.
(ii) $\Gamma : \Re^n_+ \to \Re^n_+$ *satisfies the cyclic small-gain conditions*.
(iii) *The following implication holds*: $\Gamma(x) \geq x \Rightarrow x = 0$.
(iv) (iii) *holds, and for all* $k \geq 1$ *and* $x \in \Re^n_+$, *it holds that* $\Gamma^{(k)}(x) \leq Q(x) = \mathrm{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\}$.

Proof We prove implications (iv) ⇒ (i), (i) ⇒ (iii), (iii) ⇒ (ii), and (ii) ⇒ (iv).

(iv) ⇒ (i): Since $\Gamma : \Re^n_+ \to \Re^n_+$ is MAX-preserving, we have

$$\Gamma\big(Q(x)\big) = \mathrm{MAX}\big\{\Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n)}(x)\big\} \quad \text{for all } x \in \Re^n_+.$$

Moreover, since $\Gamma^{(k)}(x) \leq Q(x)$ for all $k \geq 1$ and $x \in \Re^n_+$, it follows that $\Gamma(Q(x)) \leq Q(x)$ for all $x \in \Re^n_+$. Therefore, using induction, it follows that $0 \leq \Gamma^{(k+1)}(Q(x)) \leq \Gamma^{(k)}(Q(x))$ for all $k \geq 0$ and $x \in \Re^n_+$. It follows that the limit $\lim_{k\to\infty} \Gamma^{(k)}(Q(x))$ exists and satisfies $q := \lim_{k\to\infty} \Gamma^{(k)}(Q(x)) \geq 0$. The continuity of the mapping $\Gamma : \Re^n_+ \to \Re^n_+$ shows that $\Gamma(q) = q$. By virtue of implication (iii) it follows that $q = 0$. Using the fact $x \leq Q(x)$ for all $x \in \Re^n_+$ (which implies $0 \leq \Gamma^{(k)}(x) \leq \Gamma^{(k)}(Q(x))$ for all $k \geq 1$ and $x \in \Re^n_+$), we obtain $\lim_{k\to\infty} \Gamma^{(k)}(x) = 0$ for all $x \in \Re^n_+$.

(i) ⇒ (iii): The proof of implication (iii) is made by contradiction. Suppose that there exists $x \in \Re^n_+$ with $x \neq 0$ such that $\Gamma(x) \geq x$. Using induction, we can show that $\Gamma^{(k)}(x) \geq x$ for all $k \geq 1$. Consequently, we must have $0 = \lim_{k\to\infty} \Gamma^{(k)}(x) \geq x$, which contradicts the assumption $x \neq 0$.

(iii) $\Rightarrow$ (ii): If there exist $s > 0$ and some integer $i = 1, \ldots, n$ such that $\gamma_{i,i}(s) \geq s$, then the nonzero vector $x \in \Re^n_+$ with $x_i = s$ and $x_j = 0$ for $j \neq i$ will violate (iii). Consequently, $\gamma_{i,i}(s) < s$ for all $s > 0$ and $i = 1, \ldots, n$.

Next suppose that $n > 1$. Suppose that there exist some $s > 0$, $r \in \{2, \ldots, n\}$, indices $i_j \in \{1, \ldots, n\}$, $j = 1, \ldots, r$, with $i_j \neq i_k$ if $j \neq k$ such that $(\gamma_{i_1,i_2} \circ \gamma_{i_2,i_3} \circ \cdots \circ \gamma_{i_r,i_1})(s) \geq s$. Without loss of generality we may assume that $i_j = j$ for $j = 1, \ldots, r$, and consequently $(\gamma_{1,2} \circ \gamma_{2,3} \circ \cdots \circ \gamma_{r,1})(s) \geq s$. The nonzero vector $x \in \Re^n_+$ with $x_1 = s$, $x_j = (\gamma_{j,j+1} \circ \gamma_{j+1,j+2} \circ \cdots \circ \gamma_{r,1})(s)$ for $j = 2, \ldots, r$, and $x_j = 0$ for $j > r$ satisfies $\Gamma(x) \geq x$, and consequently hypothesis (iii) is violated. Therefore (ii) must hold.

(ii) $\Rightarrow$ (iv): The proof of this implication is a direct consequence of the fact that

$$\Gamma_i^{(k)}(x) = \max\bigl\{(\gamma_{i,j_1} \circ \gamma_{j_1,j_2} \circ \cdots \circ \gamma_{j_{k-1},j_k})(x_{j_k}) : (j_1, \ldots, j_k) \in \{1, \ldots, n\}^k\bigr\}$$

for all $k \geq 1$, $x \in \Re^n_+$, and $i = 1, \ldots, n$. Using (ii), it can be shown that $\Gamma^{(n)}(x) \leq Q(x) = \mathrm{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\}$ for all $x \in \Re^n_+$. Since $\Gamma : \Re^n_+ \to \Re^n_+$ is MAX-preserving, we have $\Gamma(Q(x)) = \mathrm{MAX}\{\Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n)}(x)\}$ for all $x \in \Re^n_+$. As a result, we obtain $\Gamma(Q(x)) \leq Q(x)$ for all $x \in \Re^n_+$. By induction, it follows that $\Gamma^{(k)}(Q(x)) \leq Q(x)$ for all $k \geq 1$ and $x \in \Re^n_+$. Since $x \leq Q(x)$, we obtain $\Gamma^{(k)}(x) \leq Q(x)$ for all $k \geq 1$ and $x \in \Re^n_+$.

The fact that implication (iii) holds is shown by contradiction. Suppose that there exists a nonzero $x \in \Re^n_+$ with $\Gamma(x) \geq x$. Consequently, for every $i \in \{1, \ldots, n\}$, there exists $p(i) \in \{1, \ldots, n\}$ with $\gamma_{i,p(i)}(x_{p(i)}) \geq x_i$. With these inequalities in mind, there are at least one $i \in \{1, \ldots, n\}$ with $x_i > 0$ and a closed cycle $(i, j_1, \ldots, j_r, i)$ such that $(\gamma_{i,j_1} \circ \gamma_{j_1,j_2} \circ \cdots \circ \gamma_{j_r,i})(x_i) \geq x_i$, which contradicts (ii). Therefore the implication (ii) $\Rightarrow$ (iv) holds.

The proof is thus completed. □

The following proposition provides a useful algebraic implication.

Proposition 5.3 *Suppose that $\Gamma : \Re^n_+ \to \Re^n_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ is MAX-preserving and there exist functions $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, n$, with $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$, $i = 1, \ldots, n$. Moreover, suppose that $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions and that $x \leq \mathrm{MAX}\{a, \Gamma(x)\}$ for certain $x, a \in \Re^n_+$. Then $x \leq Q(a)$, where $Q(a) = \mathrm{MAX}\{a, \Gamma(a), \Gamma^{(2)}(a), \ldots, \Gamma^{(n-1)}(a)\}$.*

Proof Suppose that $x \leq \mathrm{MAX}\{a, \Gamma(x)\}$. Then $\Gamma(x) \leq \mathrm{MAX}\{\Gamma(a), \Gamma^{(2)}(x)\}$ and $x \leq \mathrm{MAX}\{a, \Gamma(a), \Gamma^{(2)}(x)\}$. By an induction argument,

$$x \leq \mathrm{MAX}\bigl\{a, \Gamma(a), \ldots, \Gamma^{(k)}(a), \Gamma^{(k+1)}(x)\bigr\} \quad \text{for all } k \geq 1.$$

It follows from statement (iv) of Proposition 5.2 that $x \leq \mathrm{MAX}\{Q(a), \Gamma^{(k+1)}(x)\}$ for all $k \geq 1$. Since $\lim_{k\to\infty} \Gamma^{(k)}(x) = 0$, we obtain $x \leq Q(a)$. □

We end this paragraph by providing a list of useful facts that are consequences of the above definitions and propositions. The following facts will be used in the rest of the chapter.

Fact I *If $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions, then* $\lim_{k\to+\infty} \Gamma^{(k)}(x) = 0$ *for all* $x \in \Re^n_+$, *and* $\Gamma^{(k)}(x) \le Q(x)$ *for all* $k \ge 1$ *and* $x \in \Re^n_+$. *Recall that* $Q(x) = \mathrm{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\}$.

Fact II *If $\Gamma : \Re^n_+ \to \Re^n_+$ is a MAX-preserving mapping, then the mapping $Q(\cdot)$ is a MAX-preserving mapping.*

Fact III *If $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions, then* $\Gamma(Q(x)) \le Q(x)$ *and* $Q(x) \ge x$ *for all* $x \in \Re^n_+$.

Fact IV *If $p \in \mathcal{N}_n$ and $R : \Re^n_+ \to \Re^n_+$ are nondecreasing mappings, then the following inequality holds for all $s, r \in \Re^+$:* $p(\mathrm{MAX}\{R(1s), R(1r)\}) = \max(p(R(1s)), p(R(1r)))$.

Fact V *If $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions and $x, y \in \Re^n_+$ satisfy* $x \le \mathrm{MAX}\{y, \Gamma(x)\}$, *then* $x \le Q(y)$.

5.2.2 Trajectory-Based Small-Gain Theorems

We consider an abstract control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. We suppose that there exists a set-valued map $\Re^+ \ni t \to S(t) \subseteq \mathcal{X}$ with $0 \in S(t)$ for all $t \ge 0$, mappings $V_i : \bigcup_{t\ge 0}\{t\} \times S(t) \times U \to \Re^+$ $(i = 1, \ldots, n)$, $L : \bigcup_{t\ge 0}\{\{t\} \times S(t)\} \to \Re^+$ with $L(t, 0) = 0$, $V_i(t, 0, 0) = 0$ for all $t \ge 0$ $(i = 1, \ldots, n)$ and a MAX-preserving continuous map $\Gamma : \Re^n_+ \to \Re^n_+$ with $\Gamma(0) = 0$ such that the following hypotheses hold:

(SG1) (The "IOS-like" inequalities) There exist functions $\sigma \in KL$ and $\zeta \in \mathcal{N}_1$ such that for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ with $\phi(t, t_0, x_0, u, d) \in S(t)$ for all $t \in [t_0, t_{\max})$, the mappings $t \to V(t) = (V_1(t, \phi(t, t_0, x_0, u, d), u(t)), \ldots, V_n(t, \phi(t, t_0, x_0, u, d), u(t)))'$ and $t \to L(t) = L(t, \phi(t, t_0, x_0, u, d))$ are locally bounded on $[t_0, t_{\max})$ and the following estimates hold for all $t \in [t_0, t_{\max})$:

$$V(t) \le \mathrm{MAX}\left\{1\sigma\left(L(t_0), t - t_0\right), \Gamma\left([V]_{[t_0,t]}\right) 1\zeta\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\} \tag{5.4}$$

where $t_{\max}$ is the maximal existence time of the transition map of Σ.

(SG2) (Estimates during and after the transient period) For every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, there exists $\xi \in \pi(t_0, x_0, u, d)$ such that $\phi(t, t_0, x_0, u, d) \in S(t)$ for all $t \in [\xi, t_{\max})$. Moreover, there exist functions $\nu, c, \tilde{c} \in K^+$, $a, \eta, \tilde{\eta}, p^u, g^u \in \mathcal{N}_1$, and $p \in \mathcal{N}_n$ such that the following inequalities hold for every $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$:

$$L(t) \le \max\{\nu(t-t_0), c(t_0), a(\|x_0\|_{\mathcal{X}}), p([V]_{[\xi,t]}), p^u([\|u\|_{\mathcal{U}}]_{[t_0,t]})\}$$
$$\text{for all } t \in [\xi, t_{\max}] \tag{5.5}$$
$$\|\phi(t,t_0,x_0,u,d)\|_{\mathcal{X}} \le \max\{\nu(t-t_0), \tilde{c}(t_0), a(\|x_0\|_{\mathcal{X}}), \tilde{\eta}([\|u\|_{\mathcal{U}}]_{[t_0,t]})\}$$
$$\text{for all } t \in [t_0, \xi] \tag{5.6}$$
$$\xi \le t_0 + a(\|x_0\|_{\mathcal{X}}) + c(t_0) \tag{5.7}$$
$$\|H(t, \phi(t,t_0,x_0,u,d), u(t))\|_{\mathcal{Y}} \le \max\{a(c(t_0)\|x_0\|_{\mathcal{X}}), \eta([\|u\|_{\mathcal{U}}]_{[t_0,t]})\}$$
$$\text{for all } t \in [t_0, \xi] \tag{5.8}$$
$$L(\xi, \phi(\xi,t_0,x_0,u,d)) \le \max\{a(c(t_0)\|x_0\|_{\mathcal{X}}), g^u([\|u\|_{\mathcal{U}}]_{[t_0,\xi]})\}. \tag{5.9}$$

(SG3) (Bounds for the norm of the state and the norm of the output) There exist functions $b \in \mathcal{N}_1$, $q, g \in \mathcal{N}_n$, and $\mu, \kappa \in K^+$ such that the following inequalities hold for all $(t,x,u) \in \bigcup_{t\ge 0}\{\{t\} \times S(t) \times U\}$:

$$\mu(t)\|x\|_{\mathcal{X}} \le b(L(t,x) + g(V(t,x,u)) + \kappa(t)), \tag{5.10}$$
$$\|H(t,x,u)\|_{\mathcal{Y}} \le q(V(t,x,u)) \tag{5.11}$$

where $V(t,x,u) = (V_1(t,x,u), \dots, V_n(t,x,u))'$.

Discussion of Hypotheses (SG1), (SG2), (SG3) It is of interest to note that Theorem 5.1 is a new trajectory-based small-gain result for IOS because inequalities (5.4), (5.5) are not assumed to hold for all times. By combining Hypotheses (SG1) and (SG2) we can conclude that for each trajectory, there exists a time $\xi \in \pi(t_0,x_0,u,d)$ after which inequalities (5.4), (5.5) hold. On the other hand, in order to be able to conclude IOS for the system, we have to assume additional inequalities which hold for the transient period $t \in [t_0, \xi]$, i.e., inequalities (5.6), (5.7), (5.8), (5.9) are required to hold. We next discuss each hypothesis in detail.

- Hypothesis (SG1) is the hypothesis made in every small-gain result: it deals with the "IOS-like" inequalities, which are to be used and be combined, in order to prove the desired estimates. Notice that since we are using a family of n functionals, the "IOS-like" inequalities are given for each functional separately: this is why (5.4) expresses n "IOS-like" inequalities (in vector notation). The difference between Hypothesis (SG1) and similar hypotheses involved in other small-gain results is that we do not assume that the "IOS-like" inequalities (5.4) hold for every initial condition and every input: instead, we assume that (5.4) holds *only* for those initial conditions and inputs for which the state $\phi(t,t_0,x_0,u,d)$ is in the set $S(t)$ for all times $t \ge t_0$ for which the state exists.
- Hypothesis (SG2) is the key hypothesis that guarantees that the state will necessarily enter the set $S(t)$. The time needed in order to enter the set $S(t)$ is denoted by $\xi \in \pi(t_0,x_0,u,d)$. Estimates (5.6), (5.8), (5.9) are estimates for the evolution of the state and the output during the transient period $t \in [t_0, \xi]$, since during the transient period, the "IOS-like" inequalities (5.4) do not hold. Estimate (5.7) is an upper bound for the time needed in order to enter the set $S(t)$: clearly, such an estimate is needed because we have to guarantee that the transient period (for

which the state can behave erratically) is not "too long." Finally, estimate (5.5) is a key estimate for the functional $L : \bigcup_{t\geq 0}\{\{t\} \times S(t)\} \to \Re^+$ that appears in the right-hand side of inequalities (5.4). Inequality (5.5) holds for all times after the time needed in order to enter the set $S(t)$ (after the transient).

- Hypothesis (SG3) is a hypothesis made in every small-gain result (explicitly or implicitly): it provides the bound that allows us to guarantee that the state does not "blow up" and the bound that allows us to conclude that the norm of the output is related to the functionals $V_i : \bigcup_{t\geq 0}\{\{t\} \times S(t) \times U\} \to \Re^+$ $(i = 1, \ldots, n)$. The reader should notice that for every small-gain result, such a hypothesis holds.
- Finally, it should be noted that the set-valued map $S(t) \subseteq \mathcal{X}$ is not assumed to be positively invariant. Instead, the state may enter and leave this set during the transient period $t \in [t_0, \xi]$. However, after the initial transient period, the state never leaves the set $S(t) \subseteq \mathcal{X}$.

We are now ready to state the small-gain results. The first result provides sufficient conditions for the IOS property.

Theorem 5.1 (Trajectory-based small-gain result for IOS) *Consider a system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *under the above hypotheses. Assume that the MAX-preserving continuous map* $\Gamma : \Re^n_+ \to \Re^n_+$ *with* $\Gamma(0) = 0$ *satisfies the cyclic small-gain conditions. Then the system* Σ *satisfies the IOS property from the input* $u \in M_U$ *with gain* $\gamma(s) := \max\{\eta(s), q(G(s))\}$, *where* $G(s) = (G_1(s), \ldots, G_n(s))'$ *is defined by*:

$$G(s) = Q\big(1 \max\big\{\sigma\big(p^u(s), 0\big), \sigma\big(g^u(s), 0\big), \sigma\big(p\big(Q\big(1\sigma\big(g^u(s), 0\big)\big)\big), 0\big), \sigma\big(p\big(Q\big(1\zeta(s)\big)\big), 0\big), \zeta(s)\big\}\big) \tag{5.12}$$

with $Q(x) = \mathrm{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\}$ *for all* $x \in \Re^n_+$. *Moreover, if* $c \in K^+$ *is bounded, then the system* Σ *satisfies the UIOS property from the input* $u \in M_U$ *with gain* $\gamma(s) := \max\{\eta(s), q(G(s))\}$.

When we seek to prove RGAOS for a given system, then some of the above hypotheses may be relaxed further. We consider an abstract control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with $U = \{0\}$ and the BIC property for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Suppose that there exists a set-valued map $\Re^+ \ni t \to S(t) \subseteq \mathcal{X}$ with $0 \in S(t)$ for all $t \geq 0$, mappings $V_i : \bigcup_{t\geq 0}\{t\} \times S(t) \to \Re^+$ $(i = 1, \ldots, n)$, $L : \bigcup_{t\geq 0}\{\{t\} \times S(t)\} \to \Re^+$ with $L(t, 0) = 0$, $V_i(t, 0) = 0$ for all $t \geq 0$ $(i = 1, \ldots, n)$, and a MAX-preserving continuous map $\Gamma : \Re^n_+ \to \Re^n_+$ with $\Gamma(0) = 0$ such that the following hypotheses hold:

(SG4) (The "IOS-like" inequalities) There exists a function $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $\phi(t, t_0, x_0, u_0, d) \in S(t)$ for all $t \in [t_0, t_{\max})$, the mappings $t \to V(t) = (V_1(t, \phi(t, t_0, x_0, u_0, d)), \ldots, V_n(t, \phi(t, t_0, x_0, u_0, d)))'$ and $t \to L(t) = L(t, \phi(t, t_0, x_0, u_0, d))$ are locally bounded on $[t_0, t_{\max})$ and the following estimate holds:

$$V(t) \leq \text{MAX}\{1\sigma(L(t_0), t - t_0), \Gamma([V]_{[t_0,t]})\} \quad \forall t \in [t_0, t_{\max}) \tag{5.13}$$

where $t_{\max}$ is the maximal existence time of the transition map of Σ.

(SG5) (Estimates during and after the transient period) For every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, there exists $\xi \in \pi(t_0, x_0, u_0, d)$ such that $\phi(t, t_0, x_0, u_0, d) \in S(t)$ for all $t \in [\xi, t_{\max})$. Moreover, there exist functions $\nu, c \in K^+$, $a \in \mathcal{N}_1$, and $p \in \mathcal{N}_n$, such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, the following inequalities hold:

$$L(t) \leq \max\{\nu(t - t_0), c(t_0), a(\|x_0\|_{\mathcal{X}}), p([V]_{[\xi,t]})\} \quad \forall t \in [\xi, t_{\max}) \tag{5.14}$$

$$\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \max\{\nu(t - t_0), c(t_0), a(\|x_0\|_{\mathcal{X}})\} \quad \forall t \in [t_0, \xi] \tag{5.15}$$

$$\xi \leq t_0 + a(\|x_0\|_{\mathcal{X}}) + c(t_0) \tag{5.16}$$

$$L(\xi) \leq a(\|x_0\|_{\mathcal{X}}) + c(t_0). \tag{5.17}$$

Discussion of Hypothesis (SG5) Hypothesis (SG5) is almost the same with Hypothesis (SG2) applied to the case $U = \{0\}$. Nonetheless, notice the difference that the estimate for $L(\xi)$ in inequality (5.17) is less tight than the estimate needed in inequality (5.9) of Hypothesis (SG2). Indeed, when $x_0 = 0$, estimate (5.17) does not yield $L(\xi) = 0$, contrary to the estimate (5.9), which gives $L(\xi) = 0$. Finally, the analogue of inequality (5.8) for $U = \{0\}$ (estimation of the norm of the output during the transient period) is not needed in Hypothesis (SG5).

Theorem 5.2 (Trajectory-based small-gain for Robust Global Asymptotic Output Stability (RGAOS)) *Consider a system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *with* $U = \{0\}$ *under Hypotheses* (SG3), (SG4), (SG5). *Assume that the MAX-preserving continuous map* $\Gamma : \Re^n_+ \to \Re^n_+$ *with* $\Gamma(0) = 0$ *satisfies the cyclic small-gain conditions. Then the system* Σ *is RGAOS. Moreover, if* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *is* T*-periodic for some* $T > 0$, *then the system* Σ *is Uniformly RGAOS* (*URGAOS*).

It is clear that Hypotheses (SG4), (SG5) are less demanding than Hypotheses (SG1), (SG2). On the other hand, the conclusion of Theorem 5.2 is weaker than the conclusion of Theorem 5.1: Theorem 5.2 only guarantees RGAOS, while Theorem 5.1 guarantees IOS.

Proof of Theorem 5.1 The proof consists of four steps:

Step 1: We show that for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, the following inequality holds for all $t \in [\xi, t_{\max})$:

$$V(t) \leq \text{MAX}\{Q(1\sigma(L(\xi), 0)), Q(1\zeta([\|u\|_{\mathcal{U}}]_{[\xi,t]}))\} \tag{5.18}$$

where $\xi \in \pi(t_0, x_0, u, d)$ is the time such that $\phi(t, t_0, x_0, u, d) \in S(t)$ for all $t \in [\xi, t_{\max})$ (recall Hypothesis (SG2)).

Proof of Step 1: Let $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$. By virtue of Hypothesis (SG2), there exists $\xi \in \pi(t_0, x_0, u, d)$ such that $\phi(t, t_0, x_0, u, d) \in S(t)$ for all $t \in [\xi, t_{\max})$. Inequality (5.4) implies, for all $t \in [\xi, t_{\max})$,

$$[V]_{[\xi,t]} \leq \mathrm{MAX}\{1\sigma(L(\xi), 0), \Gamma([V]_{[\xi,t]}), 1\zeta([\|u\|_{\mathcal{U}}]_{[\xi,t]})\} \tag{5.19}$$

Fact V in conjunction with (5.19) implies (5.18) for all $t \in [\xi, t_{\max})$.

Step 2: We show that for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, it holds that $t_{\max} = +\infty$.

Proof of Step 2: Suppose that $t_{\max} < +\infty$. Then by virtue of the BIC property for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$. On the other hand, estimate (5.18), in conjunction with the hypothesis $t_{\max} < +\infty$, shows that there exists $M_1 \geq 0$ such that $\sup_{\xi \leq \tau < t_{\max}} |V(\tau)| \leq M_1$. The fact that $V(t)$ is bounded for $t \in [\xi, t_{\max})$, in conjunction with estimate (5.5), implies that there exists $M_2 \geq 0$ such that $\sup_{\xi \leq \tau < t_{\max}} L(\tau) \leq M_2$. It follows from (5.6), (5.10) that the transition map of Σ, i.e., $\phi(t, t_0, x_0, u, d)$, is bounded on $[t_0, t_{\max})$, and this contradicts the requirement that for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$. Hence, we must have $t_{\max} = +\infty$.

Step 3: We show that Σ is RFC from the input $u \in M_U$.

Proof of Step 3: Let arbitrary $r \geq 0$, $T \geq 0$ and arbitrary $u \in \mathcal{M}(B_U[0, r]) \cap M_U$, $\|x_0\|_{\mathcal{X}} \leq r$, $t_0 \in [0, T]$, $d \in M_D$ be given. Estimate (5.6) shows that there exists $M := M(r, T) \geq 0$ such that $\sup_{t_0 \leq \tau \leq \xi} \|\phi(\tau, t_0, x_0, u, d)\|_{\mathcal{X}} \leq M < +\infty$. This provides an upper bound for $\sup\{\|\phi(t_0 + s, t_0, x_0, u, d)\|_{\mathcal{X}}; s \in [0, T]\}$ when $t_0 + T \leq \xi$. We next assume that $t_0 + T > \xi$. By (5.9) there exists $\tilde{M} := \tilde{M}(r, T) \geq 0$ such that $L(\xi) \leq \tilde{M} < +\infty$. Estimate (5.18) shows that there exists $M_1 := M_1(r, T) \geq 0$ such that $\sup_{\xi \leq \tau < t_0 + T} |V(\tau)| \leq M_1 < +\infty$. Consequently, estimate (5.5) implies that there exists $M_2 := M_2(r, T) \geq 0$ such that $\sup_{\xi \leq \tau \leq t_0 + T} L(\tau) \leq M_2 < +\infty$. It follows from (5.10) that there exists $M_3 := M_3(r, T) \geq 0$ such that $\sup_{\xi \leq \tau \leq t_0 + T} \|\phi(\tau, t_0, x_0, u, d)\|_{\mathcal{X}} \leq M_3 < +\infty$. Hence,

$$\sup\{\|\phi(t_0 + s, t_0, x_0, u, d)\|_{\mathcal{X}}; u \in \mathcal{M}(B_U[0, r]) \cap M_U, s \in [0, T], \|x_0\|_{\mathcal{X}} \leq r, \\ t_0 \in [0, T], d \in M_D\} < +\infty$$

Therefore, we conclude that Σ is RFC from the input $u \in M_U$.

Step 4: We prove the following claim.

Claim *For all $\varepsilon > 0$, $k \in Z^+$, and $R, T \geq 0$, there exists $\tau_k(\varepsilon, R, T) \geq 0$ such that for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$, the following inequality holds for all $t \geq \xi + \tau_k$:*

$$V(t) \leq \mathrm{MAX}\{Q(\mathbf{1}\varepsilon), \Gamma^{(k)}(Q(\mathbf{1}\sigma(L(\xi), 0))), G([\|u\|_{\mathcal{U}}]_{[t_0,t]})\} \tag{5.20}$$

where G is defined by (5.12). Moreover, if $c \in K^+$ is bounded, then for all $\varepsilon > 0$, $k \in Z^+$, and $R \geq 0$, there exists $\tau_k(\varepsilon, R) \geq 0$ such that for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ with $\|x_0\|_{\mathcal{X}} \leq R$, inequality (5.20) holds.

Proof of Step 4: The proof of the claim will be made by induction on $k \in Z^+$.

First, we show inequality (5.20) for $k = 1$.

Let arbitrary $\varepsilon > 0$, $R, T \geq 0$, and $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$ be given. Inequality (5.4), in conjunction with inequality (5.18), gives, for all $t \geq \xi$,

$$V(t) \leq \mathrm{MAX}\big\{\mathbf{1}\sigma\big(L(\xi), t-\xi\big), \Gamma\big(Q\big(\mathbf{1}\sigma\big(L(\xi), 0\big)\big)\big), \Gamma\big(Q\big(1\zeta\big(\big[\|u\|_{\mathcal{U}}\big]_{[\xi,t]}\big)\big)\big), \mathbf{1}\zeta\big(\big[\|u\|_{\mathcal{U}}\big]_{[\xi,t]}\big)\big\} \tag{5.21}$$

Since $\Gamma(Q(x)) \leq Q(x)$ and $Q(x) \geq x$ for all $x \in \Re^n_+$ (see Fact III), inequality (5.21) implies, for all $t \geq \xi$,

$$V(t) \leq \mathrm{MAX}\big\{\mathbf{1}\sigma\big(L(\xi), t-\xi\big), \Gamma\big(Q\big(\mathbf{1}\sigma\big(L(\xi), 0\big)\big)\big), Q\big(\mathbf{1}\zeta\big(\big[\|u\|_{\mathcal{U}}\big]_{[\xi,t]}\big)\big)\big\} \tag{5.22}$$

Inequality (5.22), in conjunction with (5.9), implies, for all $t \geq \xi$,

$$V(t) \leq \mathrm{MAX}\big\{\mathbf{1}\sigma\big(a\big(c(t_0)\|x_0\|_{\mathcal{X}}\big), t-\xi\big), \mathbf{1}\sigma\big(g^u\big(\big[\|u\|_{\mathcal{U}}\big]_{[t_0,\xi]}\big), 0\big), \Gamma\big(Q\big(\mathbf{1}\sigma\big(L(\xi), 0\big)\big)\big), Q\big(\mathbf{1}\zeta\big(\big[\|u\|_{\mathcal{U}}\big]_{[\xi,t]}\big)\big)\big\} \tag{5.23}$$

Finally, inequality (5.23), in conjunction with the fact that $Q(x) \geq x$ for all $x \in \Re^n_+$ and definition (5.12), implies the following estimate for all $t \geq \xi$:

$$V(t) \leq \mathrm{MAX}\big\{\mathbf{1}\sigma\big(a\big(c(t_0)\|x_0\|_{\mathcal{X}}\big), t-\xi\big), \Gamma\big(Q\big(\mathbf{1}\sigma\big(L(\xi), 0\big)\big)\big), G\big(\big[\|u\|_{\mathcal{U}}\big]_{[t_0,t]}\big)\big\} \tag{5.24}$$

Using the properties of the KL functions, we can guarantee that there exists $\tau_1(\varepsilon, R, T) \geq 0$ such that $\sigma(a(R \max_{0 \leq t \leq T} c(t)), \tau_1) \leq \varepsilon$. Notice that if $c \in K^+$ is bounded, then $\tau_1 \geq 0$ is independent of T. Then, it follows from (5.24) that we have $V(t) \leq \mathrm{MAX}\{1\varepsilon, \Gamma(Q(\mathbf{1}\sigma(L(\xi), 0))), Q(\mathbf{1}\zeta([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}))\}$ for all $t \geq \xi + \tau_1$. Since $Q(\mathbf{1}\varepsilon) \geq \mathbf{1}\varepsilon$, we conclude that inequality (5.20) holds for $k = 1$.

Inequality (5.5), in conjunction with inequality (5.18) and Fact IV, gives, for all $t \geq \xi$,

$$L(t) \leq \max\big\{\nu(t-t_0), c(t_0), a\big(\|x_0\|_{\mathcal{X}}\big), p\big(Q\big(\mathbf{1}\sigma\big(L(\xi), 0\big)\big)\big), p\big(Q\big(\mathbf{1}\zeta\big(\big[\|u\|_{\mathcal{U}}\big]_{[t_0,t]}\big)\big)\big), p^u\big(\big[\|u\|_{\mathcal{U}}\big]_{[t_0,t]}\big)\big\} \tag{5.25}$$

Next, suppose that for all $\varepsilon > 0$ and $R, T \geq 0$, there exists $\tau_k(\varepsilon, R, T) \geq 0$ such that for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$, (5.20) holds for some $k \in Z^+$. Let arbitrary $\varepsilon > 0$, $R, T \geq 0$, and $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$ be given. Notice that the weak semigroup property implies that $\pi(t_0, x_0, u, d) \cap [\xi + \tau_k, \xi + \tau_k + r] \neq \emptyset$. Let $t_k \in \pi(t_0, x_0, u, d) \cap [\xi + \tau_k, \xi + \tau_k + r]$. Then (5.4) implies that, for all $t \geq t_k$,

$$V(t) \leq \mathrm{MAX}\big\{\mathbf{1}\sigma\big(L(t_k), t-t_k\big), \Gamma\big([V]_{[t_k,t]}\big), \mathbf{1}\zeta\big(\big[\|u\|_{\mathcal{U}}\big]_{[t_k,t]}\big)\big\} \tag{5.26}$$

Moreover, inequality (5.20) gives, for all $t \geq t_k$,

$$[V]_{[t_k,t]} \leq \mathrm{MAX}\left\{Q(\mathbf{1}\varepsilon),\, \Gamma^{(k)}\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right),\, G\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\} \tag{5.27}$$

Inequality (5.25) also implies

$$L(t_k) \leq \max\left\{\nu(t_k - t_0),\, c(t_0),\, a(R),\, p\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right),\right. \\ \left. p\left(Q\left(\mathbf{1}\zeta\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t_k]}\right)\right)\right),\, p^u\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t_k]}\right)\right\} \tag{5.28}$$

Using (5.9) and (5.28), we obtain

$$L(t_k) \leq \max\left\{\nu(t_k - t_0),\, c(t_0),\, a(R),\, p\left(Q\left(\mathbf{1}\sigma\left(a\left(c(t_0)R\right),0\right)\right)\right),\right. \\ p\left(Q\left(\mathbf{1}\sigma\left(g^u\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t_k]}\right),0\right)\right)\right),\, p\left(Q\left(\mathbf{1}\zeta\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t_k]}\right)\right)\right), \\ \left. p^u\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t_k]}\right)\right\} \tag{5.29}$$

Using (5.27) and the fact that $\Gamma(G(s)) \leq G(s)$ for all $s \geq 0$, we obtain

$$\Gamma\left([V]_{[t_k,t]}\right) \leq \mathrm{MAX}\left\{Q(\mathbf{1}\varepsilon),\, \Gamma^{(k+1)}\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right),\, G\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\} \\ \text{for all } t \geq t_k \tag{5.30}$$

Inequality (5.30), in conjunction with inequality (5.26), the fact that $G(s) \geq Q(\mathbf{1}\zeta(s)) \geq \mathbf{1}\zeta(s)$ for all $s \geq 0$, and the fact that $t_k \leq \xi + \tau_k + r$, implies

$$V(t) \leq \mathrm{MAX}\left\{\mathbf{1}\sigma\left(L(t_k),\, t - \xi - \tau_k - r\right),\, Q(\mathbf{1}\varepsilon),\, \Gamma^{(k+1)}\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right),\right. \\ \left. G\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\} \quad \text{for all } t \geq \xi + \tau_k + r \tag{5.31}$$

Inequality (5.29), in conjunction with (5.7) and the facts that $\mathbf{1}\sigma(p^u(s),0) \leq G(s)$, $\mathbf{1}\sigma(p(Q(\mathbf{1}\zeta(s))),0) \leq G(s)$, and $\mathbf{1}\sigma(p(Q(\mathbf{1}\sigma(g^u(s),0))),0) \leq G(s)$ for all $s \geq 0$ (see definition (5.12)) and $t_k \leq \xi + \tau_k + r$, $t_0 \in [0,T]$, and $\|x_0\|_{\mathcal{X}} \leq R$, implies that

$$\mathbf{1}\sigma\left(L(t_k),\, t - \xi - \tau_k - r\right) \\ \leq \mathrm{MAX}\left\{\mathbf{1}\sigma\left(f(\varepsilon,T,R),\, t - \xi - \tau_k - r\right),\, G\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\} \\ \text{for all } t \geq \xi + \tau_k + r \tag{5.32}$$

where

$$f(\varepsilon,T,R) := \max\left\{\max_{0 \leq t \leq a(R)+C(T)+\tau_k(\varepsilon,R,T)+r} \nu(t),\, C(T),\, a(R),\right. \\ \left. p\left(Q\left(\mathbf{1}\sigma\left(a\left(RC(T)\right),0\right)\right)\right)\right\} \tag{5.33}$$

and

$$C(T) := \max_{0 \leq t \leq T} c(t) \tag{5.34}$$

Notice that if $c \in K^+$ is bounded and τ_k is independent of T, then f can be chosen to be independent of T as well. By combining (5.31) and (5.32) we get

$$V(t) \leq \mathrm{MAX}\left\{\mathbf{1}\sigma\left(f(\varepsilon,T,R),\, t - \xi - \tau_k - r\right),\, \Gamma^{(k+1)}\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right),\right. \\ \left. Q(\mathbf{1}\varepsilon),\, G\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\} \quad \text{for all } t \geq \xi + \tau_k + r \tag{5.35}$$

Clearly, there exists $\tau(\varepsilon, R, T) \geq 0$ such that $\sigma(f(\varepsilon, T, R), \tau) \leq \varepsilon$. Define

$$\tau_{k+1}(\varepsilon, R, T) = \tau_k(\varepsilon, R, T) + r + \tau(\varepsilon, R, T) \tag{5.36}$$

Again, notice that if f and τ_k are independent of T, then τ_{k+1} is independent of T as well. Since $Q(\mathbf{1}\varepsilon) \geq \mathbf{1}\varepsilon$, we obtain from (5.35)

$$V(t) \leq \text{MAX}\{Q(\mathbf{1}\varepsilon), \Gamma^{(k+1)}(Q(\mathbf{1}\sigma(L(\xi), 0))), G([\|u\|_{\mathcal{U}}]_{[t_0,t]})\}$$
$$\text{for all } t \geq \xi + \tau_{k+1} \tag{5.37}$$

which shows that (5.20) holds for $k + 1$.

To finish the proof, let $\varepsilon > 0$, $R, T \geq 0$, and $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ be arbitrary and denote $Y(t) = H(t, \phi(t, t_0, x_0, u, d), u(t))$ for $t \geq t_0$.

Using Fact IV, (5.11), and (5.18), we obtain, for all $t \geq \xi$,

$$\|Y(t)\|_{\mathcal{Y}} \leq \max\{q(Q(\mathbf{1}\sigma(L(\xi), 0))), q(Q(\mathbf{1}\zeta([\|u\|_{\mathcal{U}}]_{[\xi,t]})))\}$$

The above inequality, in conjunction with (5.9), implies that

$$\|Y(t)\|_{\mathcal{Y}} \leq \max\{q(Q(\mathbf{1}\sigma(a(c(t_0)\|x_0\|_{\mathcal{X}}), 0))), q(Q(\mathbf{1}\zeta([\|u\|_{\mathcal{U}}]_{[t_0,t]}))),$$
$$q(Q(\mathbf{1}\sigma(g^u([\|u\|_{\mathcal{U}}]_{[t_0,t]}), 0)))\} \quad \text{for all } t \geq \xi \tag{5.38}$$

Using (5.8) and (5.38), we conclude that the following estimate holds for all $t \geq t_0$:

$$\|Y(t)\|_{\mathcal{Y}} \leq \max\{q(Q(\mathbf{1}\sigma(a(c(t_0)\|x_0\|_{\mathcal{X}}), 0))), a(c(t_0)\|x_0\|_{\mathcal{X}}),$$
$$\eta([\|u\|_{\mathcal{U}}]_{[t_0,t]}), q(Q(\mathbf{1}\sigma(g^u([\|u\|_{\mathcal{U}}]_{[t_0,t]}), 0))),$$
$$q(Q(\mathbf{1}\zeta([\|u\|_{\mathcal{U}}]_{[t_0,t]})))\} \tag{5.39}$$

Inequality (5.39) shows that Properties P1 and P2 of Lemma 4.1 hold for system Σ with $V = \|H(t, x, u)\|_{\mathcal{Y}}$, $\delta(t) \equiv 1$, and $\gamma(s) := \max\{\eta(s), q(G(s))\}$. Moreover, if $c \in K^+$ is bounded, then (5.39) implies that Properties P1 and P2 of Lemma 4.2 hold for system Σ with $V = \|H(t, x, u)\|_{\mathcal{Y}}$, $\delta(t) \equiv 1$, and $\gamma(s) := \max\{\eta(s), q(G(s))\}$.

Inequality (5.20), in conjunction with Fact III, (5.9), (5.34), and definition (5.12), guarantees that for all $\varepsilon > 0$, $k \in Z^+$, and $R, T \geq 0$, there exists $\tau_k(\varepsilon, R, T) \geq 0$ such that for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$, the following inequality holds:

$$V(t) \leq \text{MAX}\{Q(\mathbf{1}\varepsilon), \Gamma^{(k)}(Q(\mathbf{1}\sigma(a(RC(T)), 0))), G([\|u\|_{\mathcal{U}}]_{[t_0,t]})\}$$
$$\text{for all } t \geq \xi + \tau_k. \tag{5.40}$$

Notice that Fact I guarantees the existence of $k(\varepsilon, T, R) \in Z_+$ such that $Q(\mathbf{1}\varepsilon) \geq \Gamma^{(l)}(Q(\mathbf{1}\sigma(a(RC(T)), 0)))$ for all $l \geq k$. If $c \in K^+$ is bounded, then k is independent of T. Therefore, (5.40) implies that for all $\varepsilon > 0$ and $R, T \geq 0$, there exists $\tau(\varepsilon, R, T) \geq 0$ such that for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$, it holds that

$$V(t) \leq \text{MAX}\{Q(\mathbf{1}\varepsilon), G([\|u\|_{\mathcal{U}}]_{[t_0,t]})\} \quad \text{for all } t \geq \xi + \tau \tag{5.41}$$

If $c \in K^+$ is bounded, then $\tau(\varepsilon, R, T) \geq 0$ is independent of T. It follows from inequalities (5.11), (5.41), definition (5.12), and Fact IV that for every $\varepsilon > 0$ and

$R, T \geq 0$, there exists $\tau(\varepsilon, R, T) \geq 0$ such that for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$, it holds that

$$\|Y(t)\|_{\mathcal{Y}} \leq \max\{q(Q(\mathbf{1}\varepsilon)), q(G([\|u\|_{\mathcal{U}}]_{[t_0,t]}))\} \quad \text{for all } t \geq \xi + \tau \quad (5.42)$$

Therefore, by virtue of (5.42) and (5.7), Property P3 of Lemma 4.1 holds for system Σ with $V = \|H(t, x, u)\|_{\mathcal{Y}}$, $\delta(t) \equiv 1$, and $\gamma(s) := \max\{\eta(s), q(G(s))\}$. Moreover, if $c \in K^+$ is bounded, then (5.42) and (5.7) imply that Property P3 of Lemma 4.2 in Chap. 4 hold for system Σ with $V = \|H(t, x, u)\|_{\mathcal{Y}}$, $\delta(t) \equiv 1$ and $\gamma(s) := \max\{\eta(s), q(G(s))\}$.

The conclusions of Theorem 5.1 are direct consequences of Lemma 4.1 (or Lemma 4.2 for the case of bounded $c \in K^+$). □

Proof of Theorem 5.2 By virtue of Lemma 2.1 we have to show that Σ is Robustly Forward Complete (RFC) and satisfies the *Robust Output Attractivity Property*, i.e., for every $\varepsilon > 0$, $T \geq 0$, and $R \geq 0$, there exists $\tau := \tau(\varepsilon, T, R) \geq 0$ such that

$$\|x_0\|_{\mathcal{X}} \leq R \quad t_0 \in [0, T] \quad \Rightarrow \quad \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \varepsilon$$

for all $t \in [t_0 + \tau, +\infty)$ and all $d \in M_D$. Moreover, Lemma 2.2 guarantees that the system $\Sigma = (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is URGAOS if it is T-periodic for certain $T > 0$.

The proof consists of four steps:

Step 1: We show that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, the following inequality holds for all $t \in [\xi, t_{\max})$:

$$V(t) \leq Q(\mathbf{1}\sigma(L(\xi), 0)) \quad (5.43)$$

where $\xi \in \pi(t_0, x_0, u_0, d)$ is the time such that $\phi(t, t_0, x_0, u_0, d) \in S(t)$ for all $t \in [\xi, t_{\max})$ (recall Hypothesis (SG5)).

Proof of Step 1: Exactly the same as in Theorem 5.1.

Step 2: We show that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, it holds that $t_{\max} = +\infty$.

Proof of Step 2: Exactly the same as in Theorem 5.1.

Step 3: We show that Σ is RFC.

Proof of Step 3: Exactly the same as in Theorem 5.1.

Step 4: We prove the following claim.

Claim *For all $\varepsilon > 0$, $k \in Z^+$, and $R, T \geq 0$, there exists $\tau_k(\varepsilon, R, T) \geq 0$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$, the following inequality holds*:

$$V(t) \leq \mathrm{MAX}\{Q(\mathbf{1}\varepsilon), \Gamma^{(k)}(Q(\mathbf{1}\sigma(L(\xi), 0)))\} \quad \textit{for all } t \geq \xi + \tau_k \quad (5.44)$$

Proof of Step 4: The proof of the claim will be made by induction on $k \in Z^+$.

First we show inequality (5.44) for $k = 1$.

Let arbitrary $\varepsilon > 0$, $R, T \geq 0$, and $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$ be given. Inequality (5.13), in conjunction with inequality (5.43), gives, for all $t \geq \xi$,

$$V(t) \leq \text{MAX}\{\mathbf{1}\sigma(L(\xi), t - \xi), \Gamma(Q(\mathbf{1}\sigma(L(\xi), 0)))\} \tag{5.45}$$

Inequality (5.45), in conjunction with (5.17), implies, for all $t \geq \xi$,

$$V(t) \leq \text{MAX}\{\mathbf{1}\sigma(a(\|x_0\|_{\mathcal{X}}) + c(t_0), t - \xi), \Gamma(Q(\mathbf{1}\sigma(L(\xi), 0)))\} \tag{5.46}$$

Using the properties of the *KL* functions, we can guarantee that there exists $\tau_1(\varepsilon, R, T) \geq 0$ such that $\sigma(a(R) + \max_{0 \leq t \leq T} c(t), \tau_1) \leq \varepsilon$. Since $Q(\mathbf{1}\varepsilon) \geq \mathbf{1}\varepsilon$, we conclude that inequality (5.44) holds for $k = 1$.

Inequality (5.14), in conjunction with inequality (5.43) and Fact IV, gives, for all $t \geq \xi$,

$$L(t) \leq \max\{\nu(t - t_0), c(t_0), a(\|x_0\|_{\mathcal{X}}), p(Q(\mathbf{1}\sigma(L(\xi), 0)))\} \tag{5.47}$$

Next, suppose that for all $\varepsilon > 0$ and $R, T \geq 0$, there exists $\tau_k(\varepsilon, R, T) \geq 0$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$, (5.44) holds for some $k \in Z^+$. Let arbitrary $\varepsilon > 0$, $R, T \geq 0$, and $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leq R$ be given. Notice that the weak semigroup property implies that $\pi(t_0, x_0, u, d) \cap [\xi + \tau_k, \xi + \tau_k + r] \neq \emptyset$. Let $t_k \in \pi(t_0, x_0, u, d) \cap [\xi + \tau_k, \xi + \tau_k + r]$. Then (5.13) implies

$$V(t) \leq \text{MAX}\{\mathbf{1}\sigma(L(t_k), t - t_k), \Gamma([V]_{[t_k,t]})\} \quad \forall t \geq t_k \tag{5.48}$$

Moreover, inequality (5.44) gives

$$[V]_{[t_k,t]} \leq \text{MAX}\{Q(\mathbf{1}\varepsilon), \Gamma^{(k)}(Q(\mathbf{1}\sigma(L(\xi), 0)))\} \quad \forall t \geq t_k \tag{5.49}$$

Inequality (5.47) also implies:

$$L(t_k) \leq \max\{\nu(t_k - t_0), c(t_0), a(R), p(Q(\mathbf{1}\sigma(L(\xi), 0)))\} \tag{5.50}$$

Using (5.17) and (5.50), we obtain

$$L(t_k) \leq \max\{\nu(t_k - t_0), c(t_0), a(R), p(Q(\mathbf{1}\sigma(a(R) + c(t_0), 0)))\} \tag{5.51}$$

Using (5.49) and the fact that $\Gamma(Q(x)) \leq Q(x)$ for all $x \in \Re^n_+$, it holds

$$\Gamma([V]_{[t_k,t]}) \leq \text{MAX}\{Q(\mathbf{1}\varepsilon), \Gamma^{(k+1)}(Q(\mathbf{1}\sigma(L(\xi), 0)))\} \quad \forall t \geq t_k \tag{5.52}$$

Inequality (5.52), in conjunction with inequality (5.48) and the fact that $t_k \leq \xi + \tau_k + r$, implies

$$V(t) \leq \text{MAX}\{\mathbf{1}\sigma(L(t_k), t - \xi - \tau_k - r), Q(\mathbf{1}\varepsilon), \Gamma^{(k+1)}(Q(\mathbf{1}\sigma(L(\xi), 0)))\}$$
$$\text{for all } t \geq \xi + \tau_k + r \tag{5.53}$$

Inequality (5.51), in conjunction with (5.16) and the facts that $t_k \leq \xi + \tau_k + r$, $t_0 \in [0, T]$, and $\|x_0\|_{\mathcal{X}} \leq R$, implies that, for all $t \geq \xi + \tau_k + r$,

$$\sigma(L(t_k), t - \xi - \tau_k - r) \leq \sigma(f(\varepsilon, T, R), t - \xi - \tau_k - r) \tag{5.54}$$

where

$$f(\varepsilon, T, R) := \max\left\{\max_{0 \le t \le a(R)+C(T)+\tau_k(\varepsilon,R,T)+r} \nu(t), C(T), a(R), p\big(Q\big(\mathbf{1}\sigma\big(a(R)+C(T),0\big)\big)\big)\right\} \tag{5.55}$$

and

$$C(T) := \max_{0 \le t \le T} c(t) \tag{5.56}$$

Combining (5.53) and (5.54) leads to

$$V(t) \le \mathrm{MAX}\left\{\mathbf{1}\sigma\big(f(\varepsilon, T, R), t - \xi - \tau_k - r\big), Q(\mathbf{1}\varepsilon), \Gamma^{(k+1)}\big(Q\big(\mathbf{1}\sigma\big(L(\xi),0\big)\big)\big)\right\}$$
$$\text{for all } t \ge \xi + \tau_k + r \tag{5.57}$$

Clearly, there exists $\tau(\varepsilon, R, T) \ge 0$ such that $\sigma(f(\varepsilon, T, R), \tau) \le \varepsilon$. Define

$$\tau_{k+1}(\varepsilon, R, T) = \tau_k(\varepsilon, R, T) + r + \tau(\varepsilon, R, T) \tag{5.58}$$

Since $Q(\mathbf{1}\varepsilon) \ge \mathbf{1}\varepsilon$, it follows from (5.57) that

$$V(t) \le \mathrm{MAX}\left\{Q(\mathbf{1}\varepsilon), \Gamma^{(k+1)}\big(Q\big(\mathbf{1}\sigma\big(L(\xi),0\big)\big)\big)\right\} \quad \forall t \ge \xi + \tau_{k+1} \tag{5.59}$$

which shows that (5.44) holds for $k+1$.

To finish the proof, let $\varepsilon > 0$, $R, T \ge 0$, and $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ be arbitrary and denote $Y(t) = H(t, \phi(t, t_0, x_0, u_0, d), 0)$ for $t \ge t_0$.

Inequality (5.44), in conjunction with Fact III, (5.17), and (5.56), guarantees that for all $\varepsilon > 0$, $k \in Z^+$, and $R, T \ge 0$, there exists $\tau_k(\varepsilon, R, T) \ge 0$ such that, for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \le R$, the following inequality holds:

$$V(t) \le \mathrm{MAX}\left\{Q(\mathbf{1}\varepsilon), \Gamma^{(k)}\big(Q\big(\mathbf{1}\sigma\big(a(R)+C(T),0\big)\big)\big)\right\} \quad \forall t \ge \xi + \tau_k \tag{5.60}$$

Notice that Fact I guarantees the existence of $k(\varepsilon, T, R) \in Z^+$ such that $Q(\mathbf{1}\varepsilon) \ge \Gamma^{(l)}(Q(\mathbf{1}\sigma(a(R)+C(T),0)))$ for all $l \ge k$. Therefore, (5.60) implies that for all $\varepsilon > 0$, and $R, T \ge 0$, there exists $\tau(\varepsilon, R, T) \ge 0$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \le R$, it holds that

$$V(t) \le Q(\mathbf{1}\varepsilon) \quad \text{for all } t \ge \xi + \tau \tag{5.61}$$

It follows from inequalities (5.11) and (5.61) that for all $\varepsilon > 0$ and $R, T \ge 0$, there exists $\tau(\varepsilon, R, T) \ge 0$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \le R$, it holds that

$$\big\|Y(t)\big\|_{\mathcal{Y}} \le q\big(Q(\mathbf{1}\varepsilon)\big) \quad \text{for all } t \ge \xi + \tau \tag{5.62}$$

Therefore, by virtue of (5.62) and (5.16), the *Robust Output Attractivity Property* holds for system Σ. The proof is complete. □

The small-gain conditions (5.3) for $n = 2$ are equivalent to the inequalities

$$\gamma_{1,2}\big(\gamma_{2,1}(s)\big) < s \quad \text{and} \quad \gamma_{2,1}\big(\gamma_{1,2}(s)\big) < s \quad \text{for all } s > 0$$

However, it should be noted that the second inequality above is implied by the first one. For $n = 3$, conditions (5.3) are equivalent to the following twelve inequalities for all $s > 0$:

$$\gamma_{1,2}\big(\gamma_{2,1}(s)\big) < s \qquad \gamma_{2,1}\big(\gamma_{1,2}(s)\big) < s$$
$$\gamma_{1,3}\big(\gamma_{3,1}(s)\big) < s \qquad \gamma_{3,1}\big(\gamma_{1,3}(s)\big) < s$$
$$\gamma_{2,3}\big(\gamma_{3,2}(s)\big) < s \qquad \gamma_{3,2}\big(\gamma_{2,3}(s)\big) < s$$
$$\gamma_{1,2}\big(\gamma_{2,3}(\gamma_{3,1}(s))\big) < s \qquad \gamma_{3,1}\big(\gamma_{1,2}(\gamma_{2,3}(s))\big) < s, \qquad \gamma_{2,3}\big(\gamma_{3,1}(\gamma_{1,2}(s))\big) < s$$
$$\gamma_{2,1}\big(\gamma_{1,3}(\gamma_{3,2}(s))\big) < s \qquad \gamma_{1,3}\big(\gamma_{3,2}(\gamma_{2,1}(s))\big) < s \qquad \gamma_{3,2}\big(\gamma_{2,1}(\gamma_{1,3}(s))\big) < s$$

Also notice that some of the above inequalities are equivalent. The following five inequalities

$$\gamma_{1,2}\big(\gamma_{2,1}(s)\big) < s \qquad \gamma_{1,3}\big(\gamma_{3,1}(s)\big) < s\gamma_{2,3} \qquad \big(\gamma_{3,2}(s)\big) < s$$
$$\gamma_{1,2}\big(\gamma_{2,3}(\gamma_{3,1}(s))\big) < s \quad \text{and} \quad \gamma_{2,1}\big(\gamma_{1,3}(\gamma_{3,2}(s))\big) < s \quad \text{for all } s > 0$$

imply all twelve inequalities that express conditions (5.3) in this case.

Finally, it should be noticed that if the norm of the input $\|u\|_{\mathcal{U}}$ appears in inequalities (5.4)–(5.9) "weighted" by certain $\delta \in K^+$, i.e., if $[\|u\|_{\mathcal{U}}]_{[t_0,t]}$ is replaced by $\sup_{t_0 \le \tau \le t} \delta(\tau)\|u(\tau)\|_{\mathcal{U}}$, then the result of Theorem 5.1 will guarantee WIOS with weight function $\delta \in K^+$, instead of IOS.

5.3 Vector Lyapunov Functionals

In this section, we will restrict our attention to the problem of the verification of the assumptions of the small-gain results of the previous paragraph (Theorems 5.1 and 5.2) by means of Lyapunov functionals. The obtained results are termed as results involving "vector Lyapunov functionals" due to the fact that a family of Lyapunov functionals is used.

5.3.1 Vector Lyapunov Functions for Systems Described by ODEs

Consider systems described by Ordinary Differential Equations (ODEs) of the form (1.3) under Hypotheses (H1–5). Moreover, we assume that the output map H is independent of $u \in U$. Finally, we will assume that

(SG6) There exist functions $V_i \in C^1(\Re^+ \times \Re^n; \Re^+)$ $(i = 1, \dots, k)$, $W \in C^1(\Re^+ \times \Re^n; \Re^+)$, $a_1, a_2, a_3, a_4 \in K_\infty$ $\mu, \beta, \kappa \in K^+$, $\zeta \in \mathcal{N}_1$, $g \in \mathcal{N}_k, \gamma_{i,j} \in \mathcal{N}_1$, $p_i \in \mathcal{N}_1$, $i, j = 1, \dots, k$, a family of positive definite functions $\rho_i \in C^0(\Re^+; \Re^+)$ $(i = 1, \dots, k)$, and a constant $\lambda \in (0, 1)$ such that the following inequalities hold for all $(t, x, u) \in \Re^+ \times \Re^n \times U$:

$$a_1\big(|H(t,x)|\big) \le \max_{i=1,\dots,k} V_i(t,x) \le a_2\big(\beta(t)|x|\big) \tag{5.63}$$

$$a_3\big(\mu(t)|x|\big) - g\big(V_1(t,x),\dots,V_k(t,x)\big) - \kappa(t) \le W(t,x) \le a_4\big(\beta(t)|x|\big) \tag{5.64}$$

$$\sup\left\{\frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x) f(t,x,u,d) : d \in D\right\}$$
$$\le -W(t,x) + \lambda \max\left\{\zeta\big(|u|\big), \max_{j=1,\dots,k} p_j\big(V_j(t,x)\big)\right\} \tag{5.65}$$

and, for all $i = 1,\dots,k$ and $(t,x,u) \in \Re^+ \times \Re^n \times U$, the following implication holds:

$$\text{“if } \max\left\{\zeta\big(|u|\big), \max_{j=1,\dots,k} \gamma_{i,j}\big(V_j(t,x)\big)\right\} \le V_i(t,x),$$
$$\text{then } \frac{\partial V_i}{\partial t}(t,x) + \sup_{d\in D} \frac{\partial V_i}{\partial x}(t,x) f(t,x,u,d) \le -\rho_i\big(V_i(t,x)\big)\text{”} \tag{5.66}$$

Our main result concerning systems of the form (1.3) is the following result, which provides sufficient conditions for Theorem 5.1 to hold.

Theorem 5.3 (Vector Lyapunov function characterization of the IOS property) *Consider system* (1.3) *under Hypotheses* (H1–5) *and* (SG6). *Moreover, we assume that the output map H is independent of $u \in U$. If the small-gain conditions* (5.3) *hold, then system* (1.3) *satisfies the IOS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$, where*

$$\theta(s) := \max_{i=1,\dots,k} \varphi_i\Big(\max\Big\{\max_{l=1,\dots,k}\max_{j=1,\dots,k} \gamma_{l,j}\big(\varphi_j\big(\zeta(s)\big)\big),$$
$$\max_{j=1,\dots,k} p_j\big(\varphi_j\big(\zeta(s)\big)\big), \zeta(s)\Big\}\Big) \tag{5.67}$$

and

$$\varphi_i(s) := \max\Big\{s, \max_{l=1,\dots,k-1} \max\big\{(\gamma_{i,j_1} \circ \gamma_{j_1,j_2} \circ \cdots \circ \gamma_{j_{l-1},j_l})(s);$$
$$(j_1,\dots,j_l) \in \{1,\dots,n\}^l\big\}\Big\} \quad i = 1,\dots,k \tag{5.68}$$

Moreover, if $\beta \in K^+$ is bounded, then system (1.3) *with output $Y = H(t,x)$ satisfies the UIOS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$.*

Proof We want to show that all hypotheses of Theorem 5.1 hold with $S(t) \equiv \Re^n$ and

$$L(t,x) := \max\left\{W(t,x), \max_{i=1,\dots,k} V_i(t,x)\right\} \tag{5.69}$$

Notice that Hypothesis (SG3) of Theorem 5.1 is a direct consequence of inequalities (5.64), definition (5.69), and inequality (5.63) with $q(x) := a_1^{-1}(\max_{i=1,\dots,k} x_i)$ for all $x \in \Re^n_+$.

Finally, notice that since $S(t) \equiv \Re^n$, we can select the time ξ involved in Hypothesis (SG2) to be $\xi = t_0$. Consequently, inequalities (5.6), (5.7), (5.8), and (5.9) are automatically satisfied for appropriate functions. Thus, we are left with the task of proving inequalities (5.4) and (5.5).

Consider a solution $x(t)$ of (1.3) corresponding to arbitrary $(u,d) \in M_U \times M_D$ with arbitrary initial condition $x(t_0) = x_0 \in \Re^n$. Clearly, there exists a maximal existence time for the solution denoted by $t_{\max} \le +\infty$. Define $V_i(t) = V_i(t, x(t))$, $i = 1, \dots, k$, $W(t) = W(t, x(t))$ absolutely continuous functions on $[t_0, t_{\max})$, and $L(t) = L(t, x(t))$. Moreover, let $I \subset [t_0, t_{\max})$ be the zero Lebesgue measure set where $x(t)$ is not differentiable or $\dot{x}(t) \neq f(t, x(t), u(t), d(t))$. By virtue of (5.66), it follows that the following implication holds for $t \in [t_0, t_{\max})\backslash I$ and $i = 1, \dots, k$:

$$V_i(t) \ge \max\Big\{\zeta\big(|u(t)|\big), \max_{j=1,\dots,k} \gamma_{i,j}\big(V_j(t)\big)\Big\} \quad \Rightarrow \quad \dot{V}_i(t) \le -\rho_i\big(V_i(t)\big) \tag{5.70}$$

and by virtue of (5.65), we get, for $t \in [t_0, t_{\max})\backslash I$,

$$\dot{W}(t) \le -W(t) + \lambda \max\Big\{\zeta\big(|u(t)|\big), \max_{j=1,\dots,k} p_j\big(V_j(t)\big)\Big\} \tag{5.71}$$

Lemma 2.14 in Chap. 2, in conjunction with (5.70), implies that there exists a family of continuous functions σ_i $(i = 1, \dots, k)$ of class KL, with $\sigma_i(s, 0) = s$ for all $s \ge 0$ such that, for all $t \in [t_0, t_{\max})$ and $i = 1, \dots, k$, we have

$$\begin{aligned} V_i(t) \le \max\Big\{&\sigma_i\big(V_i(t_0), t - t_0\big), \sup_{t_0 \le \tau \le t} \sigma_i\Big(\max_{j=1,\dots,k} \sup_{t_0 \le s \le \tau} \gamma_{i,j}\big(V_j(s)\big), t - \tau\Big), \\ &\sup_{t_0 \le \tau \le t} \sigma_i\Big(\zeta\Big(\sup_{t_0 \le s \le \tau} \big|u(s)\big|\Big), t - \tau\Big)\Big\} \end{aligned} \tag{5.72}$$

Moreover, inequality (5.71) directly implies that, for all $t \in [t_0, t_{\max})$,

$$W(t) \le W(t_0) + \lambda \max\Big\{\zeta\Big(\sup_{t_0 \le s \le t} \big|u(s)\big|\Big), \max_{j=1,\dots,k} p_j\Big(\sup_{t_0 \le s \le t} V_j(s)\Big)\Big\} \tag{5.73}$$

Let $\sigma(s,t) := \max_{i=1,\dots,k} \sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s, 0) = s$ for all $s \ge 0$. It follows from (5.72), (5.73), and definition (5.69) that, for all $t \in [t_0, t_{\max})$ and $i = 1, \dots, k$,

$$V_i(t) \le \max\Big\{V_i(t_0), \max_{j=1,\dots,k} \gamma_{i,j}\Big(\sup_{t_0 \le s \le t} V_j(s)\Big), \zeta\Big(\sup_{t_0 \le s \le t} \big|u(s)\big|\Big)\Big\} \tag{5.74}$$

$$V_i(t) \le \max\Big\{\sigma\big(L(t_0), t - t_0\big), \max_{j=1,\dots,k} \gamma_{i,j}\Big(\sup_{t_0 \le s \le t} V_j(s)\Big), \zeta\Big(\sup_{t_0 \le s \le t} \big|u(s)\big|\Big)\Big\} \tag{5.75}$$

$$W(t) \le \max\Big\{\frac{1}{1-\lambda} W(t_0), \zeta\Big(\sup_{t_0 \le s \le t} \big|u(s)\big|\Big), \max_{i=1,\dots,k} p_i\Big(\sup_{t_0 \le s \le t} V_i(s)\Big)\Big\} \tag{5.76}$$

Clearly, inequalities (5.75) show that (5.4) holds with $\Gamma : \Re_+^k \to \Re_+^k$, $\Gamma(x) = (\Gamma_1(x), \dots, \Gamma_n(x))'$ with $\Gamma_i(x) = \max_{j=1,\dots,k} \gamma_{i,j}(x_j)$ for all $i = 1, \dots, k$ and $x \in \Re_+^n$. Furthermore, the small-gain conditions hold for the MAX-preserving mapping $\Gamma : \Re_+^k \to \Re_+^k$ as well. Moreover, inequalities (5.74) and (5.76) imply that the following estimates hold, for all $t \in [t_0, t_{\max})$,

$$\max_{i=1,\dots,k} V_i(t) \le \max\left\{ \max_{i=1,\dots,k} V_i(t_0), \max_{i=1,\dots,k} \max_{j=1,\dots,k} \gamma_{i,j}\left(\sup_{t_0\le s\le t} V_j(s)\right),\right.$$
$$\left. \zeta\left(\sup_{t_0\le s\le t} \left|u(s)\right|\right)\right\} \tag{5.77}$$

Define

$$p^u(s) := \zeta(s) \quad \forall s \ge 0 \tag{5.78}$$

$$p(x) := \max\left\{ \max_{i,j=1,\dots,k} \gamma_{i,j}(x_j), \max_{j=1,\dots,k} p_j(x_j)\right\} \quad \forall x \in \Re^n_+ \tag{5.79}$$

Combining estimates (5.76), (5.77) and exploiting definitions (5.69), (5.78), and (5.79), we get, for all $t \in [t_0, t_{\max})$,

$$L(t) \le \max\left\{ h\big(L(t_0)\big), p\left(\sup_{t_0\le s\le t} V_1(s), \dots, \sup_{t_0\le s\le t} V_k(s)\right), p^u\left(\sup_{t_0\le s\le t}\left|u(s)\right|\right)\right\} \tag{5.80}$$

where $h(s) := \frac{s}{1-\lambda}$. Inequality (5.5) is a direct consequence of (5.80), inequalities (5.63), (5.64), and Lemma 3.2 in Chap. 3 with $\nu(t) \equiv 1$, $p^u \in \mathcal{N}_1$, and $p \in \mathcal{N}_n$ as defined by (5.78), (5.79), and appropriate $a \in \mathcal{N}_1$ and $c \in K^+$. The reader should notice that if $\beta \in K^+$ is bounded, then $c \in K^+$ is bounded as well.

Consequently, all hypotheses of Theorem 5.1 hold with $\sigma(s,t) := \max_{i=1,\dots,k} \sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s,0) = s$ for all $s \ge 0$. The rest of the proof is a consequence of (5.12) in conjunction with definitions (5.78), (5.79). The proof is complete. □

For the ISS case where $H(t,x) = x$, one can set $W(t,x) \equiv 0$ in Theorem 5.3 to arrive at a corollary on the vector Lyapunov function characterization of the ISS property.

Corollary 5.1 (Vector Lyapunov function characterization of the ISS property) *Consider system* (1.3) *under Hypotheses* (H1–5) *and suppose that there exists a family of functions* $V_i \in C^1(\Re^+ \times \Re^n; \Re^+)$ $(i = 1, \dots, k)$, *functions* $a_1, a_2 \in K_\infty$ $\beta \in K^+$, $\zeta \in \mathcal{N}_1$, $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \dots, k$, *and a family of positive definite functions* $\rho_i \in C^0(\Re^+; \Re^+)$ $(i = 1, \dots, k)$ *such that*

$$a_1\big(|x|\big) \le \max_{i=1,\dots,k} V_i(t,x) \le a_2\big(\beta(t)|x|\big) \quad \textit{for all } (t,x) \in \Re^+ \times \Re^n \tag{5.81}$$

and implication (5.66) *holds for every* $i = 1, \dots, k$ *and* $(t,x,u) \in \Re^+ \times \Re^n \times U$. *If, additionally, the small-gain conditions* (5.3) *hold, then system* (1.3) *satisfies the ISS property with gain* $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ *from the input* $u \in M_U$, *where* $\theta \in \mathcal{N}_1$ *is defined by* (5.67), (5.68). *Moreover, if* $\beta \in K^+$ *is bounded, then system* (1.3) *satisfies the UISS property with gain* $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ *from the input* $u \in M_U$.

Finally, we present an additional result for autonomous systems for which the set-valued map $S(t)$ does not coincide with the whole state space. Consider the following nonlinear system described by ODEs of the form:

$$\dot{x} = f(x,d,u) \quad x \in \Re^n, d \in D, u \in U \tag{5.82}$$

where $D \subseteq \Re^l$, $U \subseteq \Re^m$ with $0 \in U$, and $f : \Re^n \times D \times U \to \Re^n$ is a continuous mapping with $f(0, d, 0) = 0$ for all $d \in D$.

Theorem 5.4 *Consider the autonomous system* (5.82) *under Hypotheses* (H1), (H3) *and suppose that there exist functions $h \in C^1(\Re^n; \Re)$ with $h(0) \leq 0$, $V_i \in C^1(\Re^n; \Re^+)$ $(i = 1, \dots, k)$, a radially unbounded function $W \in C^1(\Re^n; \Re^+)$, a function $\delta \in C^0(\Re^+; (0, +\infty))$, a nondecreasing function $K \in C^0(\Re^+; \Re^+)$, $a_1, a_2 \in K_\infty$, $\zeta \in \mathcal{N}_1$, $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \dots, k$, and positive definite functions $\rho_i \in C^0(\Re^+; \Re^+)$ $(i = 1, \dots, k)$ such that the following inequalities hold*:

$$a_1\big(|x|\big) \leq \max_{i=1,\dots,k} V_i(x) \leq a_2\big(|x|\big) \quad \textit{for all } x \in \Re^n \textit{ with } h(x) \leq 0 \tag{5.83}$$

$$\sup_{d\in D} \nabla h(x) f(x, d, u) \leq -\delta\big(h(x)\big) \quad \textit{for all } (x, u) \in \Re^n \times U \textit{ with } h(x) \geq 0 \tag{5.84}$$

$$\sup_{d\in D} \nabla W(x) f(x, d, u) \leq K\big(h(x)\big) W(x) + K\big(h(x)\big)\zeta\big(|u|\big)$$
$$\textit{for all } (x, u) \in \Re^n \times U \textit{ with } h(x) \geq 0 \tag{5.85}$$

Moreover, for all $i = 1, \dots, k$ and $x \in \Re^n$ with $h(x) \leq 0$, the following implication holds:

$$\text{“}\textit{if } \max\Big\{\zeta\big(|u|\big), \max_{j=1,\dots,k} \gamma_{i,j}\big(V_j(x)\big)\Big\} \leq V_i(x),$$
$$\textit{then } \sup_{d\in D} \nabla V_i(x) f(x, d, u) \leq -\rho_i\big(V_i(x)\big)\text{”} \tag{5.86}$$

Also suppose that the MAX-preserving mapping $\Gamma : \Re^k_+ \to \Re^k_+$ with $\Gamma(x) = (\Gamma_1(x), \dots, \Gamma_k(x))'$, $\Gamma_i(x) = \max_{j=1,\dots,k} \gamma_{i,j}(x_j)$ for all $x \in \Re^k_+$, $i = 1, \dots, k$, satisfies the cyclic small-gain conditions. Then the following statements hold:

- *If $W \in C^1(\Re^n; \Re_+)$ is positive definite, then system* (5.82) *satisfies the UISS property from the input $u \in U$.*
- *If $U = \{0\}$, then system* (5.82) *is URGAS.*

Proof of Theorem 5.4 Consider a solution $x(t)$ of (5.82) corresponding to arbitrary $(u, d) \in M_U \times M_D$ with arbitrary initial condition $x(0) = x_0 \in \Re^n$. Clearly, there exists a maximal existence time for the solution denoted by $t_{\max} \leq +\infty$. Suppose that $h(x(t)) \leq 0$ for all $t \in [0, t_{\max})$ and let $V_i(t) = V_i(x(t))$, $i = 1, \dots, k$, be absolutely continuous functions on $[0, t_{\max})$. Moreover, let $I \subset [0, t_{\max})$ be the zero Lebesgue measure set where $x(t)$ is not differentiable or $\dot{x}(t) \neq f(x(t), d(t), u(t))$. By virtue of (5.86), it follows that the following implication holds for $t \in [0, t_{\max})\backslash I$ and $i = 1, \dots, k$:

$$V_i(t) \geq \max\Big\{\zeta\big(|u(t)|\big), \max_{j=1,\dots,k} \gamma_{i,j}\big(V_j(t)\big)\Big\} \quad \Rightarrow \quad \dot{V}_i(t) \leq -\rho_i\big(V_i(t)\big) \tag{5.87}$$

Lemma 2.14 in Chap. 2, in conjunction with (5.87), implies that there exists a family of continuous functions σ_i $(i = 1, \dots, k)$ of class KL with $\sigma_i(s, 0) = s$ for all $s \geq 0$ such that, for all $t \in [0, t_{\max})$ and $i = 1, \dots, k$,

$$V_i(t) \le \max\Big\{\sigma_i\big(V_i(0),t\big),\ \sup_{0\le\tau\le t}\sigma_i\Big(\max_{j=1,\dots,k}\sup_{0\le s\le\tau}\gamma_{i,j}\big(V_j(s)\big),t-\tau\Big),$$
$$\sup_{0\le\tau\le t}\sigma_i\Big(\zeta\Big(\sup_{0\le s\le\tau}\big|u(s)\big|\Big),t-\tau\Big)\Big\} \tag{5.88}$$

Let $\sigma(s,t) := \max_{i=1,\dots,k}\sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s,0) = s$ for all $s \ge 0$. It follows from (5.88) that, if the solution $x(t)$ of (5.82) satisfies $h(x(t)) \le 0$ for all $t \in [0,t_{\max})$, then the following inequalities hold for all $t \in [0,t_{\max})$ and $i = 1,\dots,k$:

$$V_i(t) \le \max\Big\{\sigma\Big(\max_{i=1,\dots,k}V_i(0),t\Big),\ \max_{j=1,\dots,k}\gamma_{i,j}\Big(\sup_{0\le s\le t}V_j(s)\Big),\zeta\Big(\sup_{0\le s\le t}\big|u(s)\big|\Big)\Big\} \tag{5.89}$$

Since system (5.82) is autonomous, we need to consider only the case where the initial time is 0. Consequently, (5.89) implies that Hypothesis (SG1) holds with $\Gamma : \Re^k_+ \to \Re^k_+$, $\Gamma(x) = (\Gamma_1(x),\dots,\Gamma_n(x))'$ with $\Gamma_i(x) = \max_{j=1,\dots,k}\gamma_{i,j}(x_j)$ for all $i = 1,\dots,k$ and $x \in \Re^n_+$, and

$$\begin{aligned} &L(t,x) := \max_{i=1,\dots,k}V_i(x) \quad \text{for all } (t,x)\in\Re^+\times\Re^n \text{ and}\\ &S(t) := S = \big\{x\in\Re^n : h(x)\le 0\big\} \quad \text{for all } t\ge 0\end{aligned} \tag{5.90}$$

Furthermore, if $U = \{0\}$, then, inequality (5.89) and definitions (5.90) imply that Hypothesis (SG4) holds. Definitions (5.90), in conjunction with (5.83), show that Hypothesis (SG3) holds as well with $q(x) := a_1^{-1}(\max_{i=1,\dots,k}x_i)$ for all $x \in \Re^k_+$, $H(t,x) := x$ for all $(t,x) \in \Re^+\times\Re^n$, $g \equiv 0$, $\kappa(t) = \mu(t) \equiv 1$, and $b(s) := a_1^{-1}(s)$ for all $s \ge 0$.

It should be noticed that inequality (5.84) guarantees that the set $S = \{x \in \Re^n : h(x) \le 0\}$ is positively invariant for system (5.82) and for every applied input $(u,d) \in M_U \times M_D$.

We next consider the solution $x(t)$ of (5.82) corresponding to arbitrary $(u,d) \in M_U \times M_D$ with arbitrary initial condition $x(0) = x_0 \notin S$. Define

$$\xi := \sup\big\{t\in[0,t_{\max}) : h\big(x(t)\big) > 0\big\} \tag{5.91}$$

The continuity of h and the fact that $x(0) = x_0 \notin S$ imply that $\xi > 0$. Definition (5.91) and positive invariance of the set $S = \{x \in \Re^n : h(x) \le 0\}$ implies that

(a) $h(x(t)) > 0$ for all $t \in [0,\xi)$,
(b) Either $\xi = t_{\max}$, or $\xi < t_{\max}$ and $h(x(\xi)) = 0$.

Therefore, inequalities (5.84), (5.85) imply that the following differential inequalities hold:

$$\dot h(t) \le -\delta\big(h(t)\big) \quad \text{for almost all } t\in[0,\xi) \tag{5.92}$$
$$\dot W(t) \le K\big(h(t)\big)W(t) + K\big(h(t)\big)\zeta\big(\big|u(t)\big|\big) \quad \text{for almost all } t\in[0,\xi) \tag{5.93}$$

where $h(t) := h(x(t))$ and $W(t) := W(x(t))$. Inequality (5.92) implies that the mapping $t \to h(t)$ is nonincreasing on $[0,\xi)$. Using the fact that K is nondecreasing, we obtain from (5.92) and (5.93) the following inequalities:

$$0 < h(t) \leq h(x_0) - \tilde{\delta} t \quad \text{for all } t \in [0, \xi) \tag{5.94}$$

$$W(t) \leq \exp\big(K\big(h(x_0)\big)t\big)\Big[W(x_0) + \sup_{0 \leq \tau \leq t} \zeta\big(|u(\tau)|\big)\Big] \quad \text{for all } t \in [0, \xi) \tag{5.95}$$

where $\tilde{\delta} := \min_{0 \leq s \leq h(x_0)} \delta(s) > 0$.

We next show by contradiction that the case $\xi = t_{\max}$ cannot happen. Suppose that $\xi = t_{\max}$.

- If $t_{\max} < +\infty$, then necessarily $\limsup_{t \to t_{\max}^-} |x(t)| = +\infty$. Since W is radially unbounded, we must have $\limsup_{t \to t_{\max}^-} W(x(t)) = +\infty$. On the other hand, inequality (5.95) shows that there exists a finite constant $A > 0$ such that $W(x(t)) \leq A$ for all $t \in [0, t_{\max})$, a contradiction.
- If $\xi = t_{\max} = +\infty$, then inequality (5.94) shows that $0 < h(t) \leq h(x_0) - \tilde{\delta} t$ for all $t \geq 0$, again a contradiction.

Therefore, we can conclude that $\xi < t_{\max}$ and $h(x(\xi)) = 0$. Inequality (5.94) implies that $\xi \leq \frac{h(x_0)}{\tilde{\delta}}$.

Positive invariance of the set $S = \{x \in \Re^n : h(x) \leq 0\}$ implies that, for all $x(0) = x_0 \in \Re^n$ and $(u, d) \in M_U \times M_D$, there exists $\xi \in [0, t_{\max})$ with $\xi \leq \Xi(x_0)$ satisfying $x(t) \in S$ for all $t \in [\xi, t_{\max})$, where

$$\Xi(x) := \frac{\max\{0, h(x)\}}{\min_{0 \leq s \leq \max\{0, h(x)\}} \delta(s)} \tag{5.96}$$

It should be noticed that the function $\Xi : \Re^n \to \Re_+$ defined by (5.96) is a continuous mapping with $\Xi(0) = 0$. Therefore, there exists $A \in K_\infty$ such that $\Xi(x) \leq A(|x|)$ for all $x \in \Re^n$. It follows from all the above that inequalities (5.7), (5.16) hold with $c(t) \equiv 1$. Moreover, definition (5.90) and (5.95) implies that inequalities (5.5), (5.14) hold with $p(x) := \max_{i,j=1,\ldots,k} \gamma_{i,j}(x_i)$ for all $x \in \Re_+^k$, $v(t) = c(t) \equiv 1$, $p^u \equiv 0$, and arbitrary $a \in K_\infty$.

In order to finish the proof, we have to show the following.

- There exist $a, \eta \in \mathcal{N}_1$ such that (5.8) holds with $H(t, x) := x$, $c(t) \equiv 1$ for the case that $W \in C^1(\Re^n; \Re_+)$ is positive definite. Notice that inequalities (5.6), (5.9) are direct consequences of (5.8) with $H(t, x) := x$, $c(t) \equiv 1$ and (5.83), (5.90) for appropriate $a, \tilde{\eta}, g^u \in K_\infty$ and $c(t) \equiv 1$. In this case the conclusion of the theorem follows from Theorem 5.1.
- There exists $a \in K_\infty$ and $R \geq 0$ such that (5.15) holds with $\nu(t) = c(t) \equiv 2(1 + R)$ for the case $U = \{0\}$. Notice that inequality (5.17) is a direct consequence of (5.15) with $\nu(t) = c(t) \equiv 2(1 + R)$ and (5.83), (5.90) for appropriate $a \in K_\infty$, $c \in K^+$. In this case the conclusion of the theorem follows from Theorem 5.2.

If $W \in C^1(\Re^n; \Re^+)$ is positive definite and radially unbounded, then (Proposition 2.2) there exist $b_1, b_2 \in K_\infty$ such that $b_1(|x|) \leq W(x) \leq b_2(|x|)$ for all $x \in \Re^n$. Moreover, using the fact that $\xi \leq \Xi(x_0)$, from (5.95) we obtain, for all

$x(0) = x_0 \in \Re^n$, $(u,d) \in M_U \times M_D$, and $t \in [0,\xi]$,

$$\begin{aligned}
b_1\big(|x(t)|\big) &\le W(t), \\
W(t) &\le \exp\big(K\big(\max\{0, h(x_0)\}\big)t\big)\Big[W(x_0) + \sup_{0\le\tau\le t} \zeta\big(|u(\tau)|\big)\Big] \\
&\le \exp\big(K\big(\max\{0, h(x_0)\}\big)\Xi(x_0)\big)\Big[W(x_0) + \sup_{0\le\tau\le t} \zeta\big(|u(\tau)|\big)\Big] \\
&= \exp\big(K\big(\max\{0, h(x_0)\}\big)\Xi(x_0)\big)W(x_0) + \sup_{0\le\tau\le t} \zeta\big(|u(\tau)|\big) \\
&\quad + \big(\exp\big(K\big(\max\{0, h(x_0)\}\big)\Xi(x_0)\big) - 1\big) \sup_{0\le\tau\le t} \zeta\big(|u(\tau)|\big) \\
&\le \exp\big(K\big(\max\{0, h(x_0)\}\big)\Xi(x_0)\big)b_2\big(|x_0|\big) + \sup_{0\le\tau\le t} \zeta\big(|u(\tau)|\big) \\
&\quad + \frac{1}{2}\big(\exp\big(K\big(\max\{0, h(x_0)\}\big)\Xi(x_0)\big) - 1\big)^2 + \frac{1}{2} \sup_{0\le\tau\le t} \zeta^2\big(|u(\tau)|\big) \\
&\le \max\Big\{2\exp\big(K\big(\max\{0, h(x_0)\}\big)\Xi(x_0)\big)b_2\big(|x_0|\big) \\
&\quad + \frac{1}{2}\big(\exp\big(K\big(\max\{0, h(x_0)\}\big)\Xi(x_0)\big) - 1\big)^2, \sup_{0\le\tau\le t} A\big(|u(\tau)|\big)\Big\}
\end{aligned}$$

where $A(s) := \zeta^2(s) + 2\zeta(s)$. Since the mapping

$$\begin{aligned}
\Re^n \ni x \to B(x) &:= 2\exp\big(K\big(\max\{0, h(x)\}\big)\Xi(x)\big)b_2\big(|x|\big) \\
&\quad + \frac{1}{2}\big(\exp\big(K\big(\max\{0, h(x)\}\big)\Xi(x)\big) - 1\big)^2
\end{aligned}$$

is nonnegative, continuous, and vanishing at zero, by virtue of Lemma 2.4, there exists $b_3 \in K_\infty$ such that $B(x) \le b_3(|x|)$ for all $x \in \Re^n$. The above inequalities show that (5.8) holds with $H(t,x) := x$, $c(t) \equiv 1$, $\eta(s) := b_1^{-1}(A(s))$, and $a(s) := b_1^{-1}(b_3(s))$.

Finally, if $U = \{0\}$, then we can define the function $P : \Re^n \to \Re^+$ by

$$P(x) := \max\big\{|y| : y \in \Re^n, W(y) \le \exp\big(K\big(\max\{0, h(x)\}\big)\Xi(x)\big)W(x)\big\} \tag{5.97}$$

Notice that since W is continuous, nonnegative, and radially unbounded, the functions $P : \Re^n \to \Re^+$ is locally bounded. Since $P : \Re^n \to \Re^+$ is locally bounded, there exist $b_4 \in K_\infty$ and $R \ge 0$ such that $P(x) \le R + b_4(|x|)$ for all $x \in \Re^n$. From (5.95), (5.97), and the fact that $P(x) \le R + b_4(|x|)$ for all $x \in \Re^n$, the following inequality holds for all $x(0) = x_0 \in \Re^n$, $(u,d) \in M_U \times M_D$, and $t \in [0,\xi]$:

$$\big|x(t)\big| \le \max\big\{2b_4\big(|x_0|\big), 2R\big\} \tag{5.98}$$

Therefore, inequality (5.15) holds with $\nu(t) = c(t) \equiv 2(1+R)$ and $a(s) := 2b_4(s)$. The proof is complete. □

5.3.2 Vector Lyapunov Functionals for Systems Described by RFDEs

Consider system (1.10) under Hypotheses (S1–4), where the output map H is independent of $u \in U$. The following theorem provides sufficient Lyapunov-like conditions for the (U)IOS property. The gain functions of the IOS property can be determined *explicitly* in terms of the functions involved in the assumptions of the theorem. The reader should recall at this point Definition 2.4 in Chap. 2 of the *almost Lipschitz on bounded sets functionals* and the definition of the Dini derivative used in Lemma 2.16 in Chap. 2.

Theorem 5.5 *Consider system* (1.10) *under* (S1–4) *and suppose that there exist* (*almost Lipschitz on bounded sets*) *functionals* $Q_i : [-r + r_i, +\infty) \times C^0([-r_i, 0]; \Re^n) \to \Re^+$ *with* $0 \le r_i \le r$ $(i = 1, \ldots, k)$, $Q_0 : [-r + r_0, +\infty) \times C^0([-r_0, 0]; \Re^n) \to \Re^+$ *with* $0 \le r_0 \le r$, *functions* $a_1, a_2, a_3, a_4 \in K_\infty$ $\mu, \beta, \kappa \in K^+$, $\zeta \in \mathcal{N}_1$, $g \in \mathcal{N}_k, \gamma_{i,j} \in \mathcal{N}_1$, $p_i \in \mathcal{N}_1$, $i, j = 1, \ldots, k$, *positive definite functions* $\rho_i \in C^0(\Re^+; \Re^+)$ $(i = 1, \ldots, k)$, *and a constant* $\lambda \in (0, 1)$ *such that*, *for all* $(t, x, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U$, *the following inequalities hold*:

$$a_1\big(\|H(t,x)\|_{\mathcal{Y}}\big) \le \max_{i=1,\ldots,k} V_i(t,x) \le a_2\big(\beta(t)\|x\|_r\big) \tag{5.99}$$

$$a_3\big(\mu(t)\|x\|_r\big) - g\big(V_1(t,x), \ldots, V_k(t,x)\big) - \kappa(t) \le W(t,x) \le a_4\big(\beta(t)\|x\|_r\big) \tag{5.100}$$

$$\begin{aligned} &\sup_{d\in D} Q_0^0\big(t, T_{r_0}(0)x; f(t,x,u,d)\big) \\ &\quad \le -Q_0\big(t, T_{r_0}(0)x\big) + \lambda \max\Big\{\zeta\big(|u|\big), \max_{j=1,\ldots,k} p_j\big(V_j(t,x)\big)\Big\} \end{aligned} \tag{5.101}$$

where

$$\begin{aligned} V_i(t,x) &:= \sup_{\theta\in[-r+r_i,0]} Q_i\big(t+\theta, T_{r_i}(\theta)x\big) \quad i = 1, \ldots, k, \\ W(t,x) &:= \sup_{\theta\in[-r+r_0,0]} Q_0\big(t+\theta, T_{r_0}(\theta)x\big) \end{aligned} \tag{5.102}$$

and for all $i = 1, \ldots, k$ *and* $(t, x, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U$, *the following implication holds*:

$$\begin{aligned} &\textit{if } \max\Big\{\zeta\big(|u|\big), \max_{j=1,\ldots,k} \gamma_{i,j}\big(V_j(t,x)\big)\Big\} \le Q_i\big(t, T_{r_i}(0)x\big), \\ &\textit{then } \sup_{d\in D} Q_i^0\big(t, T_{r_i}(0)x; f(t,x,u,d)\big) \le -\rho_i\big(Q_i\big(t, T_{r_i}(0)x\big)\big) \end{aligned} \tag{5.103}$$

Furthermore, *suppose that the small-gain conditions* (5.2), (5.3) *hold*.

Then, *system* (1.10) *satisfies the IOS property with gain* $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ *from the input* $u \in M_U$, *where* $\theta \in \mathcal{N}_1$ *is defined by* (5.67), (5.68). *Moreover*, *if* $\beta \in K^+$ *is bounded*, *then for every* $\rho \in K_\infty$, *system* (1.10) *satisfies the UIOS property with gain* $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ *from the input* $u \in M_U$.

Proof of Theorem 5.5 We want to show that all hypotheses of Theorem 5.1 hold with $S(t) \equiv C^0([-r,0];\Re^n)$ and $L(t,x)$ as defined by (5.69).

Notice that Hypothesis (SG3) of Theorem 5.1 is a direct consequence of inequalities (5.100), definitions (5.69), (5.102), and inequality (5.99) with $q(x) := a_1^{-1}(\max_{i=1,\dots,k} x_i)$ for all $x \in \Re^n_+$.

Finally, notice that since $S(t) \equiv C^0([-r,0];\Re^n)$, we can select the time ξ involved in Hypothesis (SG2) to be $\xi = t_0$. Consequently, inequalities (5.6), (5.7), (5.8), and (5.9) are automatically satisfied for appropriate functions. Thus, we are left with the task of proving inequalities (5.4) and (5.5). The proof consists of two steps:

Step 1: We show that inequalities (5.4) and (5.5) hold for arbitrary $(t_0,u,d) \in \Re^+ \times M_U \times M_D$ and $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$.
Step 2: We show that inequalities (5.4) and (5.5) hold for arbitrary $(t_0,u,d) \in \Re^+ \times M_U \times M_D$ and $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$.

Proof of Step 1: In this setup, consider the solution $x(t)$ of (1.10) corresponding to arbitrary $(u,d) \in M_U \times M_D$ with arbitrary initial condition $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$. Clearly, there exists a maximal existence time for the solution denoted by $t_{\max} \le +\infty$. By virtue of Lemma 2.17 in Chap. 2, we can guarantee that the functions $Q_i(t) = Q_i(t, T_{r_i}(t)x)$, $i = 1,\dots,k$, and $Q_0(t) = Q_0(t, T_{r_0}(t)x)$ are absolutely continuous functions on $[t_0, t_{\max})$. Let $V_i(t) = V_i(t, T_r(t)x) = \sup_{\theta\in[-r+r_i,0]} Q_i(t+\theta)$, $i = 1,\dots,k$, $W(t) = W(t, T_r(t)x) = \sup_{\theta\in[-r+r_0,0]} Q_0(t+\theta)$, and $L(t) = L(t, T_r(t)x)$ be mappings defined on $[t_0, t_{\max})$. By virtue of (5.101), (5.103) and Lemma 2.16, there exists a zero Lebesgue measure set $I \subset [t_0, t_{\max})$ such that the following equations hold for $t \in [t_0, t_{\max})\backslash I$ and $i = 1,\dots,k$:

$$Q_i(t) \ge \max\left\{\sup_{t_0\le s\le t} \zeta\big(|u(s)|\big),\ \max_{j=1,\dots,k} \sup_{t_0\le s\le t} \gamma_{i,j}\big(V_j(s)\big)\right\}$$
$$\Rightarrow \quad \dot{Q}_i(t) \le -\rho_i\big(Q_i(t)\big) \tag{5.104}$$

$$\dot{Q}_0(t) \le -Q_0(t) + \lambda \max\left\{\zeta\big(|u(t)|\big),\ \max_{j=1,\dots,k} p_j\big(V_j(t)\big)\right\}. \tag{5.105}$$

Lemma 2.14 in Chap. 2, in conjunction with (5.104), implies that there exists a family of continuous functions $\tilde{\sigma}_i$ ($i = 1,\dots,k$) of class KL with $\tilde{\sigma}_i(s,0) = s$ for all $s \ge 0$ such that, for all $t \in [t_0, t_{\max})$ and $i = 1,\dots,k$, we have

$$Q_i(t) \le \max\left\{\tilde{\sigma}_i\big(Q_i(t_0), t-t_0\big);\ \sup_{t_0\le\tau\le t} \tilde{\sigma}_i\Big(\max_{j=1,\dots,k} \sup_{t_0\le s\le\tau} \gamma_{i,j}\big(V_j(s)\big), t-\tau\Big)\right.$$
$$\left.\sup_{t_0\le\tau\le t} \tilde{\sigma}_i\Big(\zeta\Big(\sup_{t_0\le s\le\tau} \big|u(s)\big|\Big), t-\tau\Big)\right\} \tag{5.106}$$

Moreover, inequality (5.105) directly implies that, for all $t \in [t_0, t_{\max})$,

$$Q_0(t) \le Q_0(t_0) + \lambda \max\left\{\zeta\Big(\sup_{t_0\le s\le t} \big|u(s)\big|\Big),\ \max_{j=1,\dots,k} p_j\Big(\sup_{t_0\le s\le t} V_j(s)\Big)\right\} \tag{5.107}$$

Using the fact that $\tilde{\sigma}_i(s,0) = s$ for all $s \ge 0$, from (5.106) we obtain, for all $t \in [t_0, t_{\max})$ and $i = 1,\dots,k$,

$$Q_i(t) \leq \max\left\{\tilde{\sigma}_i\left(Q_i(t_0), t - t_0\right), \max_{j=1,\ldots,k} \gamma_{i,j}\left(\sup_{t_0 \leq s \leq t} V_j(s)\right), \zeta\left(\sup_{t_0 \leq s \leq t} \left|u(s)\right|\right)\right\} \tag{5.108}$$

Let σ_i $(i = 1, \ldots, k)$ be functions of class KL defined by $\sigma_i(s,t) = s$ for all $s \geq 0$, $t \in [0,r]$ and $\sigma_i(s,t) = \tilde{\sigma}_i(s, t-r)$ for all $s \geq 0$, $t > r$. Using the fact that $V_i(t) = V_i(t, T_r(t)x) = \sup_{\theta\in[-r+r_i,0]} Q_i(t+\theta)$, $i = 1, \ldots, k$, it follows from (5.108) that, for all $t \in [t_0, t_{\max})$ and $i = 1, \ldots, k$,

$$V_i(t) \leq \max\left\{\sigma_i\left(V_i(t_0), t - t_0\right), \max_{j=1,\ldots,k} \gamma_{i,j}\left(\sup_{t_0 \leq s \leq t} V_j(s)\right), \zeta\left(\sup_{t_0 \leq s \leq t} \left|u(s)\right|\right)\right\} \tag{5.109}$$

Similarly, using (5.107) and the fact that $W(t) = W(t, T_r(t)x) = \sup_{\theta\in[-r+r_0,0]} Q_0(t+\theta)$, we conclude that (5.73) holds for all $t \in [t_0, t_{\max})$. Define $\sigma(s,t) := \max_{i=1,\ldots,k} \sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s,0) = s$ for all $s \geq 0$. It follows from (5.109) and definition (5.69) that inequalities (5.75), (5.76) hold for all $t \in [t_0, t_{\max})$ and $i = 1, \ldots, k$. Clearly, (5.75) shows that inequalities (5.4) hold with $\Gamma : \Re^k_+ \to \Re^k_+$, $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ with $\Gamma_i(x) = \max_{j=1,\ldots,k} \gamma_{i,j}(x_j)$ for all $i = 1, \ldots, k$ and $x \in \Re^n_+$. Moreover, the small-gain conditions hold for the MAX-preserving mapping $\Gamma : \Re^k_+ \to \Re^k_+$ as well. Define $p^u \in \mathcal{N}_1$ and $p \in \mathcal{N}_n$ by (5.78) and (5.79). Combining estimates (5.75), (5.76) and exploiting definitions (5.69), (5.78), and (5.79), we get inequality (5.80) for all $t \in [t_0, t_{\max})$. Inequality (5.5) is a direct consequence of (5.80), inequalities (5.99), (5.100), and Lemma 3.2 in Chap. 3 with $\nu(t) \equiv 1$, $p^u \in \mathcal{N}_1$, and $p \in \mathcal{N}_n$ as defined by (5.78), (5.79) and appropriate $a \in \mathcal{N}_1$ and $c \in K^+$. Notice that if $\beta \in K^+$ is bounded, then $c \in K^+$ is bounded as well.

Proof of Step 2: Let $(t_0, x_0, u, d) \in \Re^+ \times C^1([-r,0];\Re^n) \times M_U \times M_D$. Inequalities (5.4), in conjunction with Proposition 5.3, imply that, for the solution $x(t)$ of (1.10) corresponding to $(u,d) \in M_U \times M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$ and for all $t \in [t_0, t_{\max})$,

$$V(t) \leq \text{MAX}\left\{Q\left(1\sigma\left(L(t_0), 0\right)\right), Q\left(1\zeta\left(\left[\left\|u(\tau)\right\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right)\right\} \tag{5.110}$$

where $Q(x) = \text{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\}$. Using (5.99), (5.100), (5.69), (5.80), and (5.110), we obtain functions $\rho \in K^+$ and $a \in K_\infty$ such that, for the solution $x(t)$ of (1.10) corresponding to $(u,d) \in M_U \times M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$ and for all $t \in [t_0, t_{\max})$,

$$\left\|T_r(t)x\right\|_r \leq a\left(\rho(t) + \|x_0\|_r + \sup_{t_0 \leq s \leq t} \left|u(s)\right|\right) \tag{5.111}$$

Lemma 2.18 in Chap. 2 and (5.111) imply that system (1.10) is RFC from the input $u \in M_U$ and that inequalities (5.4) and (5.5) hold for all $(t_0, x_0, u, d) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_U \times M_D$ and $t \geq t_0$. Consequently, all hypotheses of Theorem 5.1 hold. The rest of proof is a consequence of (5.12) in conjunction with definitions (5.78), (5.79). The proof is complete. □

When $H(t,x) = x$, setting $Q_0(t,x) \equiv 0$ in Theorem 5.5 leads to a result on the ISS of system (1.10).

Corollary 5.2 *Consider system* (1.10) *under Hypotheses* (S1–4) *and suppose that there exists a family of* (*almost Lipschitz on bounded sets*) *functionals* $Q_i : [-r + r_i, +\infty) \times C^0([-r_i, 0]; \Re^n) \to \Re^+$ *with* $0 \le r_i \le r$ $(i = 1, \dots, k)$, *functions* $a_1, a_2 \in K_\infty$, $\beta \in K^+$, $\zeta \in \mathcal{N}_1$, $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \dots, k$, *and a family of positive definite functions* $\rho_i \in C^0(\Re^+; \Re^+)$ $(i = 1, \dots, k)$ *such that*, *for all* $(t, x, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U$, *the following inequality holds*:

$$a_1\big(\|x\|_r\big) \le \max_{i=1,\dots,k} V_i(t,x) \le a_2\big(\beta(t)\|x\|_r\big) \tag{5.112}$$

where

$$V_i(t,x) := \sup_{\theta \in [-r+r_i, 0]} Q_i\big(t+\theta, T_{r_i}(\theta)x\big) \quad i = 1, \dots, k \tag{5.113}$$

and implication (5.103) *holds for all* $i = 1, \dots, k$ *and* $(t, x, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U$. *If*, *moreover*, *the small-gain conditions* (5.2), (5.3) *hold*, *then system* (1.10) *satisfies the ISS property with gain* $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ *from input* $u \in M_U$, *where* $\theta \in \mathcal{N}_1$ *is defined by* (5.67), (5.68). *Moreover*, *if* $\beta \in K^+$ *is bounded*, *then system* (1.10) *satisfies the UISS property with gain* $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ *from input* $u \in M_U$.

5.3.3 Vector Lyapunov Functions for Sampled-Data Systems

We consider autonomous systems of the form (1.57) under Hypotheses (A1–5), where R is the identity mapping (no impulsive behavior), f and h are independent of $d(\tau_i)$, and H is independent of u. More specifically, we consider systems of the form

$$\begin{aligned}
&\dot{x}(t) = f\big(x(t), x(\tau_i), d(t), u_1(t), u_1(\tau_i)\big) \quad t \in [\tau_i, \tau_{i+1}) \\
&\quad \tau_0 = t_0, \tau_{i+1} = \tau_i + \exp\big(-u_2(\tau_i)\big)h\big(x(\tau_i), u_1(\tau_i)\big), \ i = 0, 1, \dots \\
&Y(t) = H\big(x(t)\big) \\
&\quad x(t) \in \Re^n, d(t) \in D, u(t) = \big(u_1(t), u_2(t)\big) \in U := U_1 \times \Re^+
\end{aligned} \tag{5.114}$$

The following theorem provides sufficient Lyapunov-like conditions for the (U)IOS property of system (5.114). The gain functions of the IOS property can be determined *explicitly* in terms of the functions involved in the assumptions of the theorem.

Theorem 5.6 (Vector Lyapunov function characterization of UIOS for sampled-data systems) *Consider system* (5.114) *under Hypotheses* (A1–5) *and suppose that there exist nonnegative functions* $V_i \in C^1(\Re^n; \Re^+)$ $(i = 1, \dots, k)$, $Q \in C^1(\Re^n; \Re^+)$, $a_1, a_2, a_3, a_4 \in K_\infty$, $\zeta \in \mathcal{N}_1$, $g \in \mathcal{N}_k$, $\gamma_{i,j} \in \mathcal{N}_1$, $p_i \in \mathcal{N}_1$, $i, j = 1, \dots, k$, *constants*

$\mu, \kappa \geq 0$, $\lambda \in (0, 1)$, *and positive definite functions* $\rho_i \in C^0(\Re^+; \Re^+)$ $(i = 1, \dots, k)$ *such that the following inequalities hold for all* $(x, x_0, u, u_0) \in \Re^n \times \Re^n \times U_1 \times U_1$:

$$a_1\big(\big|H(x)\big|\big) \leq \max_{i=1,\dots,k} V_i(x) \leq a_2\big(|x|\big) \tag{5.115}$$

$$a_3\big(|x|\big) - g\big(V_1(x), \dots, V_k(x)\big) - \kappa \leq Q(x) \leq a_4\big(|x|\big) \tag{5.116}$$

$$\sup_{d\in D} \nabla Q(x) f(x, x_0, d, u, u_0) \leq \mu Q(x) + \lambda \max\Big\{\zeta\big(|u|\big), \zeta\big(|u_0|\big), \max_{j=1,\dots,k} p_j\big(V_j(x)\big), \max_{j=1,\dots,k} p_j\big(V_j(x_0)\big)\Big\} \tag{5.117}$$

and for all $i = 1, \dots, k$ *and* $(x, u, u_0) \in \Re^n \times U_1 \times U_1$, *the following implication holds*:

$$\text{“If } \max\Big\{\zeta\big(|u|\big), \zeta\big(|u_0|\big), \max_{j=1,\dots,k} \gamma_{i,j}\big(V_j(x)\big)\Big\} \leq V_i(x) \text{ and } x_0 \in A_i\big(h(x_0, u_0), x\big),$$
$$\text{then } \sup_{d\in D} \nabla V_i(x) f(x, x_0, d, u, u_0) \leq -\rho_i\big(V_i(x)\big).\text{”} \tag{5.118}$$

where the family of set-valued maps $\Re^+ \times \Re^n \ni (T, x) \to A_i(T, x) \subseteq \Re^n$ $(i = 1, \dots, k)$ *is defined by*

$$\begin{aligned} A_i(T, x) = \bigcup_{0\leq s\leq T} \big\{&x_0 \in \Re^n : \exists (d, u) \in M_D \times M_{U_1} \text{ with } \phi(s, x_0; d, u) = x, \\ &\zeta\big(\big|u(t)\big|\big) \leq V_i(x), \gamma_{i,j}\big(V_j\big(\phi(t, x_0; d, u)\big)\big) \leq V_i(x) \\ &\text{for all } t \in [0, s] \text{ and } j = 1, \dots, k\big\} \end{aligned} \tag{5.119}$$

and $\phi(t, x_0; d, u)$ *denotes the solution of* $\dot{x}(t) = f(x(t), x_0, d(t), u(t), u(0))$ *with initial condition* $x(0) = x_0$ *corresponding to* $(d, u) \in M_D \times M_{U_1}$.

Furthermore, if the small-gain conditions (5.2), (5.3) *hold, then system* (5.114) *satisfies the UIOS property with gain* $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ *from the input* $u_1 \in M_{U_1}$ *and zero gain from the input* $u_2 \in M_{\Re^+}$, *where* $\theta \in \mathcal{N}_1$ *is defined by* (5.67), (5.68).

For the ISS case where $H(t, x) = x$, one can set $Q(x) \equiv 0$ in Theorem 5.6 and obtain a result on the vector Lyapunov characterization of UISS.

Corollary 5.3 (Vector Lyapunov function characterization of UISS for sampled-data systems) *Consider system* (5.114) *under Hypotheses* (A1–5) *and suppose that there exists a family of functions* $V_i \in C^1(\Re^n; \Re^+)$ $(i = 1, \dots, k)$, *functions* $a_1, a_2 \in K_\infty$, $\zeta \in \mathcal{N}_1$, $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \dots, k$, *and a family of positive definite functions* $\rho_i \in C^0(\Re^+; \Re^+)$ $(i = 1, \dots, k)$ *such that the following inequality holds for all* $x \in \Re^n$:

$$a_1\big(|x|\big) \leq \max_{i=1,\dots,k} V_i(x) \leq a_2\big(|x|\big) \tag{5.120}$$

and implication (5.118) *holds for all* $i = 1, \dots, k$ *and* $(x, u, u_0) \in \Re^n \times U_1 \times U_1$, *where the set-valued maps* $\Re^+ \times \Re^n \ni (T, x) \to A_i(T, x) \subseteq \Re^n$ $(i = 1, \dots, k)$ *are defined by* (5.119), *and* $\phi(t, x_0; d, u)$ *denotes the solution of* $\dot{x}(t) =$

$f(x(t), x_0, d(t), u(t), u(0))$ with initial condition $x(0) = x_0$ corresponding to $(d, u) \in M_D \times M_{U_1}$.

Under the small-gain conditions (5.2), (5.3), system (5.114) satisfies the UISS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u_1 \in M_{U_1}$ and zero gain from the input $u_2 \in M_{\Re+}$, where $\theta \in \mathcal{N}_1$ is defined by (5.67), (5.68).

Proof of Theorem 5.6 We want to show that all hypotheses of Theorem 5.1 hold with $S(t) \equiv \Re^n$, $L(t, x)$ as defined by (5.69), and

$$W(t, x) := \exp\bigl(-(\mu + 1)t\bigr) Q(x) \tag{5.121}$$

Notice that definition (5.121), in conjunction with inequalities (5.116), implies the following inequality for all $(t, x) \in \Re^+ \times \Re^n$:

$$\exp\bigl(-(\mu + 1)t\bigr) a_3\bigl(|x|\bigr) - g\bigl(V_1(x), \ldots, V_k(x)\bigr) - \kappa \le W(t, x) \le a_4\bigl(|x|\bigr)$$

Using Lemma 3.2 in Chap. 3, we can find functions $\tilde{a} \in K_\infty$ and $\eta \in K^+$ such that $a_3^{-1}(s \exp((\mu + 1)t)) \le \frac{1}{\eta(t)} \tilde{a}(s)$ for all $t, s \ge 0$. Consequently, we obtain

$$\exp\bigl(-(\mu + 1)t\bigr) a_3(s) \ge \tilde{a}^{-1}\bigl(\eta(t)s\bigr) \quad \text{for all } t, s \ge 0$$

Notice that Hypothesis (SG3) with $\beta(t) \equiv 1$ is a direct consequence of previous inequalities, definitions (5.69), (5.121), and inequality (5.115) with $q(x) := a_1^{-1}(\max_{i=1,\ldots,k} x_i)$ for all $x \in \Re^n_+$.

Finally, notice that since $S(t) \equiv \Re^n$, we can select the time ξ involved in Hypothesis (SG2) to be $\xi = t_0$. Consequently, inequalities (5.6), (5.7), (5.8), and (5.9) are automatically satisfied for appropriate functions. Thus, we are left with the task of proving inequalities (5.4) and (5.5).

Consider the solution $x(t)$ of (5.114) under Hypotheses (A1–5) corresponding to arbitrary $(u_1, d, u_2) \in M_{U_1} \times M_D \times M_{\Re+}$ with arbitrary initial condition $x(t_0) = x_0 \in \Re^n$. Notice that since system (5.114) is autonomous, it suffices to consider the case $t_0 = 0$. There exists a maximal existence time for the solution denoted by $t_{\max} \le +\infty$. Define $V_i(t) = V_i(x(t))$, $i = 1, \ldots, k$, $W(t) = W(t, x(t))$, and $L(t) = L(t, x(t))$, which are absolutely continuous functions on $[0, t_{\max})$. Moreover, let $\pi := \{\tau_0, \tau_1, \ldots\}$ be the set of sampling times (which may be finite if $t_{\max} < +\infty$), $p(t) := \max\{\tau \in \pi : \tau \le t\}$, and $q(t) := \min\{\tau \in \pi : \tau \ge t\}$. Let $I \subset [0, t_{\max})$ be the zero Lebesgue measure set where $x(t)$ is not differentiable or where $\dot{x}(t) \ne f(x(t), x(\tau_i), d(t), u_1(t), u_1(\tau_i))$. Clearly, we have $x(t) = \phi(t - p(t), x(p(t)); P_t d, P_t u_1)$ for all $t \in [0, t_{\max})$, where $(P_t u_1)(s) = u_1(p(t) + s)$, $(P_t d)(s) = d(p(t) + s)$, $s \ge 0$. Next, we show that the following implication holds for $t \in [0, t_{\max}) \backslash I$ and $i = 1, \ldots, k$:

$$\begin{aligned} &V_i(t) \ge \max\left\{\zeta\Bigl(\sup_{p(t) \le s \le t} \bigl|u_1(s)\bigr|\Bigr), \max_{j=1,\ldots,k} \gamma_{i,j}\Bigl(\sup_{p(t) \le s \le t} V_j(s)\Bigr)\right\} \\ &\quad\Rightarrow \quad \dot{V}_i(t) \le -\rho_i\bigl(V_i(t)\bigr) \end{aligned} \tag{5.122}$$

In order to prove implication (5.122), let $t \in [0, t_{\max}) \backslash I$, $i = 1, \ldots, k$, $\tau = p(t)$, and suppose that

$$V_i(t) \ge \max\left\{\zeta\Bigl(\sup_{p(t) \le s \le t} \bigl|u_1(s)\bigr|\Bigr), \max_{j=1,\ldots,k} \gamma_{i,j}\Bigl(\sup_{p(t) \le s \le t} V_j(s)\Bigr)\right\}$$

By virtue of the semigroup property, the previous inequality implies that $\zeta(|u_1(\tau+s)|)=\zeta(|(P_t u_1)(s)|)\leq V_i(x(t)), \gamma_{i,j}(V_j(\phi(s,x(\tau);P_t d, P_t u_1)))\leq V_i(x(t))$ for all $s\in[0,t-\tau]$ and $j=1,\dots,k$. In this case, by virtue of definition (5.119) and the fact that $t-\tau\leq h(x(\tau),u_1(\tau))$, it follows that $x(\tau)\in A_i(h(x(\tau),u_1(\tau)),x(t))$. Since $\dot{x}(t)=f(x(t),x(\tau),d(t),u_1(t),u_1(\tau))$, we conclude from (5.118) that $\dot{V}_i(t)\leq -\rho_i(V_i(t))$.

Lemma 2.14 in Chap. 2 implies that there exists a family of continuous functions σ_i of class KL ($i=1,\dots,k$) with $\sigma_i(s,0)=s$ for all $s\geq 0$ such that, for all $t\in[0,t_{\max})$ and $i=1,\dots,k$, we have

$$V_i(t)\leq \max\Big\{\sigma_i\big(V_i(0),t\big);\ \max_{j=1,\dots,k}\ \sup_{0\leq\tau\leq t}\sigma_i\Big(\gamma_{i,j}\Big(\sup_{p(\tau)\leq s\leq\tau}V_j(s)\Big),t-\tau\Big);$$
$$\sup_{0\leq\tau\leq t}\sigma_i\Big(\zeta\Big(\sup_{p(\tau)\leq s\leq\tau}\big|u(s)\big|\Big),t-\tau\Big)\Big\} \tag{5.123}$$

Let $\sigma(s,t):=\max_{i=1,\dots,k}\sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s,0)=s$ for all $s\geq 0$. Inequalities (5.4) with $\Gamma:\Re_+^k\to\Re_+^k$, $\Gamma(x)=(\Gamma_1(x),\dots,\Gamma_n(x))'$ with $\Gamma_i(x)=\max_{j=1,\dots,k}\gamma_{i,j}(x_j)$ for all $i=1,\dots,k$ and $x\in\Re_+^n$, are direct consequences of the previous definition, estimates (5.123), definition (5.69), and the fact that $\sigma_i(s,0)=s$ for all $s\geq 0$ and $i=1,\dots,k$.

Exploiting (5.117) and definition (5.121), we get, for $t\in[0,t_{\max})\backslash I$,

$$\dot{W}(t)\leq -W(t)+\lambda\max\Big\{\zeta\Big(\sup_{p(t)\leq s\leq t}\big|u(s)\big|\Big),\ \max_{j=1,\dots,k}p_j\Big(\sup_{p(t)\leq s\leq t}V_j(s)\Big)\Big\} \tag{5.124}$$

Inequality (5.124) directly implies that, for all $t\in[0,t_{\max})$,

$$W(t)\leq\max\Big\{\frac{1}{1-\lambda}W(0),\zeta\Big(\sup_{0\leq s\leq t}\big|u(s)\big|\Big),\ \max_{i=1,\dots,k}p_i\Big(\sup_{0\leq s\leq t}V_i(s)\Big)\Big\} \tag{5.125}$$

Moreover, (5.123) and (5.125) imply that estimates (5.75), (5.76) with $t_0=0$ hold for all $t\in[0,t_{\max})$. Define $p^u\in\mathcal{N}_1$ and $p\in\mathcal{N}_n$ by (5.78) and (5.79). Combining estimates (5.75), (5.76) with $t_0=0$ and exploiting definitions (5.78), (5.79), and (5.69), we obtain (5.80) for all $t\in[0,t_{\max})$ with $h(s):=\frac{s}{1-\lambda}$ and $t_0=0$. Inequality (5.5) is a direct consequence of (5.80) with $t_0=0$, inequalities (5.115), (5.116) with $\nu(t)=c(t)\equiv 1$, $p^u\in\mathcal{N}_1$, $p\in\mathcal{N}_n$ as defined by (5.78), (5.79), and appropriate $a\in\mathcal{N}_1$.

Consequently, all hypotheses of Theorem 5.1 hold with $\sigma(s,t):=\max_{i=1,\dots,k}\sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s,0)=s$ for all $s\geq 0$. The proof is complete. □

Remark 5.1 When we consider the special case of the feedback interconnection of two subsystems, as illustrated in Fig. 1.2, the small-gain results of this chapter are generalizations of the first nonlinear, ISS, small-gain theorem presented in [21]. In this case, the cyclic small-gain conditions (5.2) and (5.3) reduce down to $\gamma_{12}\circ\gamma_{21}<Id$, which is mathematically equivalent to $\gamma_{21}\circ\gamma_{12}<Id$. It is important to notice that the original nonlinear small-gain theorem in [21] tackles interconnected systems described by ODEs and naturally assumes that $\gamma_{11}=\gamma_{22}\equiv 0$.

5.4 Examples and Applications

In this section, we present examples and applications of the results of the previous sections. More specifically, the first two examples illustrate how we can use vector Lyapunov functions in order to prove global asymptotic stability of time-delay systems.

Example 5.4.1 Consider the time-delay system of the form

$$\dot{x}_i(t) = -a_i x_i(t) + g_i\big(d(t), T_r(t)x\big) \quad i = 1, \ldots, n \tag{5.126}$$

where $d(t) \in D \subseteq \Re^m$, $a_i > 0$ $(i = 1, \ldots, n)$ and $g_i : D \times C^0([-r, 0]; \Re^n) \to \Re$ $(i = 1, \ldots, n)$ are continuous mappings with

$$\sup_{d \in D} \big|g_i(d, x)\big| \le \max_{j=1,\ldots,n} c_{i,j} \|x_j\|_r \tag{5.127}$$

and

$$\sup_{d \in D} \big[\big(x_i(0) - y_i(0)\big)\big(g_i(d, x) - g_i(d, y)\big)\big] \le L\|x - y\|_r^2 \quad i = 1, \ldots, n$$

for certain constants $L \ge 0$, $c_{i,j} \ge 0$ $(i, j = 1, \ldots, n)$, and for all $x, y \in C^0([-r, 0]; \Re^n)$. We next show that $0 \in C^0([-r, 0]; \Re^n)$ is RGAS for (5.126) if $c_{i,i} < a_i$ for all $i = 1, \ldots, n$ and the following small-gain conditions hold for each $r = 2, \ldots, n$:

$$c_{i_1,i_2} c_{i_2,i_3} \cdots c_{i_r,i_1} < a_{i_1} a_{i_2} \cdots a_{i_r} \tag{5.128}$$

for all $i_j \in \{1, \ldots, n\}$, $i_j \ne i_k$ if $j \ne k$.

First, we notice that Hypotheses (S1–4) hold for system (5.126) under hypothesis (5.127) with output $H(t, x) := x \in C^0([-r, 0]; \Re^n)$. Define the family of functions $Q_i(x) = \frac{1}{2}x_i^2(0)$ and $V_i(x) := \sup_{\theta \in [-r,0]} Q_i(x(\theta)) = \frac{1}{2}\|x_i\|_r^2$ $(i = 1, \ldots, n)$ for $x \in C^0([-r, 0]; \Re^n)$. These mappings satisfy inequality (5.112) and definition (5.113) with $a_1(s) := \frac{1}{2n}s^2$, $a_2(s) := \frac{1}{2}s^2$, $\beta(t) \equiv 1$, and $r_i := 0$, $i = 1, \ldots, n$. Let $\lambda \in (0, 1)$ and notice that implication (5.103) holds with $\gamma_{i,j}(s) := \frac{c_{i,j}^2}{\lambda^2 a_i^2} s$ and $\rho_i(s) := 2(1 - \lambda)a_i s$. Condition (5.128) and the fact that $c_{i,i} < a_i$ for all $i = 1, \ldots, n$ implies that the small-gain conditions (5.2), (5.128) hold for $\lambda \in (0, 1)$ sufficiently close to 1. We conclude from Corollary 5.2 that $0 \in C^0([-r, 0]; \Re^n)$ is RGAS for (5.126).

It is important to notice that the conditions on the diagonal terms cannot be avoided in general if Razumikhin-like functions are used. Such a situation occurs, for example, when

$$\begin{aligned}
\dot{x}_1(t) &= -a_1 x_1(t) + c_{1,1} d_1(t) x_1(t - r) + c_{1,2} d_2(t) x_2(t - r) \\
\dot{x}_2(t) &= -a_2 x_2(t) + c_{2,1} d_3(t) x_1(t) \\
&\quad d_i(t) \in [-1, 1], i = 1, 2, 3
\end{aligned}$$

with $c_{1,1} > 0$, $c_{1,2} \ge 0$, and $c_{2,1} \ge 0$. In this case, $0 \in C^0([-r, 0]; \Re^2)$ is RGAS for the above system if $c_{1,1} < a_1$ and $c_{1,2} c_{2,1} < a_1 a_2$.

Example 5.4.2 Consider the following biochemical control circuit model:

$$\begin{aligned}\dot{X}_1(t) &= g\big(X_n(t-\tau_n)\big) - a_1 X_1(t)\\ \dot{X}_i(t) &= X_{i-1}(t-\tau_{i-1}) - a_i X_i(t) \quad i=2,\dots,n\\ X(t) &= \big(X_1(t),\dots,X_n(t)\big)' \in \Re^n_+\end{aligned} \tag{5.129}$$

where $a_i > 0$ and $\tau_i \ge 0$ ($i = 1,\dots,n$) are constants, and $g \in C^1(\Re^+;\Re^+)$ is a function with $g(X) > 0$ for all $X > 0$. This model has been studied in [46] (see pp. 58–60 and 93–94). In this book it is further assumed that $g \in C^1(\Re^+;\Re^+)$ is bounded and strictly increasing (a typical choice for $g \in C^1(\Re^+;\Re^+)$ is $g(X) = \frac{X^p}{1+X^p}$ with p being a positive integer or $g(X) = \frac{\mu X}{c+X}$ with $\mu, c > 0$). It is shown that if there is one equilibrium point for (5.129), then it attracts all solutions. If there are two equilibrium points, then all solutions are attracted to these points. Here we study (5.129) under the following assumption:

(H) There exist $X_n^* > 0$, $K > 0$, and $\lambda \in (0,1)$ with $aX_n^* = g(X_n^*)$ and such that

$$\frac{K + X_n^*}{K+X} X \le a^{-1} g(X) \le X_n^* + \lambda\left|X - X_n^*\right| \quad \text{for all } X \ge 0 \tag{5.130}$$

where $a = \prod_{j=1}^n a_j$.

The reader should notice that Hypothesis (H) is automatically satisfied for the case of Monod kinetics, i.e., $g(X) = \frac{\mu X}{c+X}$ with $c > 0$ and $\mu > ac$. Indeed, in this case inequality (5.130) holds with $K = c$ and $\lambda = \frac{c}{X_n^*+c}$, where $X_n^* = \frac{\mu - ac}{a}$. The case of Monod kinetics is typical for biochemical models.

Using small-gain analysis arguments, we are in a position to prove:

"Consider system (5.129) under Hypothesis (H) and let $r := \max_{i=1,\dots,n} \tau_i$. Then for every $X_0 \in C^0([-r,0];\mathrm{int}(\Re^n_+))$, the solution of (5.129) with initial condition $T_r(0)X = X_0$ satisfies $\lim_{t\to+\infty} X(t) = X^*$, where $X^* = (X_1^*,\dots,X_n^*)' \in \mathrm{int}(\Re^n_+)$ with $(\prod_{j=1}^i a_j)X_i^* = g(X_n^*)$ for $i = 1,\dots,n-1$."

In sharp contrast to the analysis performed in [46] for (5.129) based on the monotone dynamical system theory, we do not assume that $g \in C^1(\Re^+;\Re^+)$ is bounded or strictly increasing, Moreover, even if there are two equilibrium points (e.g., $0 \in \Re^n_+$ can be an equilibrium point, because (5.130) allows $g(0) = 0$), we prove almost global convergence to the nontrivial equilibrium.

A typical analysis of the equilibrium points of (5.129) under Hypothesis (H) shows that there exists an equilibrium point $X^* \in \mathrm{int}(\Re^n_+)$ satisfying

$$\left(\prod_{j=1}^i a_j\right) X_i^* = g\big(X_n^*\big) \tag{5.131}$$

In order to be able to study solutions of (5.129) evolving in $\mathrm{int}(\Re^n_+)$, we consider the transformation

$$X_i = X_i^* \exp(x_i) \tag{5.132}$$

Therefore system (5.129) under transformation (5.132) is expressed by the following set of differential equations:

$$\dot{x}_1 = a_1\left(\frac{g(X_n^* \exp(x_n(t-\tau_n)))}{g(X_n^*)} \exp\bigl(-x_1(t)\bigr) - 1\right) \tag{5.133}$$

$$\dot{x}_i(t) = a_i\bigl(\exp\bigl(x_{i-1}(t-\tau_{i-1}) - x_i(t)\bigr) - 1\bigr) \quad i = 2, \dots, n \tag{5.134}$$

$$x(t) = \bigl(x_1(t), \dots, x_n(t)\bigr)' \in \Re^n$$

First, we notice that Hypotheses (S1–4) hold for system (5.133), (5.134) under Hypothesis (H) with output $H(t,x) := x \in C^0([-r,0];\Re^n)$ and that $0 \in C^0([-r,0];\Re^n)$ is an equilibrium point for (5.133), (5.134).

Define the family of functions $Q_i(x) = \frac{1}{2}x_i^2(0)$ and $V_i(x) := \sup_{\theta\in[-r,0]} Q_i(x(\theta)) = \frac{1}{2}\|x_i\|_r^2$ $(i = 1, \dots, n)$ for $x \in C^0([-r,0];\Re^n)$. These mappings satisfy inequality (5.112) and definition (5.113) with $a_1(s) := \frac{1}{2n}s^2$, $a_2(s) := \frac{1}{2}s^2$, $\beta(t) \equiv 1$, and $r_i := 0$, $i = 1, \dots, n$. Set $\gamma_{1,j}(s) \equiv 0$ for $j \neq n$ and $\gamma_{1,n}(s) := \frac{1}{2}[\log(1+\theta(\exp(\sqrt{2s})-1))]^2$, where $\theta \in (\max\{\frac{b}{b+1}, \lambda\}, 1)$, $\lambda \in (0,1)$ being the constant involved in Hypothesis (H), and $b := \frac{K}{X_n^*}$. Notice that

$$\begin{aligned} &Q_1^0\left(x_1(0); a_1\left(\frac{g(X_n^* \exp(x_n(-\tau_n)))}{g(X_n^*)} \exp\bigl(-x_1(0)\bigr) - 1\right)\right) \\ &\quad = a_1 x_1(0)\left(\frac{g(X_n^* \exp(x_n(-\tau_n)))}{g(X_n^*)} \exp\bigl(-x_1(0)\bigr) - 1\right) \end{aligned}$$

We consider the following cases:

(1) $x_1(0) < 0$. In this case, the left-hand side of inequality (5.130) implies that

$$\begin{aligned} \frac{g(X_n^* \exp(x_n(-\tau_n)))}{g(X_n^*)} &\geq \frac{b+1}{b+\exp(x_n(-\tau_n))} \exp\bigl(x_n(-\tau_n)\bigr) \\ &\geq \frac{b+1}{b+\exp(-|x_n(-\tau_n)|)} \exp\bigl(-\bigl|x_n(-\tau_n)\bigr|\bigr) \end{aligned}$$

with $b := \dfrac{K}{X_n^*}$

The inequality $\gamma_{1,n}(V_n(x)) \leq Q_1(x_1(0))$ implies $\ln(1+\theta(\exp(|x_n(-\tau_n)|) - 1)) \leq -x_1(0)$, which combined with the previous inequalities, gives

$$\begin{aligned} &Q_1^0\left(x_1(0); a_1\left(\frac{g(X_n^* \exp(x_n(-\tau_n)))}{g(X_n^*)} \exp\bigl(-x_1(0)\bigr) - 1\right)\right) \\ &\quad \leq a_1 x_1(0) \frac{(b+1-b\theta^{-1})(\exp(-x_1(0))-1)}{b+1+b\theta^{-1}(\exp(-x_1(0))-1)}. \end{aligned} \tag{5.135}$$

(2) $x_1(0) \geq 0$. In this case, the right-hand side of inequality (5.130) implies that

$$\begin{aligned} \frac{g(X_n^* \exp(x_n(-\tau_n)))}{g(X_n^*)} &\leq 1 + \lambda\bigl|\exp\bigl(x_n(-\tau_n)\bigr) - 1\bigr| \\ &\leq 1 + \lambda\bigl(\exp\bigl(\bigl|x_n(-\tau_n)\bigr|\bigr) - 1\bigr) \end{aligned}$$

The inequality $\gamma_{1,n}(V_n(x)) \le Q_1(x_1(0))$ implies $\ln(1+\theta(\exp(|x_n(-\tau_n)|)-1)) \le x_1(0)$, which combined with the previous inequalities, gives

$$Q_1^0\left(x_1(0); a_1\left(\frac{g(X_n^* \exp(x_n(-\tau_n)))}{g(X_n^*)}\exp\bigl(-x_1(0)\bigr)-1\right)\right) \le a_1 x_1(0)\bigl(\lambda\theta^{-1}-1\bigr)\bigl(1-\exp\bigl(-x_1(0)\bigr)\bigr) \tag{5.136}$$

Combining these two cases, we obtain from (5.135) and (5.136) that the following implication holds:

$$\gamma_{1,n}\bigl(V_n(x)\bigr) \le Q_1\bigl(x_1(0)\bigr) \Rightarrow Q_1^0\left(x_1(0); a_1\left(\frac{g(X_n^* \exp(x_n(-\tau_n)))}{g(X_n^*)}\exp\bigl(-x_1(0)\bigr)-1\right)\right) \le -\rho_1\bigl(Q_1\bigl(x_1(0)\bigr)\bigr) \tag{5.137}$$

with

$$\rho_1(s) := a_1\sqrt{2s}\min\left\{\bigl(1-\lambda\theta^{-1}\bigr)\bigl(1-\exp\bigl(-\sqrt{2s}\bigr)\bigr), \frac{(b+1-b\theta^{-1})(\exp(\sqrt{2s})-1)}{b+1+b\theta^{-1}(\exp(\sqrt{2s})-1)}\right\}$$

For $i=2,\dots,n$, define $\gamma_{i,i-1}(s) := \frac{1}{2}[\log(1+\mu(\exp(\sqrt{2s})-1))]^2$ and $\gamma_{i,j}(s) \equiv 0$ for $j \ne i-1$, where $\mu > 1$ is to be selected. Working in a similar way as above, we obtain, for $i=2,\dots,n$,

$$\gamma_{i,i-1}\bigl(V_i(x)\bigr) \le Q_i\bigl(x_i(0)\bigr) \Rightarrow Q_i^0\bigl(x_i(0); a_i\bigl(\exp\bigl(x_{i-1}(-\tau_{i-1})-x_i(0)\bigr)-1\bigr)\bigr) \le -\rho_i\bigl(Q_i\bigl(x_i(0)\bigr)\bigr) \tag{5.138}$$

with $\rho_i(s) := (1-\mu^{-1})a_i\sqrt{2s}\frac{(1-\exp(-\sqrt{2s}))}{1+\mu^{-1}(\exp(\sqrt{2s})-1)}$ for $i=2,\dots,n$.

Therefore, we conclude from (5.137) and (5.138) that implication (5.103) holds.

Finally, we check the small-gain conditions. Exploiting the previous definitions of the functions $\gamma_{i,j}(s)$, $i,j=1,\dots,n$, we conclude that the small-gain conditions (5.2), (5.3) hold if and only if $(\gamma_{n,n-1}\circ\gamma_{n-1,n-2}\circ\cdots\circ\gamma_{1,n})(s) < s$ for all $s>0$. Since

$$(\gamma_{n,n-1}\circ\gamma_{n-1,n-2}\circ\cdots\circ\gamma_{1,n})(s) = \frac{1}{2}\bigl[\log\bigl(1+\mu^{n-1}\theta\bigl(\exp\bigl(\sqrt{2s}\bigr)-1\bigr)\bigr)\bigr]^2$$

the small-gain conditions (5.2), (5.3) hold with $\mu \in (1, \theta^{-\frac{1}{n-1}})$. Thus, Corollary 5.2 implies that $0 \in C^0([-r,0];\Re^n)$ is Uniformly Globally Asymptotically Stable for system (5.133), (5.134). Taking into account transformation (5.132), this implies that for every $X_0 \in C^0([-r,0];\mathrm{int}(\Re_+^n))$, the solution of (5.129) with initial condition $T_r(0)X = X_0$ satisfies $\lim_{t\to+\infty} X(t) = X^*$, where $X^* = (X_1^*,\dots,X_n^*)' \in \mathrm{int}(\Re_+^n)$ with $(\prod_{j=1}^i a_j)X_i^* = g(X_n^*)$ for $i=1,\dots,n-1$.

The following example indicates that the trajectory-based small-gain results of the previous section can be used to study the feedback interconnection of systems which do not necessarily satisfy the IOS property.

Example 5.4.3 Consider the system

$$\begin{aligned}\dot{x} &= f(d,x,y,u)\\ \dot{y} &= g(d,x,y)\\ &\quad x\in\Re^n, y\in\Re^k, d\in D\subset\Re^l, u\in\Re^m\end{aligned}\tag{5.139}$$

where $D\subset\Re^l$ is a nonempty compact set, $f: D\times\Re^n\times\Re^k\times\Re^m\to\Re^n$, and $g: D\times\Re^n\times\Re^k\to\Re^k$ are locally Lipschitz mappings with $f(d,0,0,0)=0$, $g(d,0,0)=0$ for all $d\in D$. Suppose that there exist positive definite, continuously differentiable, and radially unbounded functions $V_1:\Re^n\to\Re^+$, $V_2:\Re^k\to\Re^+$, a constant $a\in[0,1]$, and a function $k\in K_\infty$ satisfying the following inequalities for all $(x,y,u)\in\Re^n\times\Re^k\times\Re^m$:

$$\begin{aligned}&\max_{d\in D}\nabla V_1(x)f(d,x,y,u)\\ &\quad\le -(2+a)\frac{V_1(x)}{1+V_1(x)}+(1-a)\frac{V_2(y)}{(1+V_1(x))(1+V_2(y))}\\ &\qquad +a\frac{k(|u|)}{1+k(|u|)}\end{aligned}\tag{5.140}$$

$$\max_{d\in D}\nabla V_2(y)g(d,x,y)\le -2\frac{V_2(y)}{1+V_2(y)}+V_1(x)\tag{5.141}$$

It is clear that the subsystem $\dot{y}=g(d,x,y)$ does not necessarily satisfy the ISS property from the input $x\in\Re^n$. Consequently, the classical small-gain theorem in [21] cannot be applied because the y-subsystem in (5.139) is not ISS but integral ISS with $x\in\Re^n$ as input. Recent small-gain approaches have been used for system (5.139), where it is shown that $0\in\Re^n\times\Re^k$ is Globally Asymptotically Stable (see [1, 15–18]) for the disturbance-free case with $a=0$. Here we will show, by making use of Theorem 5.4, that system (5.139) satisfies the UISS property from the input $u\in\Re^m$.

Take arbitrary $\varepsilon\in(0,1)$ and define

$$h(x,y):=V_1(x)-\frac{1+\varepsilon}{2-\varepsilon}\qquad W(x,y):=V_1(x)+V_2(y)\tag{5.142}$$

Inequalities (5.140), (5.141) guarantee that if $h(x,y)\ge 0$, then inequalities (5.84), (5.85) hold with $\delta(s)\equiv\varepsilon$, $K(s)\equiv 1$, and $\zeta(s)\equiv 0$. Using (5.140), (5.141), (5.142), and the inequality $s+w\le\max\{(1+\mu)s,(1+\mu^{-1})w\}$, which holds for all $\mu>0$, $s,w\ge 0$, we can prove that, for all $\lambda\in(0,1)$ and $\mu>0$ with $\mu\ge\frac{a}{2-\lambda}$ and $\varepsilon\in(0,1)$ with $\varepsilon<\frac{3-2\lambda}{3-\lambda}$, the following implications hold:

$$\begin{aligned}&V_1(x)\ge\max\left\{\frac{(1+a)(1+\mu)}{2+a-\lambda}\frac{V_2(y)}{1+V_2(y)},k\big(|u|\big)\right\}\\ &\quad\Rightarrow\quad\max_{d\in D}\nabla V_1(x)f(d,x,y,u)\le-\rho\big(V_1(x)\big)\end{aligned}\tag{5.143}$$

$$\begin{aligned}&V_2(y)\ge\frac{V_1(x)}{2-\lambda-V_1(x)}\quad\text{and}\quad h(x,y)\le 0\\ &\quad\Rightarrow\quad\max_{d\in D}\nabla V_2(y)g(d,x,y)\le-\rho\big(V_2(y)\big)\end{aligned}\tag{5.144}$$

where $\rho(s) := \frac{\lambda s}{1+s}$. Therefore, implications (5.86) hold with $\zeta(s) := k(|u|)$, $\gamma_{1,1}(s) = \gamma_{2,2}(s) \equiv 0$, $\gamma_{1,2}(s) := \frac{(1-a)(1+\mu)}{2+a-\lambda}\frac{s}{1+s}$, $\gamma_{2,1}(s) := \frac{s}{2-\lambda-s}$ for $s \in [0, \frac{1+\varepsilon}{2-\varepsilon}]$, and $\gamma_{2,1}(s) := \frac{1+\varepsilon}{(3-2\lambda)-(3-\lambda)\varepsilon}$ for $s > \frac{1+\varepsilon}{2-\varepsilon}$. Finally, since $V_1 : \Re^n \to \Re^+$ and $V_2 : \Re^k \to \Re^+$ are radially unbounded, positive definite functions, it follows that inequality (5.83) holds for appropriate functions $a_1, a_2 \in K_\infty$ (Proposition 2.2).

It follows from Theorem 5.4 that system (5.139) satisfies the UISS property from the input $u \in \Re^m$, provided that the small-gain inequalities hold. In this case the small-gain inequalities are equivalent to the following inequality:

$$(1-a)(1+\mu) < (2-\lambda)(2+a-\lambda)$$

Since $a \in [0, 1]$, the above inequality as well as the inequality $\mu \geq \frac{a}{2-\lambda_2}$ holds for $\mu = \frac{2}{3}$, $\lambda = \frac{1}{2}$.

The following example deals with the robust global sampled-data stabilization of a nonlinear planar system and shows how Theorem 5.1 can be applied to systems with variable sampling partition.

Example 5.4.4 Consider the following planar system:

$$\begin{aligned} \dot{x} &= -\left(1+y^2\right)x + y \\ \dot{y} &= f(x) + g(x)y + u \\ &(x, y) \in \Re^2, u \in \Re \end{aligned} \tag{5.145}$$

where $f, g : \Re \to \Re$ are locally Lipschitz functions with $f(0) = 0$. We will show that there exist a sufficiently large constant $M > 0$ and a sufficiently small constant $r > 0$, so that system (5.145) in closed loop with the feedback law $u = -My$ applied with zero order hold, i.e., the closed-loop system

$$\begin{aligned} \dot{x}(t) &= -\left(1+y^2(t)\right)x(t) + y(t) \\ \dot{y}(t) &= -My(\tau_i) + f\left(x(t)\right) + g\left(x(t)\right)y(t) \quad t \in [\tau_i, \tau_{i+1}) \\ \tau_{i+1} &= \tau_i + \exp\left(-w(\tau_i)\right)r, \ \ w(t) \in \Re^+ \end{aligned} \tag{5.146}$$

satisfies the UISS property with zero gain when w is considered as input.

First, notice that there exists a function $\sigma \in KL$ such that, for all $(x_0, y) \in \Re^n \times \mathcal{L}^\infty_{\text{loc}}(\Re^+; \Re)$, the solution of $\dot{x} = -(1+y^2)x + y$ with initial condition $x(0) = x_0$ corresponding to inputs $y \in \mathcal{L}^\infty_{\text{loc}}(\Re^+; \Re)$ satisfies the following estimate for all $t \geq 0$:

$$\left|x(t)\right| \leq \max\left\{\sigma\left(|x_0|, t\right), \sup_{0\leq\tau\leq t} \gamma\left(\left|y(\tau)\right|\right)\right\} \tag{5.147}$$

with $\gamma(s) := \frac{\sqrt{2}s}{\sqrt{1+4s^2}}$. Indeed, inequality (5.147) can be verified by using the Lyapunov function $V(x) = x^2$ which satisfies the following implication:

$$\text{If } V(x) = x^2 \geq \frac{2y^2}{1+4y^2}, \quad \text{then } \dot{V} \leq -\frac{1}{4}V(x)$$

The above implication, in conjunction with Lemma 2.14 in Chap. 2, guarantees that (5.147) holds for appropriate $\sigma \in KL$. Next, we show the following claim.

Claim *For all* $\varepsilon, a > 0$, *there exist* $\sigma \in KL$, *sufficiently large* $M > 0$, *and sufficiently small* $r > 0$ *such that for every* $(y_0, x, w) \in \Re \times \mathcal{L}^\infty_{loc}(\Re^+; B[0,a]) \times \mathcal{L}^\infty_{loc}(\Re^+; \Re^+)$, *the solution of*

$$\begin{aligned} \dot{y}(t) &= -My(\tau_i) + f\big(x(t)\big) + g\big(x(t)\big)y(t), \quad t \in [\tau_i, \tau_{i+1}) \\ \tau_{i+1} &= \tau_i + \exp\big(-w(\tau_i)\big)r, \ w(t) \in \Re^+ \end{aligned} \tag{5.148}$$

with initial condition $y(0) = y_0$ *corresponding to inputs* $(x, w) \in \mathcal{L}^\infty_{loc}(\Re^+; B[0,a]) \times \mathcal{L}^\infty_{loc}(\Re^+; \Re^+)$ *satisfies the following inequality*:

$$\big|y(t)\big| \le \max\Big\{\sigma\big(|y_0|, t\big), \varepsilon \sup_{0\le\tau\le t}\big|x(\tau)\big|\Big\} \tag{5.149}$$

Proof of the Claim Let $\varepsilon, a > 0$ be arbitrary. Since $f, g : \Re \to \Re$ are locally Lipschitz functions with $f(0) = 0$, there exist constants P, $Q > 0$ such that

$$\big|f(x)\big| \le P|x| \quad \text{and} \quad \big|g(x)\big| \le Q \quad \text{for all } x \in B[0,a] \tag{5.150}$$

Let $M > 0$ and $r > 0$ be chosen so that

$$M \ge 2 + 2Q + \frac{9P^2}{2\varepsilon^2} \quad \text{and} \quad 3(M+Q)r\exp(Qr) \le 1 \tag{5.151}$$

Consider a solution $y(t)$ of (5.148) corresponding to arbitrary $(x, w) \in \mathcal{L}^\infty_{loc}(\Re^+; B[0,a]) \times \mathcal{L}^\infty_{loc}(\Re^+; \Re^+)$ with initial condition $y(0) = y_0 \in \Re$. There exists a maximal existence time for the solution denoted by $t_{max} \le +\infty$. Moreover, let $\pi := \{\tau_0, \tau_1, \ldots\}$ be the set of sampling times (which may be finite if $t_{max} < +\infty$), and $mp(t) := \max\{\tau \in \pi : \tau \le t\}$. Let $\|x\| := \sup_{0\le s\le t}|x(s)|$ and $\tau = mp(t)$. Inequalities (5.150), (5.151) and the fact that $t - \tau \le r$, in conjunction with the Gronwall–Bellman inequality, implies

$$\begin{aligned} \big|y(t) - y(\tau)\big| \le{}& \frac{(M+Q)r\exp(Qr)}{1-(M+Q)r\exp(Qr)}\big|y(t)\big| \\ &+ \frac{Pr}{1-(M+Q)r\exp(Qr)}\exp(Qr)\|x\| \end{aligned} \tag{5.152}$$

Define $V(t) = y^2(t)$ on $[0, t_{max})$. Let $I \subset [0, t_{max})$ be the zero Lebesgue measure set where $y(t)$ is not differentiable or where $\dot{y}(t) \ne -My(\tau_i) + f(x(t)) + g(x(t))y(t))$. Using (5.150), (5.151), and (5.152), we obtain, for all $t \in [0, t_{max})\backslash I$,

$$\dot{V} \le -2V(t) + \frac{\varepsilon^2}{2}\|x\|^2 \quad \text{for all } t \in [0, t_{max})\backslash I \tag{5.153}$$

Direct integration of the differential inequality (5.153) and the fact that $V(t) = y^2(t)$ give

$$\big|y(t)\big| \le \max\big\{\sqrt{2}\exp(-t)|y_0|, \varepsilon\|x\|\big\} \quad \text{for all } t \in [0, t_{max}) \tag{5.154}$$

Clearly, inequality (5.154) implies that as long as the solution of (5.148) exists, $y(t)$ is bounded. A standard contradiction argument, in conjunction with the BIC property for (5.148), implies that $t_{max} = +\infty$. Inequality (5.149) is a direct consequence of (5.154). The proof is complete. □

We select $M > 0$ sufficiently large and $r > 0$ sufficiently small such that inequality (5.149) holds with $\varepsilon < 1/\sqrt{2}$ and $a = 1 + \sqrt{2}/2$. The solution of the closed-loop system (5.146) exists for all $t \geq 0$. The existence of the solution is guaranteed by the following claim.

Claim *For all* $M > 0$, $r > 0$, *and* $(y_0, x_0, w) \in \Re \times \Re^n \times \mathcal{L}^\infty_{loc}(\Re^+; \Re^+)$, *the solution of* (5.146) *with initial condition* $(x(0), y(0)) = (x_0, y_0)$ *corresponding to input* $w \in \mathcal{L}^\infty_{loc}(\Re^+; \Re^+)$ *exists for all* $t \geq 0$. *Moreover, for* $M > 0$ *sufficiently large and* $r > 0$ *sufficiently small, there exist* $A \in K_\infty$ *and* $\xi \in \pi$ *such that*

$$\left|x(t)\right| + \left|y(t)\right| \leq A\big(\left|(x_0, y_0)\right|\big) \quad \text{for all } t \in [0, \xi] \tag{5.155}$$

$$\left|x(t)\right| \leq a \quad \text{for all } t \geq \xi \tag{5.156}$$

$$\xi \leq 1 + r + A\big(|x_0|\big) \tag{5.157}$$

where $a = 1 + \sqrt{2}/2$.

Proof of the Claim Let $M > 0$, $r > 0$, and $(y_0, x_0, w) \in \Re \times \Re^n \times \mathcal{L}^\infty_{loc}(\Re^+; \Re^+)$ be arbitrary. Consider a solution $(x(t), y(t))$ of (5.146) corresponding to arbitrary $w \in \mathcal{L}^\infty_{loc}(\Re^+; \Re^+)$ with initial condition $(x(0), y(0)) = (x_0, y_0)$. Denote by $t_{max} \leq +\infty$ the maximal existence time for the solution. Moreover, let $\pi := \{\tau_0, \tau_1, \dots\}$ be the set of sampling times (which may be finite if $t_{max} < +\infty$), and $mp(t) := \max\{\tau \in \pi : \tau \leq t\}$. By virtue of (5.147), we have, for all $t \in [0, t_{max})$,

$$\left|x(t)\right| \leq \max\big\{\sigma\big(|x_0|, 0\big), a\big\} \tag{5.158}$$

Define

$$\begin{aligned} P &:= \max\big\{\left|f(x)\right| : |x| \leq \max\big\{\sigma\big(|x_0|, 0\big), a\big\}\big\} \quad \text{and} \\ Q &:= \max\big\{\left|g(x)\right| : |x| \leq \max\big\{\sigma\big(|x_0|, 0\big), a\big\}\big\} \end{aligned} \tag{5.159}$$

Using (5.158), (5.159) in conjunction with Gronwall–Bellman's lemma, we obtain the following inequality for all $t \in [0, t_{max})$:

$$\left|y(t)\right| \leq \left|y(\tau)\right| \exp\big((M + 2Q)(t - \tau)\big) + P(t - \tau) \exp\big(Q(t - \tau)\big) \tag{5.160}$$

where $\tau = mp(t)$. Using (5.160) and by induction, we can show the following inequality for all $\tau_i \in \pi$:

$$\left|y(\tau_i)\right| \leq |y_0| \exp\big((M + 2Q)\tau_i\big) + P\tau_i \exp(Qr) \exp\big((M + 2Q)\tau_i\big) \tag{5.161}$$

where we have used the fact that $\tau_{i+1} - \tau_i \leq r$. Estimate (5.160), in conjunction with (5.161), gives, for all $t \in [0, t_{max})$,

$$\left|y(t)\right| \leq \big[|y_0| + Pt \exp(Qr)\big] \exp\big((M + 2Q)t\big) \tag{5.162}$$

A standard contradiction argument, in conjunction with the BIC property for (5.146), implies that $t_{max} = +\infty$.

The existence of $\xi \in \pi$ such that (5.156) holds is a direct consequence of (5.147) and definitions $\gamma(s) := \frac{\sqrt{2}s}{\sqrt{1+4s^2}}$, $a = 1 + \sqrt{2}/2$. By virtue of Theorem 3.1 in Chap. 3, there exists $\beta \in K_\infty$ such that

$$\xi \leq 1 + r + \beta\big(|x_0|\big) \tag{5.163}$$

Finally, let $M > 0$ be sufficiently large and $r > 0$ sufficiently small so that (5.149) holds for $a = 1 + \sqrt{2}/2$ and $\varepsilon < 1/\sqrt{2}$. For $x_0 \in \Re^n$ with $\sigma(|x_0|, 0) \leq a$, we obtain from (5.147) and (5.149), for all $t \geq 0$,

$$\left|x(t)\right| + \left|y(t)\right| \leq \left(2 + \varepsilon + 2/\sqrt{3}\right)\left(\sigma\left(|x_0|, 0\right) + \sigma\left(|y_0|, 0\right)\right) \tag{5.164}$$

Using (5.158), (5.159), (5.162), (5.163), and (5.164), we guarantee the existence of $\tilde{\beta} \in K_\infty$ such that

$$\left|x(t)\right| + \left|y(t)\right| \leq \tilde{\beta}\left(\left|(x_0, y_0)\right|\right) \quad \text{for all } t \in [0, \xi] \tag{5.165}$$

The existence of $A \in K_\infty$ satisfying (5.155) and (5.157) is a direct consequence of (5.163) and (5.165). The proof is complete. □

The fact that the robust global stabilization problem for (5.146) with sampled-data feedback applied with zero-order hold is solvable with $M > 0$ being sufficiently large and $r > 0$ being sufficiently small is a consequence from all the above and Theorem 5.1. Indeed, we apply Theorem 5.1 with $n = 2$, $V_1 = |x|$, $V_2 = |y|$, $L = |x| + |y|$, $H = (x, y)$, $S(t) := \{(x, y) \in \Re \times \Re : |x| \leq a\}$, $\gamma_{1,2}(s) := \sqrt{2}s$, $\gamma_{2,1}(s) := \varepsilon s$, $\gamma_{1,1} \equiv 0$, $\gamma_{2,2} \equiv 0$, $\zeta \equiv 0$, $g^u \equiv 0$, $\eta \equiv 0$, $\tilde{\eta} \equiv 0$, $p^u \equiv 0$, $c(t) = \tilde{c}(t) = \nu(t) = \mu(t) = \kappa(t) \equiv 1 + r$, $g \equiv 0$, and $p(s, w) := s + w$ for appropriate $a, b \in K_\infty$, $\sigma \in KL$, and $q \in \mathcal{N}_2$. All Hypotheses (SG1–3) are satisfied by using the above definitions and previous results. Therefore, we conclude that the closed-loop system (5.146) with $M > 0$ being sufficiently large and $r > 0$ being sufficiently small satisfies the UISS property from the input w with zero gain.

The following example is a large-scale system and shows how efficiently the small-gain results of the present work can be applied to large-scale systems.

Example 5.4.5 Consider the following system described by ODEs:

$$\dot{x}_i(t) = -a_i x_i(t) + g_i\left(d(t), y(t), x(t)\right) \quad i = 1, \ldots, n \tag{5.166}$$

$$\dot{y}(t) = -\left(\omega + P\left(x(t)\right)\right)y(t) + q\left(x(t)\right) \tag{5.167}$$

where $x(t) = (x_1(t), \ldots, x_n(t)) \in \Re^n$, $y(t) \in \Re$, $d(t) \in D \subseteq \Re^m$, $D \subseteq \Re^m$ is compact, $a_i > 0$ $(i = 1, \ldots, n)$, $\omega > 0$, and $g_i : D \times \Re \times \Re^n \to \Re$ $(i = 1, \ldots, n)$, $P : \Re^n \to \Re^+$, and $q : \Re^n \to \Re$ are locally Lipschitz mappings with $q(0) = 0$ for which there exist constants $\lambda \in (0, 1)$, $c_{i,j} \geq 0$ $(i, j = 1, \ldots, n)$, and $R > 0$ such that

$$\frac{|q(x)|}{\lambda\omega + P(x)} \leq R \quad \text{for all } x \in \Re^n \tag{5.168}$$

$$\sup_{d \in D}\left|g_i(d, y, x)\right| \leq \max_{j=1,\ldots,n} c_{i,j}|x_j| \quad \text{for all } x \in \Re^n \text{ and } y \in \Re \text{ with } |y| \leq R \tag{5.169}$$

Moreover, we assume that there exists a nondecreasing function $L : \Re^+ \to \Re^+$ such that

$$\sup_{d \in D}\left[x_i g_i(d, y, x)\right] \leq L\left(|y|\right)\left(1 + |x|^2\right) \quad i = 1, \ldots, n \tag{5.170}$$

for all $x, z \in \Re^n$ and $y \in \Re$.

We will next show that system (5.166), (5.167) is URGAS if $c_{i,i} < a_i$ for all $i = 1, \ldots, n$ and the following small-gain conditions hold for each $r = 2, \ldots, n$:

$$c_{i_1,i_2} c_{i_2,i_3} \cdots c_{i_r,i_1} < a_{i_1} a_{i_2} \cdots a_{i_r} \tag{5.171}$$

for all $i_j \in \{1, \ldots, n\}$, $i_j \neq i_k$ if $j \neq k$.

Define the family of functions $V_i(x, y) := \frac{1}{2} x_i^2$ $(i = 1, \ldots, n)$, $V_{n+1}(x, y) := \frac{1}{2} y^2$, $W(x, y) := 1 + \frac{1}{2}|x|^2 + \frac{1}{2} y^2$, and $h(x, y) := y^2 - R^2$ for $(x, y) \in \Re^n \times \Re$. By direct computation,

$$\begin{aligned} y\dot{y} &\leq -\big(\omega + P(x)\big) y^2 + \big|h(x)\big||y| \\ &\leq R\big(\lambda\omega + P(x)\big)|y| - \big(\lambda\omega + P(x)\big)|y|^2 - \omega(1 - \lambda)|y|^2 \\ &= -|y|\big(\lambda\omega + P(x)\big)\big(|y| - R\big) - \omega(1 - \lambda)|y|^2 \end{aligned}$$

It then implies that inequality (5.84) holds $\delta(s) \equiv 2\omega(1 - \lambda) R^2$. Moreover, by virtue of (5.170) and the above inequality, we obtain

$$\nabla W(x, y) \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \leq L\big(|y|\big)\big(1 + |x|^2\big)$$

Consequently inequality (5.85) holds with $K(s) := 2L(\sqrt{s + R^2})$. Inequality (5.83) holds with $a_1(s) := \frac{s^2}{2\sqrt{n+1}}$ and $a_2(s) := \frac{s^2}{2}$. Finally, notice that, for all $\mu \in (0, 1)$ and $i = 1, \ldots, n$, the following implication holds:

$$\begin{aligned} &V_i(x) \geq \frac{1}{2\mu^2 a_i^2} \big|g_i(d, y, x)\big|^2 \\ &\quad \Rightarrow \quad x_i\big(-a_i x_i + g_i(d, y, x)\big) \leq -(1 - \mu) a_i x_i^2 \end{aligned} \tag{5.172}$$

It follows from (5.172) and (5.169) that implications (5.86) for $i = 1, \ldots, n$ hold with

$$\begin{aligned} &\rho_i(s) := 2(1 - \mu) a_i s \quad \gamma_{i,j}(s) := \frac{c_{i,j}^2}{\mu^2 a_i^2} s \quad \text{and} \quad \gamma_{i,n+1}(s) := 0 \\ &\quad \text{for } s \geq 0 \text{ and } i, j = 1, \ldots, n \end{aligned} \tag{5.173}$$

Since $q : \Re^n \to \Re$ is a continuous mapping with $q(0) = 0$, it follows that there exists a function $\gamma \in K_\infty$ such that the following inequality holds:

$$\big|q(x)\big| \leq \gamma\big(|x|\big) \quad \text{for all } x \in \Re^n \tag{5.174}$$

Inequality (5.174) and the fact that $P(x) \geq 0$ for all $x \in \Re^n$ (notice that $P : \Re^n \to \Re^+$) imply that the following implication holds for every $\mu \in (0, 1)$:

$$\begin{aligned} &V_{n+1}(y) \geq \frac{1}{2\mu^2\omega^2} \big(\gamma\big(|x|\big)\big)^2 \\ &\quad \Rightarrow \quad y\big[-\big(\omega + P(x)\big) y + q(x)\big] \leq -(1 - \mu)\omega y^2 \end{aligned} \tag{5.175}$$

It follows from (5.175) that implication (5.86) holds for $i = n + 1$ with

$$\gamma_{n+1,j}(s) := \frac{1}{2\mu^2\omega^2}\left(\gamma\left(\sqrt{2ns}\right)\right)^2 \quad \text{and} \quad \gamma_{n+1,n+1}(s) := 0$$
$$\text{for } s \geq 0 \text{ and } j = 1, \ldots, n \tag{5.176}$$

Definitions (5.173) and (5.176), in conjunction with (5.171) and the fact that $c_{i,i} < a_i$ for all $i = 1, \ldots, n$, guarantee that there exists $\mu \in (0, 1)$ (sufficiently close to 1) such that the MAX-preserving mapping $\Gamma : \Re_+^{n+1} \to \Re_+^{n+1}$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_{n+1}(x))'$, $\Gamma_i(x) = \max_{j=1,\ldots,n+1} \gamma_{i,j}(x_j)$ for all $x \in \Re_+^{n+1}$ and $i = 1, \ldots, n+1$ satisfies the cyclic small-gain conditions. It follows from Theorem 5.4 that system (5.166), (5.167) is URGAS.

5.5 Application to Stability Analysis in Uncertain Dynamic Games

Consider a strategic game with n players and $S_i \subseteq \Re^{k_i}$ $(i = 1, \ldots, n)$ being the action space for each one of the players. We assume that the best reply mapping for each player is a function $f_i : S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \to S_i$ for $1 < i < n$, $n \geq 3$, and $f_1 : S_2 \times \cdots \times S_n \to S_1$, $f_n : S_1 \times \cdots \times S_{n-1} \to S_n$, satisfying the following inequalities:

$$\pi_i(q_i, q_{-i}) < \pi_i\left(f_i(q_{-i}), q_{-i}\right)$$
$$\text{for all } q_i \in S_i \text{ with } q_i \neq f_i(q_{-i}),\ i = 1, \ldots, n \tag{5.177}$$

where $\pi_i(q_i, q_{-i})$ is the payoff function of the ith player. We assume the existence of a Nash equilibrium $q^* \in S$ for the game, where $S := S_1 \times \cdots \times S_n$ is the outcome space for the game, i.e., there exists $q^* = (q_1^*, \ldots, q_n^*) \in S$ such that

$$q_i^* = f_i\left(q_{-i}^*\right) \quad i = 1, \ldots, n \tag{5.178}$$

The existence of a Nash equilibrium can be guaranteed by Brouwer's fixed point theorem when all action spaces $S_i \subseteq \Re^{k_i}$ $(i = 1, \ldots, n)$ are compact and convex and when all the best reply mappings $f_i : S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \to S_i$ for $1 < i < n$, $n \geq 3$, and $f_1 : S_2 \times \cdots \times S_n \to S_1$, $f_n : S_1 \times \cdots \times S_{n-1} \to S_n$ are continuous mappings.

Next, we assume that $S_i \subseteq \Re^{k_i}$ $(i = 1, \ldots, n)$ are closed convex sets and that the dynamics of the game are described in continuous time as follows:

- every player forms an expectation for the behavior of all other players at each time $t \geq 0$: the expectation of the ith player for the production level of the jth player at time $t \geq 0$ will be denoted by $q_{i,j}^{\exp}(t) \in S_j$ $(j \neq i,\ i, j = 1, \ldots, n)$,
- every player determines her action as a convex combination of a past action and the best reply response based on the expectations for the behavior of all other players at each time $t \geq 0$, i.e.,

$$
\begin{aligned}
q_1(t) &= \theta_1(t)\mathrm{Pr}_{S_1}\big(q_1\big(t-\tau_1(t)\big)\big) + \big(1-\theta_1(t)\big) f_1\big(q_{1,2}^{\exp}(t), \dots, q_{1,n}^{\exp}(t)\big) \\
&\ \ \vdots \\
q_i(t) &= \theta_i(t)\mathrm{Pr}_{S_i}\big(q_i\big(t-\tau_i(t)\big)\big) \\
&\quad + \big(1-\theta_i(t)\big) f_i\big(q_{i,1}^{\exp}(t), \dots, q_{i,i-1}^{\exp}(t), q_{i,i+1}^{\exp}(t), \dots, q_{i,n}^{\exp}(t)\big) \\
&\ \ \vdots \\
q_n(t) &= \theta_n(t)\mathrm{Pr}_{S_n}\big(q_n\big(t-\tau_n(t)\big)\big) + \big(1-\theta_n(t)\big) f_n\big(q_{n,1}^{\exp}(t), \dots, q_{n,n-1}^{\exp}(t)\big)
\end{aligned}
\tag{5.179}
$$

where $\theta_i : \Re^+ \to [0, \Theta]$, $\tau_i : \Re^+ \to [r, T]$, $i = 1, \dots, n$, are in general unknown functions, and $0 \le \Theta < 1$ and $0 < r \le T$ are constants (in general unknown),

- all expectation rules $q_{i,j}^{\exp}(t) \in S_j$ ($j \ne i$, $i = 1, \dots, n$) are Consistent Backward-looking expectations (see Example 1.4.1 in Chap. 1) with respect to the Nash equilibrium point $q^* \in S$, i.e., there exist constants $0 < T \le r$ such that, for all $t \ge 0$,

$$
\big|q_{i,j}^{\exp}(t) - q_j^*\big| \le \sup_{t-r \le \tau \le t-T} \big|q_j(\tau) - q_j^*\big| = \big\|q_j - q_j^*\big\|_{[t-r,t-T]} \tag{5.180}
$$

Notice the following fact for consistent backward-looking expectations:

Fact *Suppose that $S_j \subseteq \Re^{k_j}$ is a closed convex set. $q_{i,j}^{\exp}(t)$ ($j \ne i$, $i, j = 1, \dots, n$) is a Consistent Backward-looking expectation with respect to the Nash equilibrium point $q^* \in S$ if and only if there exist constants $0 < T \le r$ and a function $d_{i,j} : \Re^+ \to \{d \in \Re^{k_j} : |d| \le 1\}$ such that*

$$
q_{i,j}^{\exp}(t) = \mathrm{Pr}_{S_j}\big(q_j^* + d_{i,j}(t)\big\|q_j - q_j^*\big\|_{[t-r,t-T]}\big) \quad \textit{for all } t \ge 0 \tag{5.181}
$$

Proof of Fact Indeed, using the fact that $|\mathrm{Pr}_U(x) - \mathrm{Pr}_U(y)| \le |x - y|$ for every $x, y \in \Re^n$, where $U \subseteq \Re^n$ is a closed convex set, one can verify that for every $d_{i,j} : \Re^+ \to \{d \in \Re^{k_j} : |d| \le 1\}$, the function $q_{i,j}^{\exp}(t)$ defined by (5.181) satisfies (5.180) and $q_{i,j}^{\exp}(t) \in S_j$ for all $t \ge 0$. Hence it is a Consistent Backward-looking expectation with respect to the Nash equilibrium point $q^* \in S$. On the other hand, if $q_{i,j}^{\exp}(t) \in S_j$ is a Consistent Backward-looking expectation with respect to the Nash equilibrium point $q^* \in S$ satisfying (5.180) for all $t \ge 0$, then the function defined by

$$
\begin{aligned}
d_{i,j}(t) &= \tfrac{1}{\|q_j - q_j^*\|_{[t-r,t-T]}}\big(q_{i,j}^{\exp}(t) - q_j^*\big) && \text{if } \big\|q_j - q_j^*\big\|_{[t-r,t-T]} > 0 \\
d_{i,j}(t) &= 0 && \text{if } \big\|q_j - q_j^*\big\|_{[t-r,t-T]} = 0
\end{aligned}
$$

satisfies $d_{i,j}(t) \in \{d \in \Re^{k_j} : |d| \le 1\}$. Moreover, (5.181) holds for all $t \ge 0$.

The proof is complete. □

The above fact shows that if all action spaces $S_j \subseteq \Re^{k_j}$ ($j = 1, \dots, n$) are closed convex sets, then there exist constants $0 < T \le r$, $0 \le \Theta < 1$ and functions $\theta_i : \Re^+ \to [0, \Theta]$, $\tau_i : \Re^+ \to [T, r]$, and $d_{i,j} : \Re^+ \to \{d \in \Re^{k_j} : |d| \le 1\}$ ($j \ne i$, $i, j = 1, \dots, n$) such that

$$
\begin{aligned}
q_1(t) &= \theta_1(t)\mathrm{Pr}_{S_1}\big(q_1\big(t-\tau_1(t)\big)\big) \\
&\quad + \big(1-\theta_1(t)\big) f_1\big(\mathrm{Pr}_{S_2}\big(q_2^* + d_{1,2}(t)\big\|q_2 - q_2^*\big\|_{[t-r,t-T]}\big), \ldots, \\
&\qquad \mathrm{Pr}_{S_n}\big(q_n^* + d_{1,n}(t)\big\|q_n - q_n^*\big\|_{[t-r,t-T]}\big)\big) \\
&\ \ \vdots \\
q_n(t) &= \theta_n(t)\mathrm{Pr}_{S_n}\big(q_n\big(t-\tau_n(t)\big)\big) \\
&\quad + \big(1-\theta_n(t)\big) f_n\big(\mathrm{Pr}_{S_1}\big(q_1^* + d_{n,1}(t)\big\|q_1 - q_1^*\big\|_{[t-r,t-T]}\big), \ldots, \\
&\qquad \mathrm{Pr}_{S_{n-1}}\big(q_{n-1}^* + d_{n,n-1}(t)\big\|q_{n-1} - q_{n-1}^*\big\|_{[t-r,t-T]}\big)\big)
\end{aligned}
\tag{5.182}
$$

In general, the constants $0 < T \le r$, $0 \le \Theta < 1$ and the functions $\theta_i : \Re^+ \to [0, \Theta]$, $\tau_i : \Re^+ \to [T, r]$, and $d_{i,j} : \Re^+ \to \{d \in \Re^{k_j} : |d| \le 1\}$ ($j \ne i$, $i, j = 1, \ldots, n$) are unknown. Therefore, the dynamical system (5.182) is an uncertain dynamical system described by Functional Difference Equations (FDEs). In order to study the behavior of the solutions of (5.182), we define the deviation variables $x_i(t) = q_i(t) - q_i^*$ ($i = 1, \ldots, n$), and from (5.182) we obtain

$$
\begin{aligned}
x_1(t) &= \theta_1(t)\big(\mathrm{Pr}_{S_1}\big(x_1\big(t-\tau_1(t)\big) + q_1^*\big) - q_1^*\big) \\
&\quad + \big(1-\theta_1(t)\big)\big(f_1\big(\mathrm{Pr}_{S_2}\big(q_2^* + d_{1,2}(t)\|x_2\|_{[t-r,t-T]}\big), \ldots, \\
&\qquad \mathrm{Pr}_{S_n}\big(q_n^* + d_{1,n}(t)\|x_n\|_{[t-r,t-T]}\big)\big) - q_1^*\big) \\
&\ \ \vdots \\
x_n(t) &= \theta_n(t)\big(\mathrm{Pr}_{S_n}\big(x_n\big(t-\tau_n(t)\big) + q_n^*\big) - q_n^*\big) \\
&\quad + \big(1-\theta_n(t)\big)\big(f_n\big(\mathrm{Pr}_{S_1}\big(q_1^* + d_{n,1}(t)\|x_1\|_{[t-r,t-T]}\big), \ldots, \\
&\qquad \mathrm{Pr}_{S_{n-1}}\big(q_{n-1}^* + d_{n,n-1}(t)\|x_{n-1}\|_{[t-r,t-T]}\big)\big) - q_n^*\big)
\end{aligned}
\tag{5.183}
$$

Finally, we assume that there exist functions $\tilde{\gamma}_{i,j} \in \mathcal{N}$ ($j \ne i$, $i, j = 1, \ldots, n$) such that the following inequalities hold for all $q \in S$:

$$
\big|f_i(q_{-i}) - q_i^*\big| \le \max_{j \ne i} \tilde{\gamma}_{i,j}\big(\big|q_j - q_j^*\big|\big) \quad i = 1, \ldots, n \tag{5.184}
$$

Using again the fact that $|\mathrm{Pr}_U(x) - \mathrm{Pr}_U(y)| \le |x - y|$ for every $x, y \in \Re^n$, where $U \subseteq \Re^n$ is a closed convex set, and inequalities (5.184), we obtain from (5.183), for all $t \ge 0$, $\mu > \Theta$, and $i = 1, \ldots, n$,

$$
\big|x_i(t)\big| \le \max\left\{\mu\|x_i\|_{[t-r,t-T]}, \max_{j \ne i} \frac{\mu - \mu\Theta}{\mu - \Theta} \tilde{\gamma}_{i,j}\big(\|x_j\|_{[t-r,t-T]}\big)\right\} \tag{5.185}
$$

Remarks and Examples about systems (5.182), (5.183)

(a) It is worth noting that system (5.183) is an infinite-dimensional dynamical system with state space $\mathcal{X}$ being the normed linear space of bounded functions $x : [-r, 0] \to \Re^N$, where $N = k_1 + \cdots + k_n$ with norm $\|x\|_{\mathcal{X}} = \sup_{-r \le \tau \le 0} |x(\tau)|$. It is described by FDEs and satisfies Hypotheses (Q1), (Q2) of Sect. 1.2.4 in Chap. 1. The Dynamic Cournot oligopoly game (Example 1.4.1 in Chap. 1) satisfies all above hypotheses.

(b) The equilibrium point $0 \in \mathcal{X}$ of system (5.183) corresponds to the Nash equilibrium point $q^* \in S$, noting that the deviation variables have been defined by $x_i(t) = q_i(t) - q_i^*$ for $i = 1, \ldots, n$.

(c) All discrete-time models of the form

$$\begin{aligned} q_1(k+1) &= \theta_1(k)q_1(k) + \big(1-\theta_1(k)\big) f_1\big(q_{1,2}^{\exp}(k+1), \ldots, q_{1,n}^{\exp}(k+1)\big) \\ &\vdots \\ q_i(k+1) &= \theta_i(k)q_i(k) + \big(1-\theta_i(k)\big) \\ &\quad \times f_i\big(q_{i,1}^{\exp}(k+1), \ldots, q_{i,i-1}^{\exp}(k+1), q_{i,i+1}^{\exp}(k+1), \ldots, \\ &\qquad q_{i,n}^{\exp}(k+1)\big) \\ &\vdots \\ q_n(k+1) &= \theta_n(k)q_n(k) + \big(1-\theta_n(k)\big) f_n\big(q_{n,1}^{\exp}(k+1), \ldots, q_{n,n-1}^{\exp}(k+1)\big) \end{aligned} \tag{5.186}$$

with

$$q_{i,j}^{\exp}(k+1) = a_{i,j}(k) \sum_{l=0}^{m} w_{i,j,l}(k) q_j(k-l) + \big(1 - a_{i,j}(k)\big) q_j^* \tag{5.187}$$

where k, m are nonnegative integers, $a_{i,j}(k) \in [0,1]$ $(i, j = 1, \ldots, n)$, $\theta_i(k) \in [0, \Theta]$ $(i = 1, \ldots, n)$ with $\Theta \in [0,1)$, $w_{i,j,l}(k) \geq 0$ with $1 = \sum_{l=0}^{m} w_{i,j,l}(k)$ for all $k \geq 0$ and $l = 0, \ldots, m$ $(i, j = 1, \ldots, n)$, are included in the uncertain model (5.182) and its equivalent expression (5.183) in the sense that for every model of the form (5.186), (5.187), one can give functions $\theta_i : \Re^+ \to [0, \Theta]$, $\tau_i : \Re^+ \to [T, r]$, and $d_{i,j} : \Re^+ \to \{d \in \Re^{k_j} : |d| \leq 1\}$ $(j \neq i,\ i, j = 1, \ldots, n)$ such that the solution of (5.182) coincides with the solution obtained by the discrete-time model (5.186), (5.187).

(d) Similarly, as shown in Example 1.4.1, if $S_j \subseteq [0, +\infty)$ for all $j = 1, \ldots, n$ and if all expectation rules $q_{i,j}^{\exp}(t)$ $(j \neq i,\ i, j = 1, \ldots, n)$ are Consistent Backward-looking expectations with respect to the Nash equilibrium point $q^* \in S$ and all mappings $t \to q_{i,j}^{\exp}(t)$ $(j \neq i,\ i, j = 1, \ldots, n)$ are continuous, then all continuous-time models

$$\begin{aligned} \dot{q}_1(t) &= \mu_1\big(f_1\big(q_{1,2}^{\exp}(t), \ldots, q_{1,n}^{\exp}(t)\big) - q_1(t)\big) \\ &\vdots \\ \dot{q}_n(t) &= \mu_n\big(f_n\big(q_{n,1}^{\exp}(t), \ldots, q_{n,n-1}^{\exp}(t)\big) - q_n(t)\big) \end{aligned}$$

where $\mu_i > 0$ are constants, are included in the uncertain model (5.182).

(e) The reader should notice that no continuity assumption is made for the best reply mappings of the players $f_i : S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \to S_i$ for $1 < i < n$, $n \geq 3$, and $f_1 : S_2 \times \cdots \times S_n \to S_1$, $f_n : S_1 \times \cdots \times S_{n-1} \to S_n$. Moreover, we have not assumed that the action spaces $S_j \subseteq \Re^{k_j}$ $(j = 1, \ldots, n)$ are compact sets: we simply require that the action spaces are closed convex sets. However, we have assumed the existence of a Nash equilibrium point $q^* \in S$ and the existence of functions $\tilde{\gamma}_{i,j} \in \mathcal{N}$ $(j \neq i, i = 1, \ldots, n)$ satisfying (5.184).

The crucial question that can be posed is the question of robust asymptotic stability of the Nash equilibrium $q^* \in S$ for system (5.182) or equivalently the question of robust asymptotic stability of $0 \in \mathcal{X}$ for system (5.183). The following theorem is an application of Theorem 5.2 and shows that robust global stability can be determined by the functions $\tilde{\gamma}_{i,j} \in \mathcal{N}$ ($j \neq i$, $i = 1, \ldots, n$) satisfying (5.184).

Theorem 5.7 *$0 \in \mathcal{X}$ is Robustly Globally Asymptotically Stable for system* (5.183), *if there exists $\omega > 1$ such that the following set of conditions holds for each $p = 2, \ldots, n$:*

$$(\gamma_{i_1,i_2} \circ \gamma_{i_2,i_3} \circ \cdots \circ \gamma_{i_p,i_1})(s) < s \quad \text{for all } s > 0 \tag{5.188}$$

for all $i_j \in \{1, \ldots, n\}$, $i_j \neq i_k$ if $j \neq k$, where $\gamma_{i,j}(s) := \omega \tilde{\gamma}_{i,j}(\omega s)$.

In other words, if conditions (5.188) hold, then the Nash equilibrium point $q^* \in S$ is robustly globally asymptotically stable with respect to all possible Consistent Backward-looking expectation rules with respect to the Nash equilibrium point $q^* \in S$.

It should be noticed that for the Cournot oligopoly game studied in Example 1.4.1, the best reply mappings f_i ($i = 1, \ldots, n$) are defined by (1.45). Consequently, using the convexity of the sets $S_i = [0, Q_i]$ ($i = 1, \ldots, n$), we obtain the following inequalities for $i = 1, \ldots, n$:

$$\left| f_i(q_{-i}) - q_i^* \right| \leq \frac{b}{2b + K_i} \sum_{j \neq i} \left| q_j - q_j^* \right| \leq \frac{b}{2b + K_i} (n-1) \max_{j \neq i} \left| q_j - q_j^* \right|$$

The above inequalities imply that inequalities (5.188) hold with $\tilde{\gamma}_{i,j}(s) := R_i(n-1)s$, where $R_i := \frac{b}{2b+K_i}$. Theorem 5.7 and the above definitions guarantee robust global asymptotic stability of the Nash equilibrium, provided that there exists $\omega > 1$ such that the following set of conditions holds for each $p = 2, \ldots, n$:

$$R_{i_1} R_{i_2} \cdots R_{i_p} (n-1)^p \omega^{2p} < 1$$

for all $i_j \in \{1, \ldots, n\}$, $i_j \neq i_k$ if $j \neq k$. The conditions

$$R_{i_1} R_{i_2} \cdots R_{i_p} (n-1)^p < 1 \quad \text{for all } i_j \in \{1, \ldots, n\},\ i_j \neq i_k \text{ if } j \neq k \tag{5.189}$$

are necessary and sufficient conditions for the existence of a (sufficiently small) constant $\omega > 1$ satisfying the above inequalities for each $p = 2, \ldots, n$ and for all $i_j \in \{1, \ldots, n\}$, $i_j \neq i_k$ if $j \neq k$. A more careful analysis similar to the above analysis reveals that the Nash equilibrium for the Cournot oligopoly game described in Sect. 5.2 will be asymptotically stable, provided that there exist n sets of positive real numbers $A_i = \{a_{i,j}, j \neq i\}$ ($i = 1, \ldots, n$) with $\sum_{j \neq i} x_j \leq \max_{j \neq i}(a_{i,j} x_j)$ for all $x = (x_1, \ldots, x_n)' \in (\Re^+)^n$ and $i = 1, \ldots, n$ such that the following set of conditions holds for each $p = 2, \ldots, n$:

$$a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_p,i_1} R_{i_1} R_{i_2} \cdots R_{i_p} < 1$$

for all $i_j \in \{1, \ldots, n\}$, $i_j \neq i_k$ if $j \neq k$. The above conditions are less restrictive than conditions (5.189); indeed, conditions (5.189) are implied by the above conditions for the special case $a_{i,j} = n-1$ for all $i, j = 1, \dots, n$ with $j \neq i$. For example, for $n = 3$, the above small-gain conditions are equivalent to the existence of $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that

$$\begin{aligned}
&R_1 R_2 (1+\varepsilon_1)(1+\varepsilon_2) < 1\\
&R_1 R_3 \left(1+\varepsilon_1^{-1}\right)(1+\varepsilon_3) < 1\\
&R_2 R_3 \left(1+\varepsilon_2^{-1}\right)\left(1+\varepsilon_3^{-1}\right) < 1\\
&R_1 R_2 R_3 (1+\varepsilon_1)\left(1+\varepsilon_2^{-1}\right)(1+\varepsilon_3) < 1\\
&R_1 R_2 R_3 \left(1+\varepsilon_1^{-1}\right)(1+\varepsilon_2)\left(1+\varepsilon_3^{-1}\right) < 1
\end{aligned}$$

For the above inequalities, we have used $a_{1,2} = 1+\varepsilon_1$, $a_{1,3} = 1+\varepsilon_1^{-1}$, $a_{2,1} = 1+\varepsilon_2$, $a_{2,3} = 1+\varepsilon_2^{-1}$, $a_{3,1} = 1+\varepsilon_3$, and $a_{3,2} = 1+\varepsilon_3^{-1}$. By selecting $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, we obtain inequalities (5.189).

It should be emphasized that the parameters $r \geq T > 0$, which are involved in the Consistent Backward-looking expectations, play no role in the small-gain conditions. Consequently, the small-gain conditions can help us to decide whether the Nash equilibrium point is robustly stable *without any knowledge of the expectation rules*. The small-gain conditions (5.188) demand knowledge of the Nash equilibrium point $q^* \in S$ and the best reply mappings $f_i : S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \to S_i$ for $1 < i < n$, $n \geq 3$, and $f_1 : S_2 \times \cdots \times S_n \to S_1$, $f_n : S_1 \times \cdots \times S_{n-1} \to S_n$ for which inequalities (5.184) hold.

Proof of Theorem 5.7 We first notice that system (5.183) is an autonomous dynamical system. By virtue of Theorem 1.2, $0 \in \mathcal{X}$ is a robust equilibrium point for system (5.183).

Let $\sigma > 0$ and define the family of functionals $V_i : \mathcal{X} \to \Re^+$, $i = 1, \dots, n$, by

$$V_i(x) = \sup_{-r \leq \tau \leq 0} \left|x_i(\tau)\right| \exp(\sigma\tau) \tag{5.190}$$

Using definition (5.190) and (5.185), we obtain, for all $h \in (0, T)$, $i = 1, \dots, n$, $\mu > \Theta$, and $t \geq 0$,

$$\begin{aligned}
&V_i(x_{t+h})\\
&\quad= \sup_{-r \leq \tau \leq 0} \left|x_i(t+h+\tau)\right| \exp(\sigma\tau)\\
&\quad= \sup_{t+h-r \leq s \leq t+h} \left|x_i(s)\right| \exp\bigl(\sigma(s-t-h)\bigr)\\
&\quad\leq \max\left\{ \sup_{t+h-r \leq s \leq t} \left|x_i(s)\right| \exp\bigl(\sigma(s-t-h)\bigr), \sup_{t \leq s \leq t+h} \left|x_i(s)\right| \exp\bigl(\sigma(s-t-h)\bigr) \right\}\\
&\quad\leq \max\Biggl\{ \exp(-\sigma h) V_i(x_t), \sup_{t \leq s \leq t+h} \mu \|x_i\|_{[s-T,s]} \exp\bigl(\sigma(s-t-h)\bigr),\\
&\qquad\qquad \sup_{t \leq s \leq t+h} \max_{j \neq i} \frac{\mu - \mu\Theta}{\mu - \Theta} \tilde{\gamma}_{i,j}\left(\|x_j\|_{[s-T,s]}\right) \exp\bigl(\sigma(s-t-h)\bigr) \Biggr\}
\end{aligned}$$

$$
\begin{aligned}
&= \max\Big\{\exp(-\sigma h)V_i(x_t), \\
&\quad \sup_{t\le s\le t+h} \mu \sup_{-r\le w\le 0} \big|x_i(w+s)\big| \exp(\sigma w)\exp(-\sigma w)\exp\big(\sigma(s-t-h)\big), \\
&\quad \sup_{t\le s\le t+h} \max_{j\ne i} \frac{\mu-\mu\Theta}{\mu-\Theta}\tilde{\gamma}_{i,j}\Big(\sup_{-r\le w\le 0}\big|x_j(w+s)\big|\exp(\sigma w)\exp(-\sigma w)\Big) \\
&\quad \times \exp\big(\sigma(s-t-h)\big)\Big\} \\
&\le \max\Big\{\exp(-\sigma h)V_i(x_t), \sup_{t\le s\le t+h} \mu\exp(\sigma r)V_i(x_s), \\
&\quad \sup_{t\le s\le t+h} \max_{j\ne i} \frac{\mu-\mu\Theta}{\mu-\Theta}\tilde{\gamma}_{i,j}\big(\exp(\sigma r)V_j(x_s)\big)\Big\} \\
&\le \max\Big\{\exp(-\sigma h)V_i(x_t), \mu\exp(\sigma r)\sup_{t\le s\le t+h} V_i(x_s), \\
&\quad \max_{j\ne i} \frac{\mu-\mu\Theta}{\mu-\Theta}\tilde{\gamma}_{i,j}\Big(\exp(\sigma r)\sup_{t\le s\le t+h} V_j(x_s)\Big)\Big\}
\end{aligned}
$$

where $(x_t)(\tau) = x(\tau)$ for $t - r \le \tau \le t$. Consequently, for all $i = 1,\dots,n$, $\sigma > 0$, $h \in (0, T)$, $\mu > \Theta$, and $t \ge 0$, it holds that

$$
\begin{aligned}
V_i(x_{t+h}) \le \max\Big\{&\exp(-\sigma h)V_i(x_t), \mu\exp(\sigma r)\sup_{t\le s\le t+h} V_i(x_s), \\
&\max_{j\ne i} \frac{\mu-\mu\Theta}{\mu-\Theta}\tilde{\gamma}_{i,j}\Big(\exp(\sigma r)\sup_{t\le s\le t+h} V_j(x_s)\Big)\Big\}
\end{aligned} \tag{5.191}
$$

Using induction and (5.191), we can show that, for all $i = 1,\dots,n$, $\sigma > 0$, $h \in (0, T)$, $\mu > \Theta$, and $t \ge 0$ and for every nonnegative integer $k \ge 0$, it holds that

$$
\begin{aligned}
V_i(x_{t+kh}) \le \max\Big\{&\exp(-\sigma kh)V_i(x_t), \mu\exp(\sigma r)\sup_{t\le s\le t+kh} V_i(x_s), \\
&\max_{j\ne i} \frac{\mu-\mu\Theta}{\mu-\Theta}\tilde{\gamma}_{i,j}\Big(\exp(\sigma r)\sup_{t\le s\le t+kh} V_j(x_s)\Big)\Big\}
\end{aligned} \tag{5.192}
$$

Therefore, (5.192) implies that, for all $i = 1,\dots,n$, $\mu > \Theta$, $\sigma > 0$, and $t \ge 0$, the following inequality holds:

$$
\begin{aligned}
V_i(x_t) \le \max\Big\{&\exp(-\sigma t)V_i(x_0), \mu\exp(\sigma r)\sup_{0\le s\le t} V_i(x_s), \\
&\max_{j\ne i} \frac{\mu-\mu\Theta}{\mu-\Theta}\tilde{\gamma}_{i,j}\Big(\exp(\sigma r)\sup_{0\le s\le t} V_j(x_s)\Big)\Big\}
\end{aligned} \tag{5.193}
$$

Notice that definition (5.190) implies that $2V_i(x_t) \geq \sup_{-r\leq\tau\leq 0}|x_i(t+\tau)|$ for $\sigma \leq r^{-1}\ln(2)$ and, consequently,

$$\|x_t\|_{\mathcal{X}} = \|x\|_{[t-r,t]} \leq \sum_{i=1}^{n} \sup_{-r\leq\tau\leq 0} \left|x_i(t+\tau)\right| \leq 2\sum_{i=1}^{n} V_i(x_t) \tag{5.194}$$

Without loss of generality we may assume $\Theta > 0$. Define $\mu := \frac{\omega\Theta}{\omega-1+\Theta}$ and let the constant $\sigma > 0$ satisfy the inequalities $\sigma \leq r^{-1}\ln(2)$, $\sigma < r^{-1}\ln(\omega)$, and $\sigma < r^{-1}\ln(\frac{\omega-1+\Theta}{\omega\Theta})$, where $\omega > 1$ is the constant involved in the hypotheses of the theorem. Notice that the hypothesis $\Theta < 1$ and previous definitions imply that $\mu\exp(\sigma r) < 1$, $\mu > \Theta$, $\exp(\sigma r) \leq \omega$, and $\frac{\mu-\mu\Theta}{\mu-\Theta} \leq \omega$. It follows from (5.193), (5.194), and definition (5.190) (which implies $V_i(x_t) \leq \|x_t\|_{\mathcal{X}}$) that the following inequalities hold for all $i = 1, \dots, n$ and $t \geq 0$:

$$V_i(x_t) \leq \max\left\{\exp(-\sigma t)L(x_0),\, B \sup_{0\leq s\leq t} V_i(x_s),\, \max_{j\neq i} \omega\tilde{\gamma}_{i,j}\left(\omega \sup_{0\leq s\leq t} V_j(x_s)\right)\right\} \tag{5.195}$$

$$L(x_t) \leq \max\left\{2n\|x_0\|_{\mathcal{X}},\, 2\sum_{i=1}^{n} \max_{j\neq i} \omega\tilde{\gamma}_{i,j}\left(\omega \sup_{0\leq s\leq t} V_j(x_s)\right) + 2B\sum_{i=1}^{n} \sup_{0\leq s\leq t} V_i(x_s)\right\} \tag{5.196}$$

where $L(x) := \|x\|_{\mathcal{X}}$ and $B := \mu\exp(\sigma r) < 1$. It follows from (5.194), (5.195), (5.196), and Theorem 5.2 that $0 \in \mathcal{X}$ is Robustly Globally Asymptotically Stable for system (5.183), provided that the set of conditions (5.188) holds for each $p = 2, \dots, n$ and for all $i_j \in \{1, \dots, n\}$, $i_j \neq i_k$ if $j \neq k$. The proof is complete. □

Theorem 5.7 may be used to derive conditions for uniqueness of a fixed point. Indeed:

Corollary 5.4 *Let $S_i \subseteq \Re^{k_i}$ $(i = 1, \dots, n)$ be closed convex sets, and let functions $f_i : S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \to S_i$ for $1 < i < n$, $n \geq 3$, and $f_1 : S_2 \times \cdots \times S_n \to S_1$, $f_n : S_1 \times \cdots \times S_{n-1} \to S_n$ for which there exists $q^* = (q_1^*, \dots, q_n^*) \in S$, where $S := S_1 \times \cdots \times S_n$ satisfies (5.178). Furthermore, suppose that there exist functions $\tilde{\gamma}_{i,j} \in \mathcal{N}$ $(j \neq i,\ i, j = 1, \dots, n)$ such that inequalities (5.184) hold for all $q \in S$ and that there exists $\omega > 1$ such that conditions (5.188) hold for each $p = 2, \dots, n$ and for all $i_j \in \{1, \dots, n\}$, $i_j \neq i_k$ if $j \neq k$. Then $q^* = (q_1^*, \dots, q_n^*) \in S$ is the unique fixed point of the mapping $S \ni q \to F(q) := (f_1(q_{-1}), \dots, f_n(q_{-n})) \in S$.*

It is worth noting that Corollary 5.4 does not guarantee the existence of a fixed point for the mapping $S \ni q \to F(q) := (f_1(q_{-1}), \dots, f_n(q_{-n})) \in S$. Corollary 5.4 can be used in conjunction with classical fixed-point theorems (e.g., Brouwer's fixed point theorem when all action spaces $S_i \subseteq \Re^{k_i}$ $(i = 1, \dots, n)$ are compact and convex and when the mapping $S \ni q \to F(q) := (f_1(q_{-1}), \dots, f_n(q_{-n})) \in S$ is continuous) in order to guarantee the uniqueness of the fixed point.

Another issue, which is important for the economic literature, is the fact that the hypothesis of the consistent backward-looking expectations does not allow rational expectations. This point motivates the following definition for the strategic game described above.

Definition 5.4 An expectation rule $q_{i,j}^{\exp}(t)$ $(j \neq i,\ i,j = 1,\dots,n)$ is called a Rational-Consistent Backward-looking expectation with respect to the Nash equilibrium point $q^* \in S$ if there exists a constant $0 < r$ such that

$$\left|q_{i,j}^{\exp}(t) - q_j^*\right| \leq \sup_{t-r \leq \tau \leq t} \left|q_j(\tau) - q_j^*\right| = \left\|q_j - q_j^*\right\|_{[t-r,t]} \text{ for all } t \geq 0 \tag{5.197}$$

Clearly, rational expectations are Rational-Consistent Backward-looking expectations with respect to the Nash equilibrium point $q^* \in S$. Moreover, a Consistent Backward-looking expectation is a Rational-Consistent Backward-looking expectation with respect to the Nash equilibrium point $q^* \in S$.

Can we consider system (5.179) where all expectation rules $q_{i,j}^{\exp}(t) \in S_j$ $(j \neq i$, $i, j = 1,\dots,n)$ are Rational-Consistent Backward-looking expectations with respect to the Nash equilibrium point $q^* \in S$? The key mathematical problem that arises in this case is whether we can obtain a well-defined dynamical system if Hypotheses (Q1) and (Q2) of Sect. 1.2.4 in Chap. 1 do not hold. However, we can extend the analysis of the previous section under the following hypothesis:

(H′) There exist m index sets $J_l \subseteq \{1,\dots,n\}$, $l = 1,\dots,m$, with $J_l \cap J_k = \emptyset$ for $l \neq k$ and $\bigcup_{l=1,\dots,m} J_l = \{1,\dots,n\}$ such that the following statements hold:

- All players with $i \in J_m$ are using Consistent Backward-looking expectations with respect to the Nash equilibrium point $q^* \in S$.
- For every $k = 1,\dots,m-1$, all players with $i \in J_k$ are using Rational-Consistent Backward-looking expectations, $q_{i,j}^{\exp}(t) \in S_j$ if $j \in J_l$ with $l > k$ and Consistent Backward-looking expectations, and $q_{i,j}^{\exp}(t) \in S_j$ otherwise.

Indeed, if Hypothesis (H′) holds, then system (5.179) is expressed in deviation variables $x_i(t) = q_i(t) - q_i^*$ $(i = 1,\dots,n)$ by the following equations:

$$\begin{aligned}
x_1(t) &= \theta_1(t)\big(\Pr_{S_1}\big(x_1\big(t-\tau_1(t)\big)+q_1^*\big)-q_1^*\big)+\big(1-\theta_1(t)\big)\\
&\quad\times\big(f_1\big(\Pr_{S_2}\big(q_2^*+d_{1,2}(t)s_{1,2}(t)\big),\dots,\Pr_{S_n}\big(q_n^*+d_{1,n}(t)s_{1,n}(t)\big)\big)-q_1^*\big)\\
&\ \ \vdots\\
x_n(t) &= \theta_n(t)\big(\Pr_{S_n}\big(x_n\big(t-\tau_n(t)\big)+q_n^*\big)-q_n^*\big)+\big(1-\theta_n(t)\big)\\
&\quad\times\big(f_n\big(\Pr_{S_1}\big(q_1^*+d_{n,1}(t)s_{n,1}(t)\big),\dots,\\
&\qquad\Pr_{S_{n-1}}\big(q_{n-1}^*+d_{n,n-1}(t)s_{n,n-1}(t)\big)\big)-q_n^*\big)\\
s_{i,j}(t) &:= \|x_j\|_{[t-r,t]} \quad \text{if } i \in J_k \text{ and } j \in J_l \text{ with } l > k\\
s_{i,j}(t) &:= \|x_2\|_{[t-r,t-T]} \quad \text{if } i \in J_k \text{ and } j \in J_l \text{ with } l \leq k
\end{aligned} \tag{5.198}$$

where $\theta_i : \Re^+ \to [0, \Theta]$, $\tau_i : \Re^+ \to [T, r]$, $d_{i,j} : \Re^+ \to \{d \in \Re^{k_j} : |d| \le 1\}$ ($j \ne i$, $i, j = 1, \ldots, n$).

Let us explain next why system (5.198) is an infinite-dimensional dynamical system with state space $\mathcal{X}$ being the normed linear space of bounded functions $x : [-r, 0] \to \Re^N$, where $N = k_1 + \cdots + k_n$ with norm $\|x\|_{\mathcal{X}} = \sup_{-r \le \tau \le 0} |x(\tau)|$. Indeed, by using the method of steps, given an initial condition $x_0 \in \mathcal{X}$ and functions $\theta_i : \Re^+ \to [0, \Theta]$, $\tau_i : \Re^+ \to [T, r]$, and $d_{i,j} : \Re^+ \to \{d \in \Re^{k_j} : |d| \le 1\}$ ($j \ne i$, $i, j = 1, \ldots, n$), one can in principle determine from (5.198) the solution $x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^m$ for $t \in (0, T]$ with $x(\tau) = (x_1(\tau), \ldots, x_n(\tau))' = x_0(\tau)$ for all $\tau \in [-r, 0]$ using the following procedure:

Step 1: First, determine the solution $x_i(t)$ for $t \in (0, T]$ and for all players with $i \in J_m$ that are using Consistent Backward-looking expectations with respect to the Nash equilibrium point $q^* \in S$. In this case, we are in a position to determine the components of the solution $x_j(t)$ for $j \in J_m$ and $t \in (0, T]$ by means of (5.198). Furthermore, in this case inequality (5.185) holds for $t \in (0, T]$, $i \in J_m$, and consequently there exists a function $G_m \in \mathcal{N}$ such that

$$\sup_{t \in [0,T]} \left|x_i(t)\right| \le G_m\left(\|x\|_{[-r,0]}\right) \quad \text{for } i \in J_m \tag{5.199}$$

Step 2: Next, determine the solution $x_i(t)$ for $t \in (0, T]$ and for all players with $i \in J_{m-1}$ that are using Rational-Consistent Backward-looking expectations, $q_{i,j}^{\exp}(t) \in S_j$ if $j \in J_m$ and Consistent Backward-looking expectations, and $q_{i,j}^{\exp}(t) \in S_j$ otherwise. In this case, (5.184) implies that the following inequality holds for all $t \in (0, T]$:

$$\left|x_i(t)\right| \le \max_{j \ne i, j \notin J_m} \tilde{\gamma}_{i,j}\left(\|x_j\|_{[t-r,t-T]}\right) + \max_{j \ne i, j \in J_m} \tilde{\gamma}_{i,j}\left(\|x_j\|_{[t-r,t]}\right) \tag{5.200}$$

However, the components of the solution $x_j(t)$ for $j \in J_m$ and $t \in (0, T]$ have been determined by Step 1. Therefore, we are in a position to determine the components of the solution $x_j(t)$ for $j \in J_{m-1}$ and $t \in (0, T]$ by means of (5.198). Using (5.199) and (5.200), we obtain the existence of a function $G_{m-1} \in \mathcal{N}$ such that

$$\sup_{t \in [0,T]} \left|x_i(t)\right| \le G_{m-1}\left(\|x\|_{[-r,0]}\right) \quad \text{for } i \in J_m \cup J_{m-1} \tag{5.201}$$

Step k ($3 \le k \le m$): We determine the solution $x_i(t)$ for $t \in (0, T]$ and for all players with $i \in J_{m+1-k}$ that are using Rational-Consistent Backward-looking expectations, $q_{i,j}^{\exp}(t) \in S_j$ if $j \in J_l$ with $l > m + 1 - k$ and Consistent Backward-looking expectations, and $q_{i,j}^{\exp}(t) \in S_j$ otherwise. In this case, (5.184) implies that the following inequality holds for all $t \in (0, T]$:

$$\begin{aligned} \left|x_i(t)\right| \le & \max_{j \ne i, j \in J_l, l \le m+1-k} \tilde{\gamma}_{i,j}\left(\|x_j\|_{[t-r,t-T]}\right) \\ & + \max_{j \ne i, j \in J_l, l > m+1-k} \tilde{\gamma}_{i,j}\left(\|x_j\|_{[t-r,t]}\right) \end{aligned} \tag{5.202}$$

However, the components of the solution $x_j(t)$ for $j \in J_l$ with $l > m + 1 - k$ and $t \in (0, T]$ have been determined by previous steps. Therefore, we are in a position to determine the components of the solution $x_j(t)$ for $j \in J_{m+1-k}$ and $t \in (0, T]$ by means of (5.198). Moreover, by virtue of previous steps, there exists a function $G_{m+2-k} \in \mathcal{N}$ such that

$$\sup_{t\in[0,T]} \left|x_i(t)\right| \le G_{m+2-k}\left(\|x\|_{[-r,0]}\right) \quad \text{for } i \in \bigcup_{l=m+2-k,\dots,m} J_l \tag{5.203}$$

Using (5.202) and (5.203), we obtain the existence of a function $G_{m+1-k} \in \mathcal{N}$ such that

$$\sup_{t\in[0,T]} \left|x_i(t)\right| \le G_{m+1-k}\left(\|x\|_{[-r,0]}\right) \quad \text{for } i \in \bigcup_{l=m+1-k,\dots,m} J_l \tag{5.204}$$

After the completion of the m steps, we have determined all components of the solution $x_j(t)$ for $j = 1, \dots, n$ and $t \in (0, T]$. Moreover, we have also constructed a function $G \in \mathcal{N}$ such that

$$\sup_{t\in[0,T]} \left|x(t)\right| \le G\left(\|x\|_{[-r,0]}\right) \tag{5.205}$$

Without loss of generality, we may assume that $G(s) \ge s$ for all $s \ge 0$. Moreover, by using (5.205) we may establish that $0 \in \mathcal{X}$ is a robust equilibrium point for system (5.198) and conclude exactly as previously in that estimates (5.194), (5.195), and (5.196) hold.

The proof of the following theorem is exactly the same with the proof of the Theorem 5.7 and therefore is omitted.

Theorem 5.8 *$0 \in \mathcal{X}$ is Robustly Globally Asymptotically Stable for system* (5.198) *under Hypothesis* (H′) *if there exists $\omega > 1$ such that the set of conditions* (5.188) *holds for each $p = 2, \dots, n$ and for all $i_j \in \{1, \dots, n\}$, $i_j \ne i_k$ if $j \ne k$, where $\gamma_{i,j}(s) := \omega\tilde{\gamma}_{i,j}(\omega s)$.*

It should be emphasized that the parameters $r \ge T > 0$ that are involved in the definition of the Consistent Backward-looking expectation play no role in the small-gain conditions. Moreover, the number m of the index sets $J_l \subseteq \{1, \dots, n\}$ involved in Hypothesis (H′) or the particular members of each index set play absolutely no role in the small-gain conditions (5.188). Furthermore, all these parameters are allowed to change with time: there is no need to assume that these parameters remain constant. Consequently, the small-gain conditions can help us to decide whether the Nash equilibrium point is robustly stable *without any knowledge of the expectation rules*. Again, the small-gain conditions (5.188) demand knowledge of the Nash equilibrium point $q^* \in S$ and the best reply mappings $f_i : S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \to S_i$ for $1 < i < n$, $n > 3$, and $f_1 : S_2 \times \cdots \times S_n \to S_1$, $f_n : S_1 \times \cdots \times S_{n-1} \to S_n$ for which inequalities (5.184) hold.

Finally, it should be noted that, for a specific strategic game, even less demanding hypotheses than Hypothesis (H′) can be used in order to guarantee that system (5.179) gives an infinite-dimensional dynamical system with state space $\mathcal{X}$ being the normed linear space of bounded functions $x : [-r, 0] \to \Re^N$, where $N =$

$k_1 + \cdots + k_n$ with norm $\|x\|_{\mathcal{X}} = \sup_{-r \le \tau \le 0} |x(\tau)|$. This can be done by exploiting special properties of the best reply mappings $f_i : S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \to S_i$ for $1 < i < n$, $n > 3$, and $f_1 : S_2 \times \cdots \times S_n \to S_1$, $f_n : S_1 \times \cdots \times S_{n-1} \to S_n$ (e.g., if some of the functions are independent of certain arguments).

5.6 Historical and Bibliographical Notes

1. The small-gain theorem has been widely recognized as an important tool for robustness analysis and robust controller design within the control systems community. For instance, classical small-gain theorems [9, 52, 53] have played a crucial role for linear robust control of uncertain systems subject to dynamic uncertainties [54]. As introduced in the framework of classical small-gain, an essential condition for input–output stability of a feedback system is that the loop gain is less than one. This condition relying upon on the concept of linear finite-gain was first relaxed by Hill [14] and then by Mareels and Hill [38] using the notions of monotone gain and nonlinear operators. Quickly after the birth of the notion of input-to-state stability (ISS) originally introduced by Sontag [47], a nonlinear, generalized small-gain theorem was developed in [21]. This nonlinear ISS small-gain theorem differs from classical small-gain theorems and the nonlinear small-gain theorem of [14] and [38] in several aspects. One of them is that both internal and external stability properties are discussed in a single framework, while only input–output stability is addressed in previous small-gain theorems. As demonstrated in [21] and the subsequent work of many others, nonlinear small-gain has led to new solutions of several challenging problems in robust nonlinear control, such as stabilization by partial-state and output feedback, robust adaptive tracking, and nonlinear observers.
2. The nonlinear small-gain theory presented in [21] has stimulated the follow-up research by several researchers (see [1, 4–8, 11, 15–20, 22, 25–28, 31, 37, 44, 48, 49, 51]).

   Nonlinear small-gain theorems for discrete-time systems can be found in [23, 24]. Extensions of small-gain results were presented recently in the literature. In [1, 15–18] less conservative small-gain conditions were presented for finite-dimensional systems. In [5–8, 20, 27, 28] matrix gain functions were used for the study of large-scale finite-dimensional systems. A nonuniform in time small-gain theorem for continuous-time finite-dimensional systems was presented in [31]. Moreover, in [25, 48] small-gain results for wide classes of systems were provided. The classes of systems considered in [25, 48] satisfy the classical "semigroup property." The first small-gain result for systems satisfying the weak semigroup property was presented in [26]. Small-gain results for hybrid systems satisfying the classical "semigroup property" were recently presented in [37].
3. It is of interest to note that this perspective of nonlinear small-gain has found useful applications in monotone systems, an important class of systems in mathematical biology (see [2, 10]).

4. One of the most important obstacles in applying nonlinear small-gain results is the representation of the original composite system as the feedback interconnection of subsystems which satisfy the Input-to-Output Stability (IOS) property. More specifically, sometimes the subsystems do not satisfy the IOS property: there is a transient period after which the solution enters a certain region of the state space. Within this region of the state space, the subsystems satisfy the small-gain requirements. In other words, the essential inequalities, utilized by small-gain results in order to prove stability properties, do not hold for all times: this feature excludes almost all available small-gain results from possible application. Particularly, this feature is important in systems of Mathematical Biology and Population Dynamics. Indeed, the idea of developing stability results which utilize certain Lyapunov-like conditions after an initial transient period was used in [30, 34] with primary motivation from addressing robust feedback stabilization problems for certain chemostat models. Small-gain results presented in this chapter can allow a transient period during which the solutions do not satisfy the IOS inequalities. The obtained results are direct extensions of the recent vector small-gain result in [27], and if the initial transient vanishes, then the results coincide with Theorem 3.1 in [27]. Notice that the small-gain results presented here are included in [28]. To better position the generality of the small-gain results of this chapter, some highlights are given below.

   - They can be applied to systems that satisfy the weak semigroup property instead of the classical semigroup property.
   - They can be applied to large-scale systems composed of multiple interacting subsystems.
   - They allow the explicit computation of the gain function for the composite system.
   - They allow for nonuniform in time stability phenomena.
   - They allow nonzero diagonal gains for each interacting system. This generality is important for studying nonlinear uncertain and time-delay systems.
   - They allow vector Lyapunov function (or functional) characterizations for various stability properties.
   - They allow the application of the small-gain perspective to various systems which satisfy less demanding stability notions than ISS or IOS.

5. Hypotheses (SG1), (SG2) hold automatically when Hypotheses (H1–3) of Theorem 3.1 in [27] hold (Hypotheses (H1–3) in [27] correspond to the special case $S(t) = \mathcal{X}$ and $\xi = t_0$). Consequently, Hypotheses (SG1), (SG2) are less restrictive than previous ones. Indeed, inequalities (5.4), (5.5) are not assumed to hold for all times $t \in [t_0, t_{\max})$ but only after the solution map $\phi(t, t_0, x_0, u, d)$ has entered the set $S(t) \subseteq \mathcal{X}$.
6. It should be mentioned that alternative sampled-data feedback designs for system (5.145) applied with zero-order hold and positive sampling rate can be obtained by using the results [41, 42], which, however, achieve semiglobal and practical stabilization. However, the feedback design obtained by using the

trajectory-based small-gain results of the present chapter guarantee global and asymptotic stabilization. Moreover, robustness to perturbations of the sampling schedule is guaranteed (that is the reason for introducing the input w in the closed-loop system (5.146)).

7. The advantage of vector Lyapunov function versus single Lyapunov function in nonlinear stability analysis has been well documented in past literature [12, 36, 40]. Recent work in [29, 32] provides further evidence on the usefulness of vector Lyapunov functions for the case of input-to-state stability.
8. The results on monotone operators of Sect. 5.2 are closely related to results presented in [44, 45, 51].
9. It is worth noting that Theorem 4.1 and Corollary 4.3 in [29] can be easily derived from Theorem 5.6 and Corollary 5.3 with $\gamma_{i,j}(s) = a(s)$ for all $i, j = 1, \dots, k$ and $Q(x) \equiv 0$, where $a \in \mathcal{N}_1$ with $a(s) < s$ for $s > 0$. Moreover, it is assumed in [29] that there exist a constant $R \geq 0$ and a function $p \in K_\infty$ such that $|x| \leq R + p(|H(x)|)$. In the presented framework, such a hypothesis is not needed. The interpretation of the family of set-valued maps $\Re^+ \times \Re^n \ni (T, x) \to A_i(T, x) \subseteq \Re^n$ $(i = 1, \dots, k)$ defined in (5.119) is the following (the same with [29]): each $A_i(T, x) \subseteq \Re^n$ is the set of all states $x_0 \in \Re^n$ so that the solution of $\dot{x}(t) = f(x(t), x_0, d(t), u(t), u(0))$ with initial condition $x(0) = x_0$ can be controlled to $x \in \Re^n$ in time s less than or equal to T by means of appropriate inputs $(d, u) \in M_D \times M_{U_1}$ that satisfy $\zeta(\sup_{t\in[0,s]} |u(t)|) \leq V_i(x)$ and such that the trajectory of the solution satisfies the constraint $\max_{j=1,\dots,k} \sup_{t\in[0,s]} \gamma_{i,j}(V_j(x(t))) \leq V_i(x)$. In general, it is very difficult to obtain an accurate description of the set-valued maps $\Re^+ \times \Re^n \ni (T, x) \to A_i(T, x) \subseteq \Re^n$ defined by (5.119). However, for every $g \in C^1(\Re^n; \Re)$, we have $A_i(T, x) \subseteq B_i^g(T, x) = \{x_0 \in \Re^n : |g(x_0) - g(x)| \leq T b_i^g(x)\}$ for all $(T, x) \in \Re^+ \times \Re^n$, where
$$b_i^g(x) := \max\Big\{ \big|\nabla g(\xi) f(\xi, x_0, d, u, u_0)\big| : d \in D, \zeta\big(\max\{|u|, |u_0|\}\big) \leq V_i(x), \max_{j=1,\dots,k} \gamma_{i,j}\big(\max\{V_j(\xi), V_j(x_0)\}\big) \leq V_i(x) \Big\} < +\infty$$
and $V_i \in C^1(\Re^n; \Re^+)$ $(i = 1, \dots, k)$ are the functions involved in the hypotheses of Theorem 5.6.
10. Examples 5.4.1 and 5.4.2 have appeared previously in [27]. Examples 5.4.3, 5.4.4, and 5.4.5 have appeared in [28]. The application of the small-gain results to dynamic strategic games was presented in [35].
11. It should be noted that the method of proving stability by means of fixed-point theorems developed by Burton (see [3]) is very closely related to the small-gain methods. Another connection between small-gain methods and fixed-point theorems is provided by Corollary 5.4.
12. It is of interest to note that some of the functionals $Q_i : [-r + r_i, +\infty) \times C^0([-r_i, 0]; \Re^n) \to \Re^+$ in Theorem 5.5 and Corollary 5.2 are allowed to be functions (case of $r_i = 0$). This reminds the case of Razumikhin functions, which are used frequently for the proof of stability properties of systems described by RFDEs (see [13, 33, 39, 43, 50]). Consequently, Theorem 5.5 and

Corollary 5.2 allow the flexibility of using Lyapunov-like functionals with Razumikhin-like functions in order to prove desired stability properties. Teel [50] was the first to observe that Razumikhin results are essentially small-gain results.

13. (Comparison of Theorem 5.3 and Corollary 5.1 with existing results): The reader should compare the result of Corollary 5.1 with Theorem 3.4 in [32]. It is clear that Theorem 3.4 in [32] is a special case of Corollary 5.1 with $\gamma_{i,j}(s) = a(s)$ for all $i, j = 1, \ldots, k$, where $a \in \mathcal{N}_1$ with $a(s) < s$ for $s > 0$. Alternative vector Lyapunov characterizations are based on the main results in [5, 7, 8, 32] or on the cyclic small-gain condition in [51] (see, for example, Theorem 2 in [20]). In order to demonstrate the applicability of our results to large-scale interconnected systems, consider the case

$$\dot{x}_i = f_i(d, x, u) \quad i = 1, \ldots, k$$
$$x = (x_1', \ldots, x_k')' \in \Re^N, d \in D, u \in U$$

where $x_i \in \Re^{n_i}$, $i = 1, \ldots, k$, $N = n_1 + \cdots + n_k$, $D \subset \Re^l$ is a nonempty compact set, $U \subseteq \Re^m$ is a nonempty set with $0 \in U$, $f_i : D \times \Re^N \times U \to \Re$, $i = 1, \ldots, k$, are locally Lipschitz mappings with $f_i(d, 0, 0) = 0$ for all $d \in D$, $i = 1, \ldots, k$. We assume that the UISS property holds for each subsystem $\dot{x}_i = f_i(d, x, u)$ with input $(u, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ $(i = 1, \ldots, k)$. Let $V_i \in C^1(\Re^{n_i}; \Re^+)$ $(i = 1, \ldots, k)$ be ISS-Lyapunov functions for each of the subsystems, i.e., positive definite and radially unbounded functions for which the following inequalities hold for $i = 1, \ldots, k$:

$$\sup_{d \in D} \Big\{ \nabla V_i(x_i) f_i(d, x, u) : u \in U, x = \big(x_1', \ldots, x_k'\big)' \in \Re^N,$$
$$\max\Big\{ \zeta\big(|u|\big), \max_{j \neq i} \gamma_{i,j}\big(V_j(x_j)\big)\Big\} \leq V_i(x_i) \Big\} \leq -\rho_i\big(V_i(x_i)\big)$$
$$\text{for all } x_i \neq 0$$

for certain functions $\zeta \in \mathcal{N}_1$, $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, k$, and certain positive definite functions $\rho_i \in C^0(\Re^+; \Re^+)$ $(i = 1, \ldots, k)$. Working with the Lyapunov-like functions $V_i \in C^1(\Re^{n_i}; \Re^+)$ $(i = 1, \ldots, k)$ and exploiting Corollary 5.1, we can guarantee that the UISS property holds for the above system if the small-gain conditions (5.3) hold. It should be clear that the functions $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, k$, are the actual gain functions, i.e., the following inequalities hold for all $i = 1, \ldots, k$, $t \geq 0$, $x(0) \in \Re^N$, and $u \in M_U$:

$$V_i\big(x_i(t)\big)$$
$$\leq \max\Big\{ \sigma_i\big(V_i\big(x_i(0)\big), t\big), \zeta\Big(\sup_{0 \leq \tau \leq t} \big|u(\tau)\big|\Big), \max_{j \neq i} \gamma_{i,j}\Big(\sup_{0 \leq \tau \leq t} V_j\big(x_j(\tau)\big)\Big)\Big\}$$

for certain $\sigma_i \in KL$ $(i = 1, \ldots, k)$. Since $\gamma_{i,i}(s) \equiv 0$ for $i = 1, \ldots, k$, then the above inequalities are nothing else but the inequalities of the max-formulation of the UISS property for each subsystem $\dot{x}_i = f_i(d, x, u)$ with input $(u, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ $(i = 1, \ldots, k)$ and $V_i(x_i)$ replacing $|x_i|$ for $i = 1, \ldots, k$.

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Chapter 6
Robust Output Feedback Stabilization

6.1 Introduction

This chapter is devoted to the analysis of the problem of the Robust Output Feedback Stabilization Problem. In order to be able to pose the problem rigorously, we need the notions described in previous chapters and particularly:

- the notion of a control system (see Chap. 1),
- the notion of the equilibrium point of a control system (see Chap. 1),
- the notion of the feedback interconnection of control systems (see Chap. 1),
- the internal and external stability notions for dynamical and control systems (see Chaps. 2 and 4).

The emphasis is placed on systems described by ODEs and RFDEs. However, it will be clear that most of the obtained results can be generalized to wider classes of control systems with uncertainties.

Another issue which is important in this chapter is the introduction of errors. Although feedback and control have been proved to be extremely useful for the stabilization of many practical systems, it is also a well-known fact that the application of feedback control does induce a variety of errors to the closed-loop system. Examples of such errors may take the form of measurement noise, actuator noise, and parameter variations. Therefore the behavior of the system under feedback control can be analyzed using the external stability notions developed in Chap. 4.

The existence of stabilizing feedback laws and the feedback design problem are not yet fully understood for nonlinear systems. During the last decades, significant progress has been made for the solution of these problems. The main idea of the present chapter is that for every method of proving stability, there is a method of stabilizing feedback design. The feedback design for linear control systems is presented as a method based on analytical solutions. The feedback linearization method is presented as a method based on transformation results. The Control Lyapunov Function (CLF) methodology is presented as a method based on Lyapunov functionals. Finally, small-gain methods are also described for feedback systems. However, as noted in previous chapters, since methods of proving stability can be (and usually

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are) combined, many methods of feedback design can be combined as well when one confronts real-world control system design problems.

6.2 Description of the Robust Output Feedback Stabilization Problems

In order to introduce the feedback stabilization problems, we start with an example which illustrates all main issues in feedback stabilization.

Consider the chemostat with one microbial population and one nutrient described by the system of ODEs (1.4) or its equivalent form (1.7). Clearly, there are two inputs $(u_1, u_2) \in [-1, +\infty) \times [-M, +\infty)$ that correspond to the dilution rate D and the inlet concentration of nutrient s_{in}. As shown in Chap. 2 (Example 2.7.3), the origin $0 \in \Re^2$ is URGAS for (1.7) under the following three assumptions: (i) both inputs are zero (i.e., the dilution rate D and the inlet concentration of the nutrient s_{in} assume constant values, equal to the nominal ones); (ii) the specific growth rate $\mu(s)$ is an increasing, locally Lipschitz, bounded function, and (iii) the mortality rate is zero, and the yield coefficient is constant.

However, in many cases, one of the above assumptions does not hold. For example, there are situations where (see [38] and references therein) the specific growth rate is not an increasing function: there is substrate inhibition in the growth of the microbe. In such cases, system (1.4) may have two (or more) equilibrium points except the trivial equilibrium point $(s, x) = (s_{in}, 0)$. This implies that $0 \in \Re^2$ is not URGAS for (1.7). In fact, the region of attraction of $0 \in \Re^2$ for (1.7) can be a very "small" subset in $\Re^2$ (see [38] and references therein). Furthermore, the assumption that the inlet concentration of the nutrient s_{in} assumes a constant value, equal to the nominal one, is usually an unrealistic assumption.

All the above considerations lead us to the conclusion that the operation of the chemostat without control may drive the states to an operating point different from the nominal point. How can we force the states of the system (1.7) to approach the desired position $0 \in \Re^2$?

Since the inlet concentration of the nutrient s_{in} is usually determined by other processes, we can think of the following idea: use the input $u_1 \in [-1, +\infty)$ (or the dilution rate D) in order to force the states of the system (1.7) to approach the desired position $0 \in \Re^2$. Mathematically, there are two ways of achieving such a goal:

- Given the initial state $x_0 = (x_{1,0}, x_{2,0}) \in \Re^2$ and assuming knowledge of the input $u_2 \in [-M, +\infty)$ on a time interval $[0, T]$, calculate an admissible input $u_1 : [0, T] \to [-1, +\infty)$ that will bring the states the system (1.7) "close" to the desired position $0 \in \Re^2$ in "an optimal way."
- Given a continuous measurement of the state vector $x(t) = (x_1(t), x_2(t)) \in \Re^2$, find a "law" which will adjust "automatically" the value of the input $u_1(t) \in [-1, +\infty)$ depending on the current position of the state. The automatic "adjustment law" must guarantee that the states of system (1.7) will be "close" to

the desired position $0 \in \Re^2$ after an initial transient period. In other words, we are looking for a function $k : \Re^2 \to [-1, +\infty)$ such that the "adjustment law" $u_1(t) = k(x_1(t), x_2(t))$ for all $t \geq 0$ will achieve the objective.

The first approach is the approach followed in the literature of Optimal Control. The second approach is the robust feedback stabilization approach. The advantages and disadvantages of each approach are discussed in many books (see the discussion in [64]). In this chapter, we discuss the second approach.

The reader should notice that the function $k : \Re^2 \to [-1, +\infty)$ that we have to find in the feedback stabilization approach is called "the controller" or the "static state-feedback stabilizer." However, what we have described so far does not constitute a well-posed problem. The following questions can be asked:

Question 1 What do we mean by "close" to the desired position $0 \in \Re^2$?

The answer to the question is crucial and is strongly connected to the following question:

Question 2 What about the input $u_2 \in [-M, +\infty)$?

There are two possible answers to the above questions:

(a) In the case where the input u_2 is identically zero (i.e., the inlet concentration of the nutrient s_{in} assumes a constant value, equal to the nominal one), we can seek a static state-feedback stabilizer $k : \Re^2 \to [-1, +\infty)$ which will guarantee that $0 \in \Re^2$ is URGAS for the dynamical system

$$\begin{aligned} \dot{x}_1 &= D^*\big(g(x_2) - k(x_1, x_2)\big) \\ \dot{x}_2 &= D^* \exp(-x_2)\big[M\big(1 - p(x_2)\exp(x_1)\big) - RMp(x_2)g(x_2)\exp(x_1) \\ &\quad + 1 - \exp(x_2) + k(x_1, x_2)\big(M + 1 - \exp(x_2)\big)\big] \end{aligned} \tag{6.1}$$

Notice that the dynamical system (6.1) is obtained by the substitution of $k(x_1, x_2)$ in place of u_1: this is justified since the value of the input $u_1(t) \in [-1, +\infty)$ follows the "adjustment law" $u_1(t) = k(x_1(t), x_2(t))$ for all $t \geq 0$. System (6.1) is termed as "*the closed-loop system*" (1.7) with feedback law $u_1 = k(x_1, x_2)$. Of course, system (6.1) is a system described by ODEs, and consequently the function $k : \Re^2 \to [-1, +\infty)$ must satisfy certain regularity properties in such a way that system (6.1) is well defined.

(b) In the case where the input u_2 may assume nonzero values and is unknown, we have to take into account the effect of the external input u_2. In this case it is natural to search for a static state-feedback stabilizer which will guarantee that the control system

$$\begin{aligned} \dot{x}_1 &= D^*\big(g(x_2) - k(x_1, x_2)\big) \\ \dot{x}_2 &= D^* \exp(-x_2)\big[M\big(1 - p(x_2)\exp(x_1)\big) \\ &\quad - RMp(x_2)g(x_2)\exp(x_1) + 1 - \exp(x_2)\big] \\ &\quad + D^* \exp(-x_2)\big[u_2 + k(x_1, x_2)\big(M + 1 + u_2 - \exp(x_2)\big)\big] \end{aligned} \tag{6.2}$$

satisfies the (W)ISS property from the input u_2. Notice that it may not be possible to guarantee the (W)ISS property for all measurable and locally essentially bounded inputs $u_2 : \Re^+ \to [-M, +\infty)$. However, it may be possible to guarantee the (W)ISS property for all measurable and locally essentially bounded inputs $u_2 : \Re^+ \to [-L, L]$, where $L < M$. Again, the function $k : \Re^2 \to [-1, +\infty)$ must satisfy certain regularity properties so that system (6.2) is well defined. In this case the feedback stabilization problem is termed as "robust" since the (W)ISS property from the input u_2 will guarantee robustness properties for the solution of system (6.2) for all initial states $x_0 = (x_{1,0}, x_{2,0}) \in \Re^2$ and for all measurable and locally essentially bounded inputs $u_2 : \Re^+ \to [-L, L]$. Finally, again system (6.2) is termed as "*the closed-loop system*" (1.7) with the feedback law $u_1 = k(x_1, x_2)$.

An additional comment is needed at this point. We have described the control objective above as the satisfaction of the (W)ISS property from the input u_2. However, the weight function $\delta \in K^+$ involved in the WISS property is important: if the weight function $\delta \in K^+$ is bounded from above, then the closed-loop system will satisfy the Bounded-Input-Bounded-State property and the Converging-Input-Converging-State property. In general it will be better to be able to guarantee that the weight function $\delta \in K^+$ is as small as possible: for example, if the weight function $\delta \in K^+$ is given by $\delta(t) = \exp(-t)$, then clearly the state will converge to zero even if the absolute value of the input u_2 increases exponentially. Unfortunately, as one can imagine, this is not always achievable.

We next continue with some other significant questions.

Question 3 Do we have accurate measurements of the full state vector $x(t) = (x_1(t), x_2(t)) \in \Re^2$ at every time instant?

In many problems, it is impossible to measure accurately and continuously the state variables. Instead, we can measure only some components of the state vector or more generally a function of the state vector denoted by $y = h(x)$, where $h : \Re^n \to \Re^k$. In this case, the vector $y = h(x)$ is called the "measured output," or simply the output.

More specifically, for the chemostat problem, we can measure the concentration of the nutrient s and thus obtain x_2. However, in many problems it is impossible (or very expensive) to measure continuously the concentration of the microbial species. Therefore, in such cases the "adjustment law" $u_1(t) = k(x_1(t), x_2(t))$ for all $t \ge 0$ cannot be implemented.

There are two possible ways to address the above challenge.

(a) Try to find a static *output-feedback* law which is independent of x_1, i.e., try to find a function $k : \Re \to [-1, +\infty)$ which will guarantee that the closed-loop system (1.7) with $u_1 = k(x_2)$ satisfies the (W)ISS property from the input u_2.
(b) Instead of looking for a static output-feedback stabilizer, one can think of a dynamic output feedback stabilizer. In this case, we are looking for functions

$f : \Re^2 \to \Re$ and $k : \Re^2 \to [-1, +\infty)$ so that the "adjustment law" $u_1(t) = k(w(t), x_2(t))$ for all $t \geq 0$, where $w(t)$ is the solution of the system

$$\dot{w}(t) = f\big(w(t), x_2(t)\big) \quad w(t) \in \Re \tag{6.3}$$

can guarantee the (W)IOS property for the closed-loop system (1.7), (6.3) with $u_1 = k(w, x_2)$ from the input u_2 to the redefined output $Y = (x_1, x_2) \in \Re^2$. In this case the controller is a control system with input x_2 and output u_1 which has internal dynamics. The available information (the measurement of x_2) is fed to the controller which calculates the control action. Moreover, in this case the state vector of the closed-loop system (1.7), (6.3) with $u_1 = k(w, x_2)$ is $(x_1, x_2, w) \in \Re^3$, and the control objective is the (W)IOS property from the input u_2 for the new output $Y = (x_1, x_2) \in \Re^2$.

Question 4 Can we adjust "continuously" the value of the input $u_1(t) \in [-1, +\infty)$?

In many cases the control mechanism cannot adjust the value of the input continuously: instead, we can only adjust the value of the input every T time units. For the chemostat, the value of the input u_1 is adjusted in practice by opening a valve. It is possible to adjust the position of the valve by an automatic (analog) mechanism. However, this strategy is impractical when the feedback law is intricate (e.g., dynamic). When this happens, the control action is computed by means of a digital mechanism which determines the control action every T time units. Taking the nonlinear state-feedback $k(x_1, x_2)$ as an example, the closed-loop system will become

$$\begin{aligned}
\dot{x}_1(t) &= D^*\big(g\big(x_2(t)\big) - k\big(x_1(\tau_i), x_2(\tau_i)\big)\big) \\
\dot{x}_2(t) &= D^* \exp\big(-x_2(t)\big)\big[M\big(1 - p\big(x_2(t)\big)\exp\big(x_1(t)\big)\big) \\
&\quad - RMp\big(x_2(t)\big)g\big(x_2(t)\big)\exp\big(x_1(t)\big) + 1 - \exp\big(x_2(t)\big)\big] \\
&\quad + D^* \exp\big(-x_2(t)\big)\big[u_2(t) + k\big(x_1(\tau_i), x_2(\tau_i)\big)\big(M + 1 + u_2(t) \\
&\quad - \exp\big(x_2(t)\big)\big)\big] \\
\tau_{i+1} &= \tau_i + T
\end{aligned} \tag{6.4}$$

Notice that system (6.4) is a system with fixed sampling partition. The control input $u_1(t)$ is kept constant during two consecutive members of the sampling partition, i.e., is constant on $[\tau_i, \tau_{i+1})$. This is usually called "the application of the feedback law $u_1 = k(x_1, x_2)$ with zero-order hold."

Another issue that can lead to a closed-loop system with variable sampling partition is that the measurement of the state vector may not be available "continuously" but rather every T time units. For the chemostat, we can only measure the concentration of the nutrient s. It is possible that the measurement of the nutrient can be given in an irregular manner: two consecutive measurements can be given in times τ_i and τ_{i+1} which satisfy $\tau_{i+1} \leq \tau_i + T$ for a positive constant $T > 0$. In this case, we are looking for functions $f : \Re^2 \to \Re$ and $k : \Re^2 \to [-1, +\infty)$ such that the (W)IOS property from the input (u_2, u_3) holds for the control system

$$\dot{x}_1(t) = D^*\big(g\big(x_2(t)\big) - k\big(w(\tau_i), x_2(\tau_i)\big)\big)$$
$$\dot{x}_2(t) = D^* \exp\big(-x_2(t)\big)\big[M\big(1 - p\big(x_2(t)\big) \exp\big(x_1(t)\big)\big)$$
$$- RMp\big(x_2(t)\big)g\big(x_2(t)\big) \exp\big(x_1(t)\big) + 1 - \exp\big(x_2(t)\big)\big]$$
$$+ D^* \exp\big(-x_2(t)\big)\big[u_2(t) + k\big(w(\tau_i), x_2(\tau_i)\big)\big(M + 1 + u_2(t)$$
$$- \exp\big(x_2(t)\big)\big)\big]$$
$$\dot{w}(t) = f\big(w(t), x_2(\tau_i)\big)$$
$$\tau_{i+1} = \tau_i + T \exp\big(-u_3(\tau_i)\big) \tag{6.5}$$

where the input $u_3 \in \Re^+$ is introduced in order to represent the uncertainty of the sampling instances.

There are many other questions which can be significant for this problem. For example, the following questions are also important, and their answers may be crucial for a given control problem:

Question 5 Are there any delays and measurement error in the measurement process?

Question 6 Can we use the past values of the measured output?

From the above it is clear that the feedback stabilization problem is theoretically important and challenging, and practically relevant. Now, we are ready to state mathematically the Robust Output Feedback Stabilization Problem.

The Robust Output Feedback Stabilization Problems

Consider

(i) a deterministic control system $\Sigma_1 = (\mathcal{X}_1, Y_1, M_{S_2 \times U}, M_D, \tilde{\phi}_1, \pi_1, h)$ with outputs and the BIC property, for which $h : \Re^+ \times \mathcal{X}_1 \times U \to S_1 \subseteq \mathcal{Y}_1$, $S_2 \subseteq \mathcal{Y}_2$, $\mathcal{Y}_2$ being a normed linear space, and $0 \in \mathcal{X}_1$ is a robust equilibrium point from the input $(v_2, u) \in M_{S_2 \times U}$,

(ii) a normed linear space $\mathcal{Y}$ and a continuous map $H : \Re^+ \times \mathcal{X}_1 \times S_2 \times U \to \mathcal{Y}$ that maps bounded sets of $\Re^+ \times \mathcal{X}_1 \times S_2 \times U$ into bounded sets of $\mathcal{Y}$, with $H(t, 0, 0, 0) = 0$ for all $t \geq 0$, called the "stabilized output map".

For reasons that will become clear below, the output map $H : \Re^+ \times \mathcal{X}_1 \times S_2 \times U \to \mathcal{Y}$ will be called the "*stabilized output*," while the output $h : \Re^+ \times \mathcal{X}_1 \times S_2 \times U \to S_1 \subseteq \mathcal{Y}_1$ will be called the "*measured output*."

The *Robust Output Feedback Stabilization* problems consist of:

(1) (The existence problem) Is there a deterministic control system of the form $\Sigma_2 = (\mathcal{X}_2, Y_2, M_{S_1 \times U}, M_D, \tilde{\phi}_2, \pi_2, H_2)$, where $H_2 : \Re^+ \times \mathcal{X}_2 \times S_1 \times U \to S_2 \subseteq \mathcal{Y}_2$, with outputs and the BIC property for which $0 \in \mathcal{X}_2$ is an equilibrium point such that the *feedback connection* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, \tilde{H})$

of systems Σ_1 and Σ_2 with output $\tilde{H} : \Re^+ \times \mathcal{X}_1 \times \mathcal{X}_2 \times U \to \mathcal{Y}$ defined by $\tilde{H}(t, x_1, x_2, u) := H(t, x_1, H_2(t, x_2, h(t, x_1, u), u), u)$ is well defined and satisfies the IOS property from the input $u \in M_U$?

(2) (The feedback design problem) Design a deterministic control system $\Sigma_2 = (\mathcal{X}_2, Y_2, M_{S_1 \times U}, M_D, \tilde{\phi}_2, \pi_2, H_2)$, where $H_2 : \Re^+ \times \mathcal{X}_2 \times S_1 \times U \to S_2 \subseteq \mathcal{Y}_2$, with outputs and the BIC property for which $0 \in \mathcal{X}_2$ is an equilibrium point such that the *feedback connection* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, \tilde{H})$ of systems Σ_1 and Σ_2 with output $\tilde{H} : \Re^+ \times \mathcal{X}_1 \times \mathcal{X}_2 \times U \to \mathcal{Y}$ defined by $\tilde{H}(t, x_1, x_2, u) := H(t, x_1, H_2(t, x_2, h(t, x_1, u), u), u)$ is well defined and satisfies the IOS property from the input $u \in M_U$.

Particularly, the deterministic system $\Sigma_2 = (\mathcal{X}_2, Y_2, M_{S_1 \times U}, M_D, \tilde{\phi}_2, \pi_2, H_2)$ is called the "feedback stabilizer" or "the controller." If the deterministic system $\Sigma_2 = (\mathcal{X}_2, Y_2, M_{S_1 \times U}, M_D, \tilde{\phi}_2, \pi_2, H_2)$ is required to be a static map, then we refer to the corresponding feedback control stabilization problem as "static." If the deterministic system $\Sigma_2 = (\mathcal{X}_2, Y_2, M_{S_1 \times U}, M_D, \tilde{\phi}_2, \pi_2, H_2)$ is *not* required to be a static map, then we refer to the corresponding feedback control stabilization problem as "dynamic."

It should be clear that the controller exploits the output $h : \Re^+ \times \mathcal{X}_1 \times S_2 \times U \to S_1 \subseteq \mathcal{Y}_1$ as an input in order to produce the output $H_2 : \Re^+ \times \mathcal{X}_2 \times S_1 \times U \to S_2 \subseteq \mathcal{Y}_2$. In practice, the controller can only take information of certain measurements of the variables involved in the system $\Sigma_1 = (\mathcal{X}_1, Y_1, M_{S_2 \times U}, M_D, \tilde{\phi}_1, \pi_1, h)$. This is the reason that the output $h : \Re^+ \times \mathcal{X}_1 \times S_2 \times U \to S_1 \subseteq \mathcal{Y}_1$ is called the "*measured output*."

In general, the measured output and the regulated output do not coincide. However, special attention will be given to the problem where the measured output and the regulated output coincide to the state of $\Sigma_1 = (\mathcal{X}_1, Y_1, M_{S_2 \times U}, M_D, \tilde{\phi}_1, \pi_1, h)$. In this case, the output feedback stabilization problem reduces down to the state feedback stabilization problem.

6.3 Robustness with Respect to Errors

Every mathematical model of a physical system has a certain degree of accuracy. Therefore, modeling errors are always present to every description of a process. The effect of modeling errors can be detrimental for systems stability and performance and must always be taken into account in control systems design. For example, the chemostat stabilization problem described in the previous section is based on the mathematical model (1.7). The input u_2 represents the deviation of the inlet concentration of the nutrient from its nominal value. In other words, the input u_2 quantifies the effect of modeling errors with respect to the inlet concentration of the nutrient.

When feedback control is applied to a system, two additional sources of errors for the closed-loop system may arise, as stated previously:

(1) *Actuator errors*

The actuator errors represent the deviation between the applied control action and the nominal control action which is determined by the feedback law. Actuator errors are often present to control mechanisms and must not be neglected. For example, in order to quantify the effect of actuator errors for the closed-loop chemostat model (6.2), we can introduce an additional input $u_4 : \Re^+ \to \Re$ so that the actual control action $u_1(t)$ is determined by the equation

$$u_1(t) = \max\{-1, k(x(t)) + u_4(t)\} \tag{6.6}$$

Notice the nonlinear character of (6.6): it reflects the fact that the input $u_1(t)$ must always take values in the set $[-1, +\infty)$ or, equivalently, that the dilution rate D must be nonnegative. Therefore, the closed-loop system (1.7) with the feedback law (6.6) is described by the following nonlinear system:

$$\begin{aligned}
\dot{x}_1 &= D^*\big(g(x_2) - \max\{-1, k(x) + u_4\}\big)\\
\dot{x}_2 &= D^* \exp(-x_2)\big[M\big(1 - p(x_2)\exp(x_1)\big)\\
&\quad - RMp(x_2)g(x_2)\exp(x_1) + 1 - \exp(x_2)\big]\\
&\quad + D^* \exp(-x_2)\big[u_2 + \max\{-1, k(x) + u_4\}\big(M + 1 + u_2 - \exp(x_2)\big)\big]
\end{aligned} \tag{6.7}$$

(2) *Measurement errors*

The measurement error, or sensor noise, represents the deviation between the actual value of a measured output and the value which is provided by a measurement device. Unfortunately, measurement errors are always present to every measuring mechanism and can significantly affect the behavior of the closed-loop system. For example, in order to quantify the effect of measurement errors for the closed-loop chemostat model closed-loop system, i.e., (1.7), (6.3) with $u_1 = k(w, x_2)$, we can introduce an additional input $u_5 : \Re^+ \to \Re$ so that the measured value of the output $x_2(t)$ is corrupted by the additive error $u_5(t)$. Therefore, the closed-loop system (1.7), (6.3) with $u_1 = k(w, x_2)$ is described by the following nonlinear system:

$$\begin{aligned}
\dot{x}_1 &= D^*\big(g(x_2) - \max\{-1, k(w, x_2 + u_5) + u_4\}\big)\\
\dot{x}_2 &= D^* \exp(-x_2)\big[M\big(1 - p(x_2)\exp(x_1)\big)\\
&\quad - RMp(x_2)g(x_2)\exp(x_1) + 1 - \exp(x_2)\big]\\
&\quad + D^* \exp(-x_2)\big[u_2 + \max\{-1, k(w, x_2 + u_5) + u_4\}\\
&\quad \times \big(M + 1 + u_2 - \exp(x_2)\big)\big]\\
\dot{w} &= f(w, x_2 + u_5)
\end{aligned} \tag{6.8}$$

Again, the input $u_4 : \Re^+ \to \Re$ is introduced in order to quantify the effect of the actuator error. The control objective in this case is to find functions $f : \Re^2 \to \Re$ and $k : \Re^2 \to [-1, +\infty)$ and constants $L_1 \in (0, M]$, $L_i > 0$ $(i = 2, \dots, 6)$ so that system (6.8) satisfies the (W)IOS property from the inputs $(u_2, u_4, u_5) \in [-L_1, L_2] \times [-L_3, L_4] \times [-L_5, L_6]$ for the regulated output $Y = (x_1, x_2)$.

6.4 Analytical Solutions

In Chap. 2, we have already encountered the method of proving stability by means of analytical expressions of the solutions of a system. This method is particularly useful for linear systems. The usefulness of the method still holds for the feedback stabilization problem.

A linear autonomous control system described by ODEs takes the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ x &\in \Re^n, u \in \Re^m \end{aligned} \tag{6.9}$$

where $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ are constant matrices. For linear systems of the form (6.9), it can be shown that there exists a continuous stabilizing feedback that guarantees that $0 \in \Re^n$ is UGAS if and only if there exists a linear stabilizing feedback, i.e., there exists a matrix $K \in \Re^{m\times n}$ such that $(A + BK)$ is Hurwitz (see [64]). In the latter case, the pair (A, B) is said to be stabilizable. Conditions for the pair of matrices $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ to be stabilizable are given in the literature (see [64]).

The same comments are applicable to the linear autonomous discrete-time case

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ x(t) &\in \Re^n, u(t) \in \Re^m, t \in Z^+ \end{aligned} \tag{6.10}$$

where the objective is the existence of a matrix $K \in \Re^{m\times n}$ such that $(A + BK)$ has all its eigenvalues strictly inside the unit circle (see [64]).

A less obvious application of the method of analytical solutions is the control design for homogeneous systems described by ODEs. A control system described by ODEs of the form

$$\begin{aligned} \dot{x}(t) &= f\big(x(t), u(t)\big) \\ x(t) &\in \Re^n, u(t) \in \Re \end{aligned} \tag{6.11}$$

where $f : \Re^n \times \Re \to \Re^n$ is locally Lipschitz with $f(0, 0) = 0$, is characterized as homogeneous if there exist numbers $r_i > 0$ $(i = 1, \ldots, n)$, $p > 0$, and $q \in \Re$ such that the equality

$$f(x, u) = \delta_a^{-r}\big(a^{-q} f\big(\delta_a^r(x), a^p u\big)\big) \tag{6.12}$$

holds for all $(x, u) \in \Re^n \times \Re$ and $a > 0$, where

$$\begin{aligned} \delta_a^r(x) &= \big(a^{r_1}x_1, \ldots, a^{r_n}x_n\big)' \in \Re^n \\ \delta_a^{-r}(x) &= \big(a^{-r_1}x_1, \ldots, a^{-r_n}x_n\big)' \in \Re^n \end{aligned} \tag{6.13}$$

Although the solution $\phi(t, x_0; u)$ of system (6.11) at time $t \geq 0$ with initial condition $x(0) = x_0 \in \Re^n$ and corresponding to a measurable and locally essentially bounded input $u : \Re^+ \to \Re$ cannot in general be computed by means of elementary functions, it can be shown that the following equality holds:

$$\delta_a^r\big(\phi\big(a^q t, \delta_a^{-r}(x_0); a^{-p}u\big)\big) = \phi(t, x_0; u) \tag{6.14}$$

for all $a > 0$ and $x_0 \in \Re^n$, for all measurable and locally essentially bounded input $u : \Re^+ \to \Re$, and for all $t \geq 0$ for which $\phi(t, x_0; u)$ and $\phi(a^q t, \delta_a^{-r}(x_0); a^{-p}u)$ exist, where $a^{-p}u : \Re^+ \to \Re$ denotes the measurable and locally essentially bounded input that satisfies $(a^{-p}u)(t) = a^{-p}u(t)$ for almost all $t \geq 0$.

Define $N(x) := |x_1|^{1/r_1} + \cdots + |x_n|^{1/r_n}$ for $x = (x_1, \ldots, x_n) \in \Re^n$. Suppose that we know the solution $\phi(t, x_0; u)$ of system (6.11) at every time $t \geq 0$ with initial condition $x(0) = x_0 \in \Re^n$ satisfying $N(x_0) = 1$ and corresponding to every measurable and locally essentially bounded input $u : \Re^+ \to \Re$. Then, using formulae (6.14), we can compute the solution $\phi(t, x_0; u)$ of system (6.11) at every time $t \geq 0$ with initial condition $x(0) = x_0 \in \Re^n$ and corresponding to every measurable and locally essentially bounded input $u : \Re^+ \to \Re$. Indeed, formula (6.14) gives

$$\phi(t, x_0; u) = \delta^r_{N(x_0)}\big(\phi\big(N^q(x_0)t, \delta^{-r}_{N(x_0)}(x_0); N^{-p}(x_0)u\big)\big) \quad \text{for all } x_0 \neq 0$$

Since $y = \delta^{-r}_{N(x_0)}(x_0)$ satisfies $N(y) = 1$, $\phi(N^q(x_0)t, \delta^{-r}_{N(x_0)}(x_0); N^{-p}(x_0)u)$ is known by assumption.

Formula (6.14) is an essential feature of homogeneous systems which can be utilized for the solution of the feedback stabilization problem (see [5, 10, 16, 17] and references therein).

6.5 Transformation Methods: Feedback Linearization

One method for the design of stabilizing feedback for nonlinear autonomous systems described by ODEs is the method of feedback linearization (see [20, 52]). Since the method is analyzed rigorously in many other books, we will only discuss briefly the scope of the method and its relation to the transformation methods of proving stability.

The problem studied by feedback linearization methods can be described as follows. Consider the nonlinear control system

$$\begin{gathered}\dot{x} = f(x) + g(x)u \\ x \in \Re^n, u \in \Re^m\end{gathered} \tag{6.15}$$

where $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^{n\times m}$ are locally Lipschitz mappings with $f(0) = 0$. If we are in a position to find a diffeomorphism $\Phi : \Re^n \to \Re^n$ and a locally Lipschitz mapping $k : \Re^n \to \Re^m$ such that

$$\begin{gathered}D\Phi(x)(f(x) + g(x)k(x)) = A\Phi(x) \\ D\Phi(x)g(x) = B\end{gathered} \tag{6.16}$$

for all $x \in \Re^n$, where $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ are constant matrices, then the closed-loop system (6.15) with $u = k(x) + v$ under the change of coordinates $\Phi : \Re^n \to \Re^n$ will be the linear control system

$$\begin{gathered}\dot{z} = Az + Bv \\ z \in \Re^n, v \in \Re^m\end{gathered} \tag{6.17}$$

The design of a stabilizing feedback for the linear system (6.17) can be performed by using linear control theory as long as the transformed system (6.17) is stabilizable. Therefore, if the feedback law

$$v = Kz \tag{6.18}$$

guarantees that $0 \in \Re^n$ is GAS for the closed-loop system (6.17) with (6.18), then Proposition 2.1 in Chap. 2 implies that $0 \in \Re^n$ is GAS for the closed-loop system (6.15) with

$$u = k(x) + K\Phi^{-1}(x) \tag{6.19}$$

The theory of feedback linearization has dealt with necessary and sufficient conditions for the existence of a diffeomorphism $\Phi : \Re^n \to \Re^n$, a locally Lipschitz mapping $k : \Re^n \to \Re^m$, and a stabilizable pair of constant matrices $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ such that (6.16) holds.

In many cases we can guarantee the existence of a local diffeomorphism $\Phi : \Re^n \to \Re^n$, a locally Lipschitz mapping $k : \Re^n \to \Re^m$, and constant matrices $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ such that (6.16) holds locally. The theory of global feedback linearization is far from being complete. The interested reader should consult the rich and mathematically sophisticated feedback linearization literature for many deep and interesting results.

6.6 Lyapunov Functionals: The Control Lyapunov Functional

The feedback design methodology by means of a Control Lyapunov Functional is the method that corresponds to the method of proving stability by means of Lyapunov functionals. It should be noticed that the Control Lyapunov Functional methodology has been developed mostly for control systems described by ODEs and control systems described by RFDEs.

Exactly as the method of proving stability by means of Lyapunov functionals admits two different approaches (the discretization approach and the Lyapunov approach, see Chap. 2), the feedback design methodology by means of a Control Lyapunov functional admits two different approaches:

1. The Coron–Rosier approach
2. The Artstein–Sontag approach.

Next, we study each approach separately for systems described by ODEs.

6.6.1 Control Systems Described by ODEs: The Coron–Rosier Approach

This methodology is based on the following idea: given a Control Lyapunov Function (CLF), design a feedback law so that the difference of the values of the CLF evaluated along the solutions of the closed-loop system at time instances which differ by a constant quantity becomes negative. Notice that this methodology does not guarantee that the time derivative of the CLF along the solutions of the closed-loop system is negative definite.

Using this methodology in their pioneering work [12], Coron and Rosier showed that the existence of a time-independent CLF satisfying the "small-control" property is a sufficient condition for the existence of a continuous time-periodic stabilizing feedback for general nonaffine disturbance-free autonomous nonlinear control systems with $U = \Re^m$ and output Y being identically the state of the system (see also [10]). The result has been generalized in [37], where it was shown that the existence of a CLF is a necessary and sufficient condition for the existence of a continuous time-varying stabilizing feedback for systems of the form (1.3) with $U \subseteq \Re^m$ being a positive cone (not necessarily convex).

Here, we will only present results which allow the construction of locally Lipschitz feedback laws. We consider systems of the form (1.3) under the following hypotheses:

(HH) The mappings $f : \Re^+ \times D \times \Re^n \times U \to \Re^n$ and $H : \Re^+ \times \Re^n \to \Re^k$ are continuous, and for every bounded interval $I \subset \Re^+$ and every compact set $S \subset \Re^n \times U$, there exists a constant $L \geq 0$ such that

$$\big|f(t,d,x,u) - f(t,d,y,v)\big| \leq L|x-y| + L|u-v|$$
$$\forall (t,d) \in I \times D,\ (x,u) \in S,\ (y,v) \in S$$

Moreover, the set $D \subset \Re^l$ is compact, and U is a closed positive cone, i.e., $U \subseteq \Re^m$ is a closed set, and if $u \in U$, then $(\lambda u) \in U$ for all $\lambda \in [0,1]$.

Definition 6.1 We say that a function $V \in C^1(\Re^+ \times \Re^n; \Re^+)$ is an *Output Robust Control Lyapunov Function* (*ORCLF*) *for* (1.3) under hypotheses (H1–4) and (HH) if there exist functions $a_1, a_2 \in K_\infty$, $\mu, \beta \in K^+$, $\rho \in C^0(\Re^+; \Re^+)$ that are locally Lipschitz and positive definite, and $b \in C^0(\Re^+ \times (\Re^n \backslash \{0\}); \Re^+)$ such that

1. for every $(t,x) \in \Re^+ \times \Re^n$,

$$a_1\big(\big|H(t,x)\big| + \mu(t)|x|\big) \leq V(t,x) \leq a_2\big(\beta(t)|x|\big), \tag{6.20}$$

2. for every $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$,

$$\min_{u \in U, |u| \leq b(t,x)} \max_{d \in D} \left(\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,d,x,u) \right) \leq -\rho\big(V(t,x)\big). \tag{6.21}$$

For the case $H(t,x) := x$, the corresponding $V \in C^1(\Re^+ \times \Re^n; \Re^+)$ is called *State Robust Control Lyapunov Function* (*SRCLF*).

We are now in a position to state the main result that guarantees the existence of a stabilizing feedback law.

Theorem 6.1 *Consider system* (1.3) *under hypotheses* (H1–4) *and* (HH) *and assume that* (1.3) *admits an ORCLF which satisfies* (6.20), (6.21). *Then there exists* *a continuous mapping* $K : \Re^+ \times \Re^n \to U$ *with* $K(t,0) = 0$ *for all* $t \ge 0$, *which is continuously differentiable with respect to* $x \in \Re^n$ *on* $\Re^+ \times \Re^n$, *such that the closed-loop system* (1.3) *with* $u = K(t,x)$ *is RGAOS.*

If in addition we assume that there exist a function $a \in K_\infty$ *and a function* $q \in C^0(\Re^+ \times \Re^n; \Re^+)$ *satisfying*

$$0 < q(t,x) \le \min\left\{1, \frac{|x|}{2}\right\} \quad \forall (t,x) \in \Re^+ \times \left(\Re^n \backslash \{0\}\right) \tag{6.22}$$

such that, for all $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$,

$$\max\left\{b(\tau,y) : (\tau,y) \in \Re^+ \times \Re^n, |\tau - t| + |y - x| \le q(\tau,y)\right\} \le a\left(V(t,x)\right) \tag{6.23}$$

where $V \in C^1(\Re^+ \times \Re^n; \Re^+)$ *is the ORCLF for* (1.3), *and* $b \in C^0(\Re^+ \times (\Re^n \backslash \{0\}); \Re^+)$ *is the function involved in* (6.21), *then the closed-loop system* (1.3) *with* $u = K(t,x)$ *and output* $\tilde{Y} = K(t,x)$ *is RGAOS. Particularly, this implies that for all* $(t_0, x_0, d) \in \Re^+ \times \Re^n \times M_D$, *the control action* $[t_0, +\infty) \ni t \to u(t) = K(t, x(t)) \in U$ *is bounded and converges to zero as* $t \to +\infty$, *where* $x(\cdot)$ *denotes the solution of* (1.3) *with* $u = K(t,x)$, *initial condition* $x(t_0) = x_0 \in \Re^n$, *corresponding to* $d \in M_D$.

The proof of Theorem 6.1 is based on three lemmas below. Particularly, Lemma 6.1 is a preparatory result for the construction of the desired feedback stabilizer. It constitutes a time-varying extension of Lemma 2.7 in [12], but its constructive proof differs from the corresponding proof of the previously mentioned result.

Lemma 6.1 *Consider system* (1.3) *under hypotheses* (H1–4) *and* (HH) *and assume that* (1.3) *admits an ORCLF which satisfies* (6.20), (6.21). *Then, for every function* $q \in C^0(\Re^+ \times \Re^n; \Re^+)$ *satisfying* (6.22), *there exists a* C^1 *function* $k : [0,1] \times \Re^+ \times (\Re^n \backslash \{0\}) \to U$ *with*

$$k(0,t,x) = k(1,t,x) = 0 \tag{6.24}$$

$$\frac{\partial k}{\partial s}(0,t,x) = \frac{\partial k}{\partial t}(0,t,x) = 0; \qquad \frac{\partial k}{\partial x}(0,t,x) = 0 \tag{6.25}$$

$$\frac{\partial k}{\partial s}(1,t,x) = \frac{\partial k}{\partial t}(1,t,x) = 0; \qquad \frac{\partial k}{\partial x}(1,t,x) = 0 \tag{6.26}$$

for all $t \ge 0$, $x \in \Re^n \backslash \{0\}$ *and such that*

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) \int_0^1 f\left(t, d(s), x, k(s,t,x)\right) ds \le -\frac{1}{2}\rho\left(V(t,x)\right) \tag{6.27}$$

for all $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$, $d \in M_D$. *Moreover, the following inequality holds for all* $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$:

$$\max_{s \in [0,1]} \left|k(s,t,x)\right| \le \tilde{b}(t,x) \tag{6.28}$$

where, for all $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$,

$$\tilde{b}(t,x) := \max\left\{b(\tau,y) : (\tau,y) \in \Re^+ \times \Re^n, |\tau - t| + |y - x| \le q(\tau,y)\right\} \quad (6.29)$$

with $b(\cdot,\cdot)$ *the function involved in* (6.21).

Proof Let $q \in C^0(\Re^+ \times \Re^n; \Re^+)$ satisfying (6.22), let $\tilde{b} : \Re^+ \times (\Re^n \backslash \{0\}) \to \Re^+$ as defined by (6.29), which is locally bounded on $\Re^+ \times (\Re^n \backslash \{0\})$, and let $\varphi : \Re^+ \times (\Re^n \backslash \{0\}) \to [1, +\infty)$ be any smooth (C^∞) function such that, for all $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$,

$$\max_{u \in U, |u| \le \tilde{b}(t,x)} \max_{d \in D} \left(\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,d,x,u) \right) \le \varphi(t,x) \quad (6.30)$$

Moreover, let $\varepsilon : \Re^+ \times (\Re^n \backslash \{0\}) \to (0,1)$ be a smooth function such that

$$0 < \varepsilon(t,x) \le \frac{\rho(V(t,x))}{4(\rho(V(t,x)) + \varphi(t,x))} \quad \forall (t,x) \in \Re^+ \times \left(\Re^n \backslash \{0\}\right) \quad (6.31)$$

and define

$$\Psi(t,x,u) := \max_{d \in D} \left(\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,d,x,u) + \frac{3}{4}\rho\left(V(t,x)\right)\right)$$
$$\text{for all } (t,x,u) \in \Re^+ \times \Re^n \times U \quad (6.32)$$

$$\Psi(t,x,u) := \Psi(0,x,u) \quad \text{for all } (t,x,u) \in (-1,0) \times \Re^n \times U \quad (6.33)$$

By virtue of (6.21), the continuity of Ψ, and compactness of $D \subset \Re^l$, it follows that for each $(t,x) \in (-1,+\infty) \times (\Re^n \backslash \{0\})$, there exist $u = u(t,x) \in U$ with $|u| \le b(t,x)$ and $\delta = \delta(t,x) \in (0,1]$ with $\delta(t,x) \le q(t,x)$ such that

$$\Psi\left(\tau, y, u(t,x)\right) \le 0$$
$$\forall (\tau,y) \in \left\{(\tau,y) \in (-1,+\infty) \times \Re^n : |\tau - t| + |y - x| < \delta\right\} \quad (6.34)$$

Using (6.34) and standard partition of unity arguments, we can determine sequences $\{(t_i,x_i) \in (-1,+\infty) \times (\Re^n \backslash \{0\})\}_{i=1}^\infty$, $\{u_i \in U\}_{i=1}^\infty$, and $\{\delta_i \in (0,1]\}_{i=1}^\infty$ with $|u_i| \le b(\max(0,t_i), x_i)$ and $\delta_i = \delta(t_i,x_i) \le q(t_i,x_i)$ associated with a sequence of open sets $\{\Omega_i\}_{i=1}^\infty$ with

$$\Omega_i \subseteq \left\{(\tau,y) \in (-1,+\infty) \times \Re^n : |\tau - t_i| + |y - x_i| < \delta_i\right\} \quad (6.35)$$

forming a locally finite open covering of $(-1,+\infty) \times (\Re^n \backslash \{0\})$ and that

$$\Psi(\tau,y,u_i) \le 0 \quad \forall (\tau,y) \in \Omega_i \quad (6.36)$$

Also, a family of smooth functions $\{\theta_i\}_{i=1}^\infty$ with $\theta_i(t,x) \ge 0$ for all $(t,x) \in (-1,+\infty) \times (\Re^n \backslash \{0\})$ can be determined with

$$\operatorname{supp} \theta_i \subseteq \Omega_i \quad (6.37)$$

$$\sum_{i=1}^\infty \theta_i(t,x) = 1 \quad \forall (t,x) \in (-1,+\infty) \times \left(\Re^n \backslash \{0\}\right) \quad (6.38)$$

Next, define recursively the following mappings for each $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$:

$$\begin{aligned} &T_i(t,x) = T_{i-1}(t,x) + \theta_i(t,x) \quad i \geq 1 \\ &T_0(t,x) = 0 \\ &(t,x) \in \Re^+ \times \left(\Re^n \backslash \{0\}\right) \end{aligned} \tag{6.39}$$

Notice that definition (6.39) implies $T_n(x) = \sum_{i=1}^{n} \theta_i(t,x)$ for all $n \geq 1$. Since the open sets $\{\Omega_i\}_{i=1}^{\infty}$ form a locally finite open covering of $\Re^+ \times (\Re^n \backslash \{0\})$, it follows from (6.37) and (6.39) that for every $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$ there exists $m = m(t,x) \in \{1,2,3,\ldots\}$ such that

$$T_i(t,x) = 1 \quad \text{for } i \geq m \tag{6.40}$$

Define the index set

$$J(t,x) := \left\{ j \in \mathbb{N} : \theta_j(t,x) > 0 \right\} \tag{6.41}$$

which by virtue of (6.40) is a nonempty finite set. It follows from definitions (6.39) and (6.41) that

$$\bigcup_{j \in J(t,x)} \left[T_{j-1}(t,x), T_j(t,x)\right) = [0,1) \quad \forall (t,x) \in [0,1] \times \Re^+ \times \left(\Re^n \backslash \{0\}\right) \tag{6.42}$$

Let $h : \Re \to [0,1]$ be any smooth nondecreasing function with

$$h(s) = 0 \quad \text{for } s \leq 0 \quad \text{and} \quad h(s) = 1 \quad \text{for } s \geq 1 \tag{6.43}$$

and let

$$\begin{aligned} g_j(t,x) := {} & \frac{1}{2}\theta_j(t,x) + \frac{1}{2}\left(\varepsilon(t,x)2^{-j-1} - \theta_j(t,x)\right) \\ & \times h\left(\frac{\theta_j(t,x) - \varepsilon(t,x)2^{-j-2}}{\varepsilon(t,x)2^{-j-2}}\right) \quad j = 1,2,\ldots \end{aligned} \tag{6.44}$$

where $\varepsilon(\cdot,\cdot)$ is the function defined by (6.31). According to (6.43) and (6.44), it holds

$$\min\left\{\varepsilon(t,x)2^{-j-2}, \frac{1}{2}\theta_j(t,x)\right\} \leq g_j(t,x) \leq \min\left\{\varepsilon(t,x)2^{-j-2}, \theta_j(t,x)\right\} \tag{6.45}$$

$$g_j(t,x) = \varepsilon(t,x)2^{-j-2} \quad \text{for } \theta_j(t,x) \geq \varepsilon(t,x)2^{-j-1} \tag{6.46}$$

We define the following map $[0,1] \times \Re^+ \times (\Re^n \backslash \{0\}) \ni (s,t,x) \to k(s,t,x) \in \Re^m$:

$$\begin{aligned} k(s,t,x) = {} & u_j\left(\frac{g_j(t,x)}{\varepsilon(t,x)2^{-j-2}}\right)^2 h\left(\frac{s - T_{j-1}(t,x) - \frac{1}{5}g_j(t,x)}{\frac{1}{5}g_j(t,x)}\right) \\ & \text{for } s \in \left[T_{j-1}(t,x), T_{j-1}(t,x) + \frac{1}{2}\theta_j(t,x)\right), \; j \in J(t,x) \end{aligned} \tag{6.47}$$

$$\begin{aligned} k(s,t,x) = {} & u_j\left(\frac{g_j(t,x)}{\varepsilon(t,x)2^{-j-2}}\right)^2 h\left(\frac{T_j(t,x) - \frac{1}{5}g_j(t,x) - s}{\frac{1}{5}g_j(t,x)}\right) \\ & \text{for } s \in \left[T_j(t,x) - \frac{1}{2}\theta_j(t,x), T_j(t,x)\right), \; j \in J(t,x) \end{aligned} \tag{6.48}$$

$$k(1,t,x) = 0 \tag{6.49}$$

Notice that, because of (6.42), $k(\cdot,\cdot,\cdot)$ is well defined for all $(s,t,x) \in [0,1] \times \Re^+ \times (\Re^n \backslash \{0\})$. Furthermore, according to definitions (6.47), (6.48), and (6.49), hypothesis (H2) guarantees that $k(\cdot,\cdot,\cdot)$ takes values in $U \subseteq \Re^m$ and is continuously differentiable on the region $(\bigcup_{j\in J(t,x)}(T_{j-1}(t,x), T_j(t,x))) \times \Re^+ \times (\Re^n \backslash \{0\})$. Furthermore, it holds that

$$\frac{\partial k}{\partial s}(s,t,x) \to 0 \qquad \frac{\partial k}{\partial t}(s,t,x) \to 0 \qquad \frac{\partial k}{\partial x}(s,t,x) \to 0$$
$$\text{as } s \to T_j(t,x) \text{ for all } j = 0,1,2,\ldots \tag{6.50}$$

Next, we show that $k(\cdot,\cdot,\cdot)$ is continuously differentiable on the whole region $[0,1] \times \Re^+ \times (\Re^n \backslash \{0\})$ and simultaneously that (6.24), (6.25), and (6.26) are fulfilled. We distinguish the following cases:

Case 1: Let $s \in (0,1)$, $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$, and suppose that there exists a positive integer p with $s = T_p(t,x)$. Then, there exist positive integers m, l with $l \le p \le m$ such that

$$\theta_{m+1}(t,x) > 0 \qquad \theta_l(t,x) > 0 \tag{6.51}$$

$$s = T_m(t,x) = \cdots = T_l(t,x) > 0 \tag{6.52}$$

Equality (6.52), in conjunction with definition (6.39), means that

$$\theta_m(t,x) = \cdots = \theta_{l+1}(t,x) = 0 \quad \text{if } m \ge l+1 \tag{6.53}$$

Notice that definition (6.47) and (6.51) imply that in our case,

$$k(s,t,x) = 0 \tag{6.54}$$

By taking into account continuity of the mappings g_l, g_{m+1}, T_l, T_m and (6.45), it follows that there exists $\delta > 0$ such that

$$s' \in \left(T_l(\tau,y) - \frac{1}{5}g_l(\tau,y),\, T_m(\tau,y) + \frac{1}{5}g_{m+1}(\tau,y)\right)$$
$$\forall \left(s',\tau,y\right) \in [0,1] \times \Re^+ \times \left(\Re^n \backslash \{0\}\right) \text{ with } \left|s'-s\right| + |\tau - t| + |y - x| < \delta \tag{6.55}$$

By virtue of definitions (6.47), (6.48), (6.54), and (6.55), it follows that for every $(s',\tau,y) \in [0,1] \times \Re^+ \times (\Re^n \backslash \{0\})$ with $|s'-s| + |\tau - t| + |y-x| < \delta$,

$$\left|k\left(s',\tau,y\right) - k(s,t,x)\right| \le \max_{v=l+1,\ldots,m} |u_v| \left(\frac{g_v(\tau,y)}{\varepsilon(\tau,y)2^{-v-2}}\right)^2 \quad \text{if } m \ge l+1 \tag{6.56}$$

$$k\left(s',\tau,y\right) = k(s,t,x) \quad \text{if } m = l \tag{6.57}$$

If $m \ge l+1$, then by (6.45) and (6.53) we also get $g_v(t,x) = 0$ for $v = l+1,\ldots,m$; hence, since the mappings g_v are continuously differentiable, there exists a constant $L > 0$ such that

$$\frac{\max_{v=l+1,\ldots,m} g_v(\tau,y)}{\varepsilon(\tau,y)} \le L|\tau - t| + L|y-x|$$
$$\forall (\tau,y) \in \Re^+ \times \Re^n \backslash \{0\} \text{ with } |\tau - t| + |y - x| < \delta \tag{6.58}$$

It follows from (6.56), (6.57), and (6.58) that

$$\left|k\left(s', \tau, y\right) - k(s, t, x)\right| \le L'\left(|\tau - t|^2 + |y - x|^2\right) \tag{6.59}$$

for some constant $L' > 0$ and for $|s' - s| + |\tau - t| + |y - x| < \delta$. We conclude from (6.59) that the derivatives of $k(\cdot, \cdot, \cdot)$ exist for $s = T_p(t, x)$ and that $\frac{\partial k}{\partial s}(s, t, x) = \frac{\partial k}{\partial t}(s, t, x) = 0$ and $\frac{\partial k}{\partial x}(s, t, x) = 0$ for $s = T_p(t, x)$. The latter, in conjunction with (6.50), implies that $k(\cdot, \cdot, \cdot)$ is continuously differentiable in a neighborhood of (s, t, x) with $s = T_p(t, x) \in (0, 1)$.

Case 2: Let $s = 0$, $(t, x) \in \Re^+ \times (\Re^n \backslash \{0\})$, and suppose that there exists an integer $p \ge 0$ with $s = T_p(t, x) = 0$. Clearly, there exists an integer $m \ge p$ such that

$$\theta_{m+1}(t, x) > 0 \tag{6.60}$$

$$s = T_m(t, x) = \cdots = T_0(t, x) = 0 \tag{6.61}$$

(note again that equality (6.61) means that $\theta_m(t, x) = \cdots = \theta_1(t, x) = 0$ for the case $m > 0$). By virtue of definition (6.47), it holds that $k(s, t, x) = 0$. The continuity of the mappings T_m and g_{m+1} implies that there exists $\delta > 0$ such that $s' \in [0, T_m(\tau, y) + \frac{1}{5} g_{m+1}(\tau, y))$ for all $(s', \tau, y) \in [0, 1] \times \Re^+ \times (\Re^n \backslash \{0\})$ with $|s' - s| + |\tau - t| + |y - x| < \delta$. Then as in Case 1, it follows from (6.47), (6.48), (6.49) that

$$\left|k\left(s', \tau, y\right) - k(s, t, x)\right| \le \max_{v=1,\ldots,m} |u_v| \left(\frac{g_v(\tau, y)}{\varepsilon(\tau, y) 2^{-v-2}}\right)^2 \quad \text{if } m > 0 \tag{6.62}$$

$$k\left(s', \tau, y\right) = k(s, t, x) \quad \text{if } m = 0 \tag{6.63}$$

for every $|s' - s| + |\tau - t| + |y - x| < \delta$, from which we get the desired conclusion, namely, that $k(\cdot, \cdot, \cdot)$ is continuously differentiable in a neighborhood of $(0, t, x)$ and that (6.25) holds.

Case 3: Let $s = 1$, $(t, x) \in \Re^+ \times (\Re^n \backslash \{0\})$, and let p be a positive integer with $s = T_p(t, x)$. Let $\{\Omega_i\}_{i=1}^{\infty}$ be the locally finite open covering of $(-1, +\infty) \times (\Re^n \backslash \{0\})$ and the associated sequence of functions in such a way that (6.35), (6.36), (6.37), and (6.38) hold. Let $N \subset (-1, +\infty) \times (\Re^n \backslash \{0\})$ be a neighborhood containing (t, x) which intersects only a finite number of the open sets $\{\Omega_i\}_{i=1}^{\infty}$ (see [18]). Consequently, by (6.38) there exists an integer $m > 1$ such that $N \cap \Omega_i = \emptyset$ for all $i > m$ and $\theta_i(\tau, y) = 0$ for all $i > m$ and $(\tau, y) \in N$. Clearly, there exists $l \in \{1, \ldots, m\}$ with

$$\theta_l(t, x) > 0 \tag{6.64}$$

$$s = T_l(t, x) = \cdots = T_m(t, x) = 1 \tag{6.65}$$

Without loss of generality, we may assume that $m > l$. By virtue of definition (6.49), we have $k(1, t, x) = 0$. This, together with the continuity of the mappings T_l and g_l, asserts the existence of a constant $\delta > 0$ such that

$$s' \in \left(T_l(\tau, y) - \frac{1}{5} g_l(\tau, y), 1\right] \quad \text{and} \quad (\tau, y) \in N$$
$$\forall \big(s', \tau, y\big) \in [0, 1] \times \Re^+ \times \big(\Re^n \backslash \{0\}\big) \text{ with } \big|s' - 1\big| + |\tau - t| + |y - x| < \delta \tag{6.66}$$

Using (6.47), (6.48), (6.49), and (6.66), we get

$$\big|k\big(s', \tau, y\big) - k(1, t, x)\big| \le \max_{v=l+1,\dots,m} |u_v| \left(\frac{g_v(\tau, y)}{\varepsilon(\tau, y)2^{-v-2}}\right)^2$$

from which it follows that (6.59) holds for all $|s' - 1| + |\tau - t| + |y - x| < \delta$ and for some constant $L' > 0$. This implies that the derivatives of $k(\cdot, \cdot, \cdot)$ exist for $s = 1$, and particularly, (6.26) holds. The latter, in conjunction with (6.50), implies that $k(\cdot, \cdot, \cdot)$ is continuously differentiable in a neighborhood of $(1, t, x)$.

We next establish (6.28). By virtue of (6.43), (6.45), and definition (6.47)–(6.49), we have $\max_{s\in[0,1]} |k(s, t, x)| \le \max_{j\in J(t,x)} |u_j|$, $J(t, x)$ being the index set defined by (6.41). For every $j \in J(t, x)$, there exist $(t_j, x_j) \in (-1, +\infty) \times (\Re^n \backslash \{0\})$ with

$$|u_j| \le b\big(\max(0, t_j), x_j\big) \tag{6.67}$$

for which $(t, x) \in \Omega_j$ and in such a way that (6.35) holds with $i = j$. The choice $\delta_j = \delta(t_j, x_j) \le q(t_j, x_j)$, in conjunction with (6.67) and definition (6.29) of $\tilde{b}(\cdot, \cdot)$, implies (6.28). Finally, we establish (6.27). Notice that by (6.41), (6.43), (6.46), (6.47), and (6.48), for any $(t, x) \in \Re^+ \times \Re^n \backslash \{0\}$ and integer $j \in J(t, x)$, it holds:

$$k(s, t, x) = u_j \quad \forall s \in \left[T_{j-1}(t, x) + \frac{2}{5} g_j(t, x), T_j(t, x) - \frac{2}{5} g_j(t, x)\right]$$
$$\text{when } \theta_j(t, x) \ge \varepsilon(t, x)2^{-j-1} \tag{6.68}$$

Hence, the set $I_{(t,x)} := \{s \in [0, 1] : k(s, t, x) \ne u_j, j \in J(t, x)\}$ has Lebesgue measure, say $|I_{(t,x)}|$, satisfying

$$|I_{(t,x)}| \le \sum_{j\in J(t,x)} \varepsilon(t, x)2^{-j-1} \le \varepsilon(t, x) \tag{6.69}$$

Then, for any $d \in M_D$, it follows by virtue of (6.32), (6.36), and (6.69) that

$$\begin{aligned}
&\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) \int_0^1 f\big(t, d(s), x, k(s, t, x)\big)\, ds \\
&\quad \le -\frac{3}{4}\big(1 - \varepsilon(t, x)\big)\rho\big(V(t, x)\big) + \varepsilon(t, x) \max_{\substack{|u|\le\tilde{b}(t,x)\\ u\in U}} \max_{d\in D} \frac{\partial V}{\partial t}(t, x) \\
&\qquad + \frac{\partial V}{\partial x}(t, x) f(t, d, x, u)
\end{aligned}$$

Inequalities (6.30) and (6.31), in conjunction with the above inequality, imply (6.27), and the proof is complete. □

The next lemmas constitute key results of the rest analysis and generalize Lemmas 2.8 and 2.9 in [12]. Their proofs are based on certain generalizations of the technique employed in [12].

Lemma 6.2 *Consider system* (1.3) *under the same hypotheses as those imposed in Lemma* 6.1. *For every function* $q \in C^0(\Re^+ \times \Re^n; \Re^+)$ *satisfying* (6.22) *and for every pair of sets* $r = \{r_i : i \in Z\}$ *and* $a = \{a_i : i \in Z\}$ *with* $r_i > 0$ *and* $a_i > 0$,

$$r_i + 2a_i < r_{i+1} - 2a_{i+1} \quad \textit{for all } i \in Z \tag{6.70}$$

$$\lim_{i \to +\infty} r_i = +\infty \qquad \lim_{i \to -\infty} r_i = 0 \tag{6.71}$$

there exists a continuous mapping $k_{r,a} : \Re^+ \times (\Re^n \backslash \{0\}) \to U$, *continuously differentiable with respect to* $x \in \Re^n \backslash \{0\}$, *with*

$$k_{r,a}(j, x) = 0 \quad \textit{and} \quad \frac{\partial k_{r,a}}{\partial x}(j, x) = 0 \quad \textit{for all } (x, j) \in \big(\Re^n \backslash \{0\}\big) \times Z^+ \tag{6.72}$$

$$\big|k_{r,a}(t, x)\big| \le \tilde{b}(t, x) \quad \forall (t, x) \in \Re^+ \times \big(\Re^n \backslash \{0\}\big) \tag{6.73}$$

where $\tilde{b}(\cdot, \cdot)$ *is defined by* (6.29), *and such that the following property holds for all* $(t_0, x_0, d, i) \in \Re^+ \times (\Re^n \backslash \{0\}) \times M_D \times Z$:

$$V(t_0, x_0) \in [r_{i-1}, r_i] \quad \Rightarrow \quad V\big(t, x(t, t_0, x_0; d)\big) \le r_i + \frac{5}{2} a_i$$
$$\textit{for all } t \in \big[t_0, \min\big([t_0] + 1, t_{\max}\big)\big) \tag{6.74}$$

where $x(\cdot, t_0, x_0; d)$ *denotes the unique solution of*

$$\dot{x} = f\big(t, d, x, k_{r,a}(t, x)\big) \quad (t, x) \in \Re^+ \times \big(\Re^n \backslash \{0\}\big) \tag{6.75}$$

with initial condition $x(t_0) = x_0 \in \Re^n \backslash \{0\}$, *corresponding to* $d \in M_D$, *and* $t_{\max} := t_{\max}(t_0, x_0, d) > t_0$ *denotes its maximal existence time. Moreover, for each* $(x_0, j, i) \in (\Re^n \backslash \{0\}) \times Z^+ \times Z$, *there exists a positive integer* $N \ge 2$ *such that*

$$V(j, x_0) \le r_i - 2a_i$$
$$\Rightarrow \quad V\left(j + \frac{s}{N}, x\left(j + \frac{s}{N}, j, x_0 d\right)\right) \le \max\left(r_{i-1} + 2a_{i-1}, V(j, x_0) - \frac{s}{N}\mu_i\right)$$
$$\textit{for all } d \in M_D \textit{ and } s \in \{0, 1, \ldots, N\} \textit{ with } j + \frac{s}{N} < t_{\max} \tag{6.76}$$

where

$$\mu_i := \frac{1}{4} \min\big\{\rho(s) : s \in [r_{i-1}, r_i]\big\} \tag{6.77}$$

Proof Let $q \in C^0(\Re^+ \times \Re^n; \Re^+)$ be a function satisfying (6.22), and $k : [0, 1] \times \Re^+ \times (\Re^n \backslash \{0\}) \to U$ a C^1 function which satisfies (6.24), (6.25), (6.26), (6.27), and (6.28) and whose existence is guaranteed by Lemma 6.1. For $i \in Z$ and $j \in Z^+$, define

$$\Omega_{i,j} = \big\{(t, x) \in [j, j + 1] \times \Re^n : V(t, x) \in [r_{i-1}, r_i]\big\} \tag{6.78}$$

$$\rho_i := \min(a_{i-2}, a_{i-1}, a_i, a_{i+1}) \tag{6.79}$$

and select $\delta_{i,j} > 0$ satisfying

$$\begin{aligned}
&\left|\frac{\partial V}{\partial t}(t,x) - \frac{\partial V}{\partial t}(t_0,x_0)\right| \\
&\qquad + \left|\frac{\partial V}{\partial x}(t,x) - \frac{\partial V}{\partial x}(t_0,x_0)\right| \max\left\{\left|f(t,x,d,u)\right| : d \in D, u \in U, |u| \le \tilde{b}(t,x)\right\} \\
&\qquad + \left|\frac{\partial V}{\partial x}(t_0,x_0)\right| \max\left\{\left|f\left(t,d,x,k(s,t_0,x)\right)\right. \right. \\
&\qquad \left.\left. - f\left(t_0,d,x_0,k(s,t_0,x_0)\right)\right| : s \in [0,1], d \in D\right\} \\
&\quad \le \frac{1}{4}\rho\left(V(t_0,x_0)\right) \\
&\quad \forall (t_0,x_0) \in \Omega_{i,j}, \forall (t,x) \in \Re^+ \times \left(\Re^n \backslash \{0\}\right) \text{ with } t \in [t_0, t_0+\delta_{i,j}], |x-x_0| \le \delta_{i,j}
\end{aligned} \tag{6.80}$$

Also, let $N_{i,j} \in Z^+$ with $N_{i,j} \ge 2$ be a family of integers which satisfies the following inequalities:

$$\begin{aligned}
&4\max\left\{\left|\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,d,x,u)\right| : t \in [j, j+2], V(t,x) \in [r_{i-3}, r_{i+2}],\right. \\
&\qquad \left. d \in D, u \in U, |u| \le \tilde{b}(t,x)\right\} \le \rho_i N_{i,j}
\end{aligned} \tag{6.81}$$

$$\begin{aligned}
&2 + 2\max\left\{\left|f(t,d,x,u)\right| : t \in [j, j+2], V(t,x) \in [r_{i-3}, r_{i+2}],\right. \\
&\qquad \left. d \in D, u \in U, |u| \le \tilde{b}(t,x)\right\} \le \delta_{i,j} N_{i,j}
\end{aligned} \tag{6.82}$$

Consider next a smooth nondecreasing function $h : \Re \to [0,1]$ with $h(s) = 0$ for $s \le 0$ and $h(s) = 1$ for $s \ge 1$ and define the desired $k_{r,a} : \Re^+ \times (\Re^n \backslash \{0\}) \to U$ as follows:

$$\begin{aligned}
&k_{r,a}(t,x) \\
&\quad := \begin{cases} h\left(2\frac{V(t,x)-r_{i-1}}{\min(a_{i-1},a_i)}\right)k\left(N_{i,j}(t-j)-l, j+\frac{l}{N_{i,j}}, x\right) & V(t,x) \in \left[r_{i-1}, \frac{r_i+r_{i-1}}{2}\right) \\ h\left(2\frac{r_i-V(t,x)}{\min(a_{i-1},a_i)}\right)k\left(N_{i,j}(t-j)-l, j+\frac{l}{N_{i,j}}, x\right) & V(t,x) \in \left[\frac{r_i+r_{i-1}}{2}, r_i\right) \end{cases} \\
&\quad (t,x) \in \Omega_{i,j}, t \in \left[j+\frac{l}{N_{i,j}}, j+\frac{l+1}{N_{i,j}}\right) \\
&\quad \text{for some } l \in \{0,1,\dots,N_{i,j}-1\}
\end{aligned} \tag{6.83}$$

Obviously, (6.73) is a consequence of (6.28), (6.29), and (6.83). Moreover, by taking into account (6.24), (6.25), (6.26), and (6.70) it follows that $k_{r,a}(\cdot,\cdot)$ above is continuous, continuously differentiable with respect to $x \in \Re^n \backslash \{0\}$, and satisfies

$$k_{r,a}(j,x) = 0 \qquad \frac{\partial k_{r,a}}{\partial x}(j,x) = 0 \quad \forall (x,j) \in \left(\Re^n \backslash \{0\}\right) \times Z^+ \tag{6.84}$$

Let $(x_0,d) \in (\Re^n \backslash \{0\}) \times M_D$ and $t_0 \in [j+\frac{l}{N_{i,j}}, j+\frac{l+1}{N_{i,j}})$ for some $l \in \{0,1,\dots,N_{i,j}-1\}$ with

$$V(t_0,x_0) \in [r_{i-2}, r_{i+1}] \tag{6.85}$$

Then, by (6.81), (6.82), and (6.85), it can be easily established that, for all $t \in [t_0, j + \frac{l+1}{N_{i,j}}]$, it holds

$$t - t_0 + \left|x(t, t_0, x_0; d) - x_0\right| \le \delta_{i,j} \tag{6.86}$$

$$\begin{aligned} &V(t_0, x_0) - \frac{1}{2}\min(a_{i-1}, a_i) \\ &\quad \le V\big(t, x(t, t_0, x_0; d)\big) \le V(t_0, x_0) + \frac{1}{2}\min(a_{i-1}, a_i) \end{aligned} \tag{6.87}$$

In fact, suppose on the contrary that there exist $(x_0, d) \in (\Re^n \backslash \{0\}) \times M_D$, $t_0 \in [j + \frac{l}{N_{i,j}}, j + \frac{l+1}{N_{i,j}})$ for some $l \in \{0, 1, \dots, N_{i,j} - 1\}$ satisfying (6.85) and $\bar{t} \in [t_0, j + \frac{l+1}{N_{i,j}}]$ such that either (6.86) or (6.87) does not hold and consider the closed set

$$\begin{aligned} A := \Bigg\{\tau \in \left[t_0, j + \frac{l+1}{N_{i,j}}\right] : \max\Bigg\{ &\frac{2|V(\tau, x(\tau, t_0, x_0; d)) - V(t_0, x_0)|}{\min(a_{i-2}, a_{i-1}, a_i, a_{i+1})}, \\ &\frac{\tau - t_0 + |x(\tau, t_0, x_0; d) - x_0|}{\delta_{i,j}}\Bigg\} \ge 1\Bigg\} \end{aligned}$$

Notice that, since $\bar{t} \in A$, the set A is nonempty. Let $t_1 := \min A$. Clearly, since $t_0 \notin A$, it holds that $t_1 > t_0$. Definition of the set A above, (6.70), and (6.85) imply that $V(\tau, x(\tau, t_0, x_0; d)) \in [r_{i-3}, r_{i+2}]$ for every $\tau \in [t_0, t_1)$. It follows from (6.73), (6.81), and (6.82) that

$$\begin{aligned} &\left|\frac{d}{d\tau} V\big(\tau, x(\tau, t_0, x_0; d)\big)\right| \le \frac{1}{4}\rho_i N_{i,j} \quad \text{and} \\ &2 + 2\left|\dot{x}(\tau)\right| \le \delta_{i,j} N_{i,j} \quad \text{a.e.} \quad \text{for } \tau \in [t_0, t_1) \end{aligned}$$

which, in conjunction with definition (6.79) and the fact that $\tau - t_0 \le \frac{1}{N_{i,j}}$, implies that, for all $\tau \in [t_0, t_1]$, we would have

$$\begin{aligned} &\left|V\big(\tau, x(\tau, t_0, x_0; d)\big) - V(t_0, x_0)\right| \\ &\quad \le \int_{t_0}^{\tau} \left|\frac{d}{ds} V\big(s, x(s, t_0, x_0; d)\big)\right| ds \le \frac{1}{4}\min(a_{i-2}, a_{i-1}, a_i, a_{i+1}) \\ &\quad \tau - t_0 + \left|x(\tau, t_0, x_0; d) - x_0\right| \le \tau - t_0 + \int_{t_0}^{\tau} \left|\dot{x}(s)\right| ds \le \frac{1}{2}\delta_{i,j} \end{aligned} \tag{6.88}$$

The previous inequalities for $\tau = t_1$ are in contradiction with the fact that $t_1 \in A$.

In order to establish properties (6.74) and (6.76), we first need the following properties: □

Property 6.1 *Let $d \in M_D$, and let*

$$t_0 = j + \frac{l}{N_{i,j}} \quad l \in \{0, 1, \dots, N_{i,j} - 1\} \tag{6.89}$$

$$V(t_0, x_0) \in [r_{i-1} + a_{i-1}, r_i - 2a_i] \tag{6.90}$$

Then the following inequality is fulfilled:

$$0 < V\left(j + \frac{l+1}{N_{i,j}}, x\left(j + \frac{l+1}{N_{i,j}}, j + \frac{l}{N_{i,j}}, x_0; d\right)\right)$$
$$\leq V\left(j + \frac{l}{N_{i,j}}, x_0\right) - \frac{1}{4N_{i,j}}\rho\left(V\left(j + \frac{l}{N_{i,j}}, x_0\right)\right) \tag{6.91}$$

Proof Using (6.87) and definition (6.83), it follows that

$$k_{r,a}\big(t, x(t, t_0, x_0; d)\big) = k\left(N_{i,j}(t - j) - l, j + \frac{l}{N_{i,j}}, x(t, t_0, x_0; d)\right)$$
$$\forall t \in \left[j + \frac{l}{N_{i,j}}, j + \frac{l+1}{N_{i,j}}\right] \tag{6.92}$$

For convenience, let us denote here $h := \frac{1}{N_{i,j}}$, $x(\cdot) = x(\cdot, t_0, x_0; d)$ and $\tilde{d}(t) := d(t_0 + ht)$ (notice that $\tilde{d} \in M_D$). From (6.92) we have

$$\begin{aligned}
&V\big(t_0 + h, x(t_0 + h)\big) - V(t_0, x_0) \\
&= \int_{t_0}^{t_0+h} \left[\frac{\partial V}{\partial t}\big(\tau, x(\tau)\big)\right. \\
&\quad \left. + \frac{\partial V}{\partial x}\big(\tau, x(\tau)\big) f\left(\tau, d(\tau), x(\tau), k\left(\frac{\tau - t_0}{h}, t_0, x(\tau)\right)\right)\right] d\tau \\
&= h\int_0^1 \left[\frac{\partial V}{\partial t}\big(t_0 + hs, x(t_0 + hs)\big) + \frac{\partial V}{\partial x}\big(t_0 + hs, x(t_0 + hs)\big)\right. \\
&\quad \left. \times f\big(t_0 + hs, d(t_0 + hs), x(t_0 + hs), k\big(s, t_0, x(t_0 + hs)\big)\big)\right] ds \\
&= h\int_0^1 \left[\frac{\partial V}{\partial t}(t_0, x_0) + \frac{\partial V}{\partial x}(t_0, x_0) f\big(t_0, \tilde{d}(s), x_0, k(s, t_0, x_0)\big)\right] ds \\
&\quad + h\int_0^1 \left[\frac{\partial V}{\partial t}\big(t_0 + hs, x(t_0 + hs)\big) - \frac{\partial V}{\partial t}(t_0, x_0)\right] ds \\
&\quad + h\int_0^1 \left[\frac{\partial V}{\partial x}\big(t_0 + hs, x(t_0 + hs)\big) - \frac{\partial V}{\partial x}(t_0, x_0)\right] \\
&\quad \times f\big(t_0 + hs, \tilde{d}(s), x(t_0 + hs), k\big(s, t_0, x(t_0 + hs)\big)\big) ds \\
&\quad + h\int_0^1 \frac{\partial V}{\partial x}(t_0, x_0)\big[f\big(t_0 + hs, \tilde{d}(s), x(t_0 + hs), k\big(s, t_0, x(t_0 + hs)\big)\big) \\
&\quad - f\big(t_0, \tilde{d}(s), x_0, k(s, t_0, x_0)\big)\big] ds
\end{aligned} \tag{6.93}$$

Using (6.27), (6.28), (6.80), (6.86), (6.87), (6.89), (6.90), and (6.93), we get the desired (6.91), and the proof of Property 6.1 is complete. □

The next property is a consequence of Property 6.1:

Property 6.2 *Suppose that*

$$0 < V\left(j + \frac{l}{N_{i,j}}, x_0\right) \leq r_i - 2a_i \quad \textit{for some } l \in \{0, 1, \ldots, N_{i,j} - 1\} \tag{6.94}$$

and assume that the solution of (6.75) *with initial condition* $x(j + \frac{l}{N_{i,j}}) = x_0 \in \Re^n \backslash \{0\}$, *corresponding to some* $d \in M_D$, *exists for* $t \in [j + \frac{l}{N_{i,j}}, j + \frac{l+1}{N_{i,j}}]$. *Then*

$$\begin{aligned} 0 &< V\left(j + \frac{l+1}{N_{i,j}}, x\left(j + \frac{l+1}{N_{i,j}}, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \\ &\leq \max\left\{r_{i-1} + 2a_{i-1}, V\left(j + \frac{l}{N_{i,j}}, x_0\right) - \frac{1}{N_{i,j}}\mu_i\right\} \end{aligned} \tag{6.95}$$

where $\mu_i > 0$ *is defined by* (6.77).

Proof Obviously, the desired (6.95) is a consequence of (6.91), provided that (6.90) is fulfilled. Consider the remaining case

$$0 < V\left(j + \frac{l}{N_{i,j}}, x_0\right) \leq r_{i-1} + a_{i-1} \tag{6.96}$$

We show by contradiction that, when (6.96) holds, then

$$0 < V\left(j + \frac{l+1}{N_{i,j}}, x\left(j + \frac{l+1}{N_{i,j}}, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \leq r_{i-1} + 2a_{i-1}$$

Indeed, suppose on the contrary that

$$V\left(j + \frac{l+1}{N_{i,j}}, x\left(j + \frac{l+1}{N_{i,j}}, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) > r_{i-1} + 2a_{i-1} \tag{6.97}$$

Then, there would exist $t_1 \in (j + \frac{l}{N_{i,j}}, j + \frac{l+1}{N_{i,j}})$ such that

$$V\left(t_1, x\left(t_1, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) = r_{i-1} + \frac{3a_{i-1}}{2}$$

Using (6.87), the latter implies

$$0 < V\left(j + \frac{l+1}{N_{i,j}}, \phi\left(j + \frac{l+1}{N_{i,j}}, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \leq r_{i-1} + 2a_{i-1}$$

which contradicts (6.97), and the proof of Property 6.2 is complete. □

The following property is a direct consequence of Property 6.2 and (6.70):

Property 6.3 *Suppose that* (6.94) *holds and assume again that the solution of* (6.75) *with initial condition* $x(j + \frac{l}{N_{i,j}}) = x_0 \in \Re^n \backslash \{0\}$ *corresponding to some* $d \in M_D$

exists for $t \in [j + \frac{l}{N_{i,j}}, j + \frac{l+s}{N_{i,j}}]$ *for certain* $s \in \{0, 1, 2, \ldots, N_{i,j} - l\}$. *Then*

$$0 < V\left(j + \frac{l+s}{N_{i,j}}, x\left(j + \frac{l+s}{N_{i,j}}, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \tag{6.98}$$

$$\leq \max\left\{r_{i-1} + 2a_{i-1}, V\left(j + \frac{l}{N_{i,j}}, x_0\right) - \frac{s}{N_{i,j}}\mu_i\right\} \tag{6.99}$$

The desired (6.76) follows from Property 6.3 with $l = 0$, $N = N_{i,j}$. We next proceed with the proof of (6.74). Combining Property 6.3 with (6.87), we obtain the following:

Property 6.4 *If* (6.94) *is fulfilled, then*

$$\begin{aligned} 0 &< V\left(t, x\left(t, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \\ &\leq \max\left\{r_{i-1} + 2a_{i-1}, V\left(j + \frac{l}{N_{i,j}}, x_0\right)\right\} + \frac{1}{2}\min(a_{i-1}, a_i) \\ &\forall t \in \left[j + \frac{l}{N_{i,j}}, \min(t_{\max}, j+1)\right) \end{aligned} \tag{6.100}$$

Proof Let $s \in \{0, 1, 2, \ldots, N_{i,j} - l - 1\}$ with $j + \frac{l+s}{N_{i,j}} < t_{\max}$. By virtue of (6.98), we distinguish the following two cases:

Case 1: Suppose that

$$V\left(j + \frac{l+s}{N_{i,j}}, x\left(j + \frac{l+s}{N_{i,j}}, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \geq r_{i-1} \tag{6.101}$$

Then by invoking (6.87), from (6.98) and (6.101) we get

$$\begin{aligned} 0 &< V\left(t, x\left(t, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \\ &\leq \max\left\{r_{i-1} + 2a_{i-1}, V\left(j + \frac{l}{N_{i,j}}, x_0\right)\right\} + \frac{1}{2}\min(a_{i-1}, a_i) \\ &\forall t \in \left[j + \frac{l+s}{N_{i,j}}, j + \frac{l+s+1}{N_{i,j}}\right] \end{aligned} \tag{6.102}$$

The desired (6.100) is a consequence of (6.102).

Case 2: Suppose that

$$0 < V\left(j + \frac{l+s}{N_{i,j}}, x\left(j + \frac{l+s}{N_{i,j}}, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) < r_{i-1} \tag{6.103}$$

We show by contradiction that, when (6.103) holds, then

$$\begin{aligned} &V\left(t, x\left(t, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \leq r_{i-1} + 2a_{i-1} + \frac{1}{2}\min(a_{i-1}, a_i) \\ &\forall t \in \left[j + \frac{l+s}{N_{i,j}}, \min\left(t_{\max}, j + \frac{l+s+1}{N_{i,j}}\right)\right) \end{aligned} \tag{6.104}$$

If (6.104) were false, then there would exist $t \in [j + \frac{l+s}{N_{i,j}}, \min(t_{\max}, j + \frac{l+s+1}{N_{i,j}}))$ such that

$$V\left(t, x\left(t, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) > r_{i-1} + 2a_{i-1} + \frac{1}{2}\min(a_{i-1}, a_i) \quad (6.105)$$

By (6.103) and (6.105), there would exist $t_1 \in [j + \frac{l+s}{N_{i,j}}, \min(t_{\max}, j + \frac{l+s+1}{N_{i,j}}))$ such that

$$\begin{aligned} &V\left(t_1, x\left(t_1, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) = r_{i-1} + 2a_{i-1} \\ &V\left(\xi, x\left(\xi, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \le r_{i-1} + 2a_{i-1} \\ &\forall \xi \in \left[j + \frac{l+s}{N_{i,j}}, t_1\right] \end{aligned} \quad (6.106)$$

By (6.106) and (6.87) we get

$$\begin{aligned} &0 < V\left(\xi, x\left(\xi, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \le r_{i-1} + 2a_{i-1} + \frac{1}{2}\min(a_{i-1}, a_i) \\ &\forall \xi \in \left[t_1, \min\left(t_{\max}, j + \frac{l+s+1}{N_{i,j}}\right)\right) \end{aligned} \quad (6.107)$$

Combining (6.106) and (6.107), we obtain $0 < V(\xi, x(\xi, j + \frac{l}{N_{i,j}}, x_0; d)) \le r_{i-1} + 2a_{i-1} + \frac{1}{2}\min(a_{i-1}, a_i)$ for all $\xi \in [j + \frac{l+s}{N_{i,j}}, \min(t_{\max}, j + \frac{l+s+1}{N_{i,j}}))$, which contradicts hypothesis (6.105).

We conclude from (6.102) and (6.104) that in both cases above we have

$$\begin{aligned} 0 &< V\left(t, x\left(t, j + \frac{l}{N_{i,j}}, x_0; d\right)\right) \\ &\le \max\left\{r_{i-1} + 2a_{i-1}, V\left(j + \frac{l}{N_{i,j}}, x_0\right)\right\} + \frac{1}{2}\min(a_{i-1}, a_i) \end{aligned}$$

for every $t \in [j + \frac{l+s}{N_{i,j}}, \min(t_{\max}, j + \frac{l+s+1}{N_{i,j}}))$ and for all $s \in \{0, 1, 2, \ldots, N_{i,j} - l - 1\}$ with $j + \frac{l+s}{N_{i,j}} < t_{\max}$, and the latter implies the desired (6.100). This completes the proof of Property 6.4. □

We are now in a position to establish (6.74). Let $(x_0, d) \in (\Re^n \backslash \{0\}) \times M_D$ and $t_0 \in [j + \frac{l}{N_{i+1,j}}, j + \frac{l+1}{N_{i+1,j}})$ for some $l \in \{0, 1, \ldots, N_{i,j} - 1\}$ with $V(t_0, x_0) \in [r_{i-1}, r_i]$. Then, exploiting inequality (6.87), we obtain

$$\begin{aligned} &V(t_0, x_0) - \frac{1}{2}a_i \le V\big(t, x(t, t_0, x_0; d)\big) \le V(t_0, x_0) + \frac{1}{2}a_i \\ &\forall t \in \left[t_0, j + \frac{l+1}{N_{i+1,j}}\right] \end{aligned} \quad (6.108)$$

Since $r_i + \frac{1}{2}a_i < r_{i+1} - 2a_{i+1}$, by virtue of (6.108) and (6.101) of Property 6.4, we get $0 < V(t, \phi(t, t_0, x_0; d)) \le r_i + \frac{5}{2}a_i$ for all $t \in [t_0, \min(t_{\max}, j+1))$, and this establishes (6.74). The proof is complete. □

Lemma 6.3 *Under the same hypotheses imposed in Lemma 6.1 for system* (1.3), *for every function $q \in C^0(\Re^+ \times (\Re^n; \Re^+)$ satisfying* (6.22), *there exists a continuous mapping $\tilde{k} : \Re^+ \times (\Re^n \backslash \{0\}) \to U$, continuously differentiable with respect to $x \in \Re^n \backslash \{0\}$, which satisfies*

$$\left|\tilde{k}(t, x)\right| \le \tilde{b}(t, x) \quad \forall (t, x) \in \Re^+ \times \left(\Re^n \backslash \{0\}\right) \tag{6.109}$$

where $\tilde{b}(\cdot, \cdot)$ is defined by (6.29), *and such that the following property holds for all $(t_0, x_0, d) \in \Re^+ \times (\Re^n \backslash \{0\}) \times M_D$:*

$$V\left(t, x(t, t_0, x_0; d)\right) \le 9V(t_0, x_0) \quad \forall t \in \left[t_0, \min\left([t_0] + 2, t_{\max}\right)\right) \tag{6.110}$$

where $x(\cdot, t_0, x_0; d)$ denotes the unique solution of

$$\dot{x} = f\left(t, d, x, \tilde{k}(t, x)\right) \quad (t, x) \in \Re^+ \times \left(\Re^n \backslash \{0\}\right) \tag{6.111}$$

with initial condition $x(t_0) = x_0 \in \Re^n \backslash \{0\}$corresponding to $d \in M_D$, and $t_{\max} := t_{\max}(t_0, x_0, d) > t_0$ denotes its maximal existence time. Moreover, there exists positive definite $\tilde{\rho} \in C^0(\Re^+; \Re^+)$ with $\tilde{\rho}(s) \le s$ for all $s \ge 0$ such that

$$\begin{aligned} &V\left(2j+2, x(2j+2, 2j, x_0; d)\right) \le V(2j, x_0) - \tilde{\rho}\left(V(2j, x_0)\right) \\ &\quad \text{for all } (x_0, d, j) \in \left(\Re^n \backslash \{0\}\right) \times M_D \times Z^+ \text{ with } 2j+2 < t_{\max} \end{aligned} \tag{6.112}$$

where $t_{\max} > 2j$ in (6.112) *is the maximal existence time of the solution $x(\cdot, 2j, x_0; d)$ of* (6.111).

Proof Let $q \in C^0(\Re^+ \times \Re^n; \Re^+)$ be a function satisfying (6.22) and let $r = \{r_i : i \in Z\}$ be a set with $r_i > 0$ and such that

$$r_{i+1} \le 2r_i \quad \text{and} \quad \lim_{i \to +\infty} r_i = +\infty \quad \lim_{i \to -\infty} r_i = 0 \tag{6.113}$$

Consider the set

$$r' = \left\{ r'_i = \frac{r_i + r_{i+1}}{2} : i \in Z \right\} \tag{6.114}$$

which by virtue of (6.114) satisfies $r'_i > 0$ and, further,

$$r'_{i+1} \le 2r'_i \quad \text{and} \quad \lim_{i \to +\infty} r'_i = +\infty \quad \lim_{i \to -\infty} r'_i = 0 \tag{6.115}$$

Define

$$\mu_i := \frac{1}{4}\min\left\{\rho(s) : s \in [r_{i-1}, r_i]\right\} \qquad \mu'_i := \frac{1}{4}\min\left\{\rho(s) : s \in \left[r'_{i-1}, r'_i\right]\right\} \tag{6.116}$$

and let $a = \{a_i : i \in Z\}$, $a' = \{a'_i : i \in Z\}$ be a pair of sets satisfying:

$$a_i > 0 \qquad a'_i > 0 \tag{6.117}$$

$$\frac{5}{2}a_i \le r_{i-1} \qquad \frac{5}{2}a'_i \le r'_{i-1} \tag{6.118}$$

$$r_i + 2a_i < r_{i+1} - 2a_{i+1} \qquad r'_i + 2a'_i < r'_{i+1} - 2a'_{i+1} \tag{6.119}$$

$$a_i + a'_i \le \frac{r_{i+1} - r_i}{8} \qquad a_i + a'_{i-1} \le \frac{r_i - r_{i-1}}{8} \tag{6.120}$$

$$a_i \le \frac{\mu'_i}{8} \qquad a'_i \le \frac{\mu_{i+1}}{8} \tag{6.121}$$

By Lemma 6.2, there exist continuous mappings $k_{r,a} : \Re^+ \times (\Re^n \backslash \{0\}) \to U$ and $k_{r',a'} : \Re^+ \times (\Re^n \backslash \{0\}) \to U$, continuously differentiable with respect to $x \in \Re^n \backslash \{0\}$, with

$$k_{r,a}(j, x) = k_{r',a'}(j, x) = 0 \quad \forall (x, j) \in \left(\Re^n \backslash \{0\}\right) \times Z^+ \tag{6.122}$$

$$\frac{\partial k_{r,a}}{\partial x}(j, x) = \frac{\partial k_{r',a'}}{\partial x}(j, x) = 0 \quad \forall (x, j) \in \left(\Re^n \backslash \{0\}\right) \times Z^+ \tag{6.123}$$

and such that properties (6.73), (6.74), and (6.76) hold . Finally, consider the map $\tilde{k} : \Re^+ \times (\Re^n \backslash \{0\}) \to U$ defined as

$$\tilde{k}(t, x) = \begin{cases} k_{r,a}(t, x) & \text{for } t \in [2j, 2j+1),\ (j, x) \in Z^+ \times (\Re^n \backslash \{0\}) \\ k_{r',a'}(t, x) & \text{for } t \in [2j+1, 2j+2),\ (j, x) \in Z^+ \times (\Re^n \backslash \{0\}) \end{cases} \tag{6.124}$$

By taking into account (6.122), (6.123), (6.124), and regularity properties of $k_{r,a}(\cdot, \cdot)$ and $k_{r',a'}(\cdot, \cdot)$, it follows that $\tilde{k}(\cdot, \cdot)$ is continuous, continuously differentiable with respect to $x \in \Re^n \backslash \{0\}$, and satisfies

$$\tilde{k}(j, x) = 0 \quad \forall (j, x) \in Z^+ \times \left(\Re^n \backslash \{0\}\right) \tag{6.125}$$

Moreover, (6.109) is an immediate consequence of definition (6.123) and inequality (6.73). Also, by (6.74), (6.113), and (6.118) it follows that for every $(t_0, x_0, d, i) \in \Re^+ \times (\Re^n \backslash \{0\}) \times M_D \times Z$, it holds:

$$V(t_0, x_0) \in [r_{i-1}, r_i] \quad \Rightarrow \quad V\left(t, x_{r,a}(t, t_0, x_0; d)\right) \le 3V(t_0, x_0)$$
$$\text{for all } t \in \left[t_0, \min\left([t_0] + 1, t_{\max}^{r,a},\right)\right) \tag{6.126}$$

$$V(t_0, x_0) \in [r_{i-1}, r_i] \quad \Rightarrow \quad V\left(t, x_{r',a'}(t, t_0, x_0; d)\right) \le 3V(t_0, x_0)$$
$$\text{for all } t \in \left[t_0, \min\left([t_0] + 1, t_{\max}^{r',a'}\right)\right) \tag{6.127}$$

where $x_{r,a}(\cdot, t_0, x_0; d)$ denotes the (unique) solution of

$$\dot{x} = f\left(t, d, x, k_{r,a}(t, x)\right) \quad (t, x) \in \Re^+ \times \left(\Re^n \backslash \{0\}\right) \tag{6.128}$$

and $x_{r',a'}(\cdot, t_0, x_0; d)$ is the (unique) solution of

$$\dot{x} = f\left(t, d, x, k_{r',a'}(t, x)\right) \quad (t, x) \in \Re^+ \times \left(\Re^n \backslash \{0\}\right) \tag{6.129}$$

with same initial condition $x(t_0) = x_0 \in \Re^n \backslash \{0\}$ and $d \in M_D$, and $t_{\max}^{r,a} > t_0$ and $t_{\max}^{r',a'} > t_0$, respectively, denote their maximal existence times. The desired inequality (6.110) is a direct consequence of (6.126), (6.127), definition (6.124), and the following obvious fact.

Fact *The solution of* (6.111) *with initial condition* $x(t_0) = x_0 \in \Re^n \backslash \{0\}$ *corresponding to some* $d \in M_D$ *is identical for* $t \in [t_0, \min([t_0] + 1, t_{\max}^{r,a}))$ *to the solution* $x_{r,a}(t, t_0, x_0; d)$ *of* (6.128) *if* $[t_0]$ *is even and is identical for* $t \in [t_0, \min([t_0] + 1, t_{\max}^{r',a'}))$ *to the solution* $x_{r',a'}(t, t_0, x_0; d)$ *of* (6.129) *if* $[t_0]$ *is odd.*

In order to show (6.112), let $(x_0, d, j) \in (\Re^n \backslash \{0\}) \times M_D \times Z^+$ be such that the unique solution $x(\cdot, 2j, x_0; d)$ of (6.111) with initial condition $x(2j) = x_0 \in \Re^n \backslash \{0\}$, corresponding to $d \in M_D$ is well defined on $[2j, 2j + 2]$. Notice that, if there is no such $(x_0, d, j) \in (\Re^n \backslash \{0\}) \times M_D \times Z^+$, then property (6.112) trivially holds for every positive definite function $\tilde{\rho} \in C^0(\Re^+; \Re^+)$. Let $i \in Z$ be the smallest integer with

$$r_{i-1} - 2a_{i-1} < V(2j, x_0) \le r_i - 2a_i \tag{6.130}$$

whose existence is guaranteed by (6.113) and (6.119). By virtue of (6.76), (6.130), and the previous fact, it follows that

$$V\big(2j + 1, x(2j + 1, 2j, x_0; d)\big) \le \max\big(r_{i-1} + 2a_{i-1}, V(2j, x_0) - \mu_i\big) \tag{6.131}$$

Notice that, by (6.120) we have $V(2j + 1, x(2j + 1, 2j, x_0; d)) \le r_i' - 2a_i'$. Consequently, there exists an integer $k \le i$ with

$$r_{k-1}' - 2a_{k-1}' < V\big(2j + 1, x(2j + 1, 2j, x_0; d)\big) \le r_k' - 2a_k' \tag{6.132}$$

We distinguish the following cases:

Case 1: $k < i$

In this case it follows from (6.132) that $V(2j + 1, x(2j + 1, 2j, x_0; d)) \le r_{i-1}' - 2a_{i-1}'$. By virtue of (6.76) and the fact above, we then obtain

$$\begin{aligned} &V\big(2j + 2, x(2j + 2, 2j, x_0; d)\big) \\ &\quad \le \max\big(r_{i-2}' + 2a_{i-2}', V(2j, x_0) - \mu_{i-1}' - \mu_i, r_{i-1} + 2a_{i-1} - \mu_{i-1}'\big) \end{aligned} \tag{6.133}$$

We now take into account (6.120), which implies

$$r_{i-2}' + 2a_{i-2}' \le r_{i-1} - 2a_{i-1} - \frac{r_{i-1} - r_{i-2}}{4} \tag{6.134}$$

From (6.133), (6.134), and the left-hand side inequality in (6.130) we get

$$\begin{aligned} &V\big(2j + 2, x(2j + 2, 2j, x_0; d)\big) \\ &\quad \le V(2j, x_0) + \max\left(-\frac{r_{i-1} - r_{i-2}}{4}, 4a_{i-1} - \mu_{i-1}'\right) \end{aligned} \tag{6.135}$$

which by (6.121) implies

$$\begin{aligned} &V\big(2j + 2, x(2j + 2, 2j, x_0; d)\big) \\ &\quad \le V(2j, x_0) - \frac{1}{4}\min\big(r_{i-1} - r_{i-2}, 2\mu_{i-1}'\big) \end{aligned} \tag{6.136}$$

Case 2: $k = i$

Notice that, since $r'_{i-1} - 2a'_{i-1} > r_{i-1} + 2a_{i-1}$ (which is a consequence of (6.120)), from (6.131) and using the left-hand side inequality (6.132) with $k = i$, we conclude that

$$r'_{i-1} - 2a'_{i-1} + \mu_i < V(2j, x_0) \tag{6.137}$$

Also, by (6.76) and the fact above we get $V(2j+2, x(2j+2, 2j, x_0; d)) \leq \max(r'_{i-1} + 2a'_{i-1}, V(2j, x_0) - \mu'_i - \mu_i)$, which, in conjunction with (6.137), gives

$$V\big(2j+2, x(2j+2, 2j, x_0; d)\big) \leq V(2j, x_0) + 4a'_{i-1} - \mu_i$$

which, in turn, by virtue of (6.131), implies

$$V\big(2j+2, x(2j+2, 2j, x_0; d)\big) \leq V(2j, x_0) - \frac{1}{2}\mu_i \tag{6.138}$$

We conclude from (6.136) and (6.138) that in both cases we have

$$\begin{aligned} & r_{i-1} - 2a_{i-1} < V(2j, x_0) \leq r_i - 2a_i \\ & \quad \Rightarrow \quad V\big(2j+2, x(2j+2, 2j, x_0; d)\big) \leq V(2j, x_0) - \gamma_i \end{aligned} \tag{6.139}$$

$$\gamma_i := \frac{1}{4}\min\big(r_{i-1} - r_{i-2}, 2\mu'_{i-1}, 2\mu_i\big) \tag{6.140}$$

Now let

$$\bar{\rho}(s) := \begin{cases} \frac{(\min(\gamma_i, \gamma_{i+1}) - \min(\gamma_{i-1}, \gamma_i))(s - r_{i-1} + 2a_{i-1})}{(r_i - 2a_i - r_{i-1} + 2a_{i-1})} + \min(\gamma_{i-1}, \gamma_i) \\ \quad \text{for } s \in (r_{i-1} - 2a_{i-1}, r_i - 2a_i] \\ 0 \quad \text{for } s = 0 \end{cases} \tag{6.141}$$

Notice that (6.116), (6.140), and (6.141) imply that $0 < \min(\gamma_{i-1}, \gamma_i, \gamma_{i+1}) \leq \bar{\rho}(s) \leq \gamma_i$ for $s \in (r_{i-1} - 2a_{i-1}, r_i - 2a_i]$ and further $\lim_{i \to -\infty} \mu_i = \lim_{i \to -\infty} \mu'_i = \lim_{i \to -\infty} \gamma_i = 0$. Thus, we may easily verify that $\bar{\rho} : \Re^+ \to \Re^+$ is positive definite and continuous. Finally, define

$$\tilde{\rho}(s) := \min\big\{\bar{\rho}(s), s\big\} \tag{6.142}$$

Property (6.139)–(6.140), in conjunction with (6.141), implies that the desired (6.112) is satisfied, and the proof is complete. □

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1 According to the statement of Lemma 6.3, for every function $q \in C^0(\Re^+ \times \Re^n; \Re^+)$ satisfying (6.22), there exists a continuous mapping $\tilde{k} : \Re^+ \times (\Re^n \backslash \{0\}) \to U$, continuously differentiable with respect to $x \in \Re^n \backslash \{0\}$, which satisfies (6.109), (6.110), and (6.112). Define

$$K(t, x) := h\left(\frac{V(t, x) - \exp(-t)}{\exp(-t)}\right)\tilde{k}(t, x) \quad \text{for } V(t, x) > \exp(-t) \tag{6.143}$$

$$K(t, x) := 0 \quad \text{for } V(t, x) \leq \exp(-t) \tag{6.144}$$

where $h : \Re \to [0, 1]$ is a smooth nondecreasing function with $h(s) = 0$ for $s \le 0$ and $h(s) = 1$ for $s \ge 1$. It can be easily verified that, according to definitions (6.143), (6.144), and the properties of $\tilde{k} : \Re^+ \times (\Re^n \backslash \{0\}) \to U$, the map K takes values in U and satisfies $K(t, 0) = 0$ for all $t \ge 0$. Moreover, $K : \Re^+ \times \Re^n \to U$ is a continuous and continuously differentiable mapping with respect to $x \in \Re^n$ on $\Re^+ \times \Re^n$.

In order to prove Theorem 6.1, we will make use of Lemma 6.3 and three facts below concerning certain properties of the solution of the closed-loop system (1.3) with $u = K(t, x)$. Let

$$T = T(t_0, x_0, d) := \begin{cases} \inf\{t \ge t_0 : \exp(t)V(t, x(t)) < 2\} & \text{if } \{t \ge t_0 : \exp(t)V(t, x(t)) < 2\} \neq \emptyset \\ +\infty & \text{if} \{t \ge t_0 : \exp(t)V(t, x(t)) < 2\} = \emptyset \end{cases} \tag{6.145}$$

where $x(\cdot) = x(\cdot, t_0, x_0; d)$ denotes the unique solution of the closed-loop system (1.3) with $u = K(t, x)$ and initial condition $x(t_0) = x_0 \in \Re^n$ corresponding to some $d \in M_D$. The following fact is an immediate consequence of (6.143), (6.144), (6.145), and the continuity of the mapping $t \to V(t, x(t))$.

Fact 1 *The unique solution* $x(\cdot) = x(\cdot, t_0, x_0; d)$ *of the closed-loop system* (1.3) *with* $u = K(t, x)$, *initial condition* $x(t_0) = x_0 \in \Re^n \backslash \{0\}$, *satisfying* $V(t_0, x_0) \ge 2\exp(-t_0)$, *corresponding to some* $d \in M_D$ *coincides with the unique solution of* (6.111) *with the same initial condition and the same* $d \in M_D$ *on the interval* $[t_0, T]$, *where* $T = T(t_0, x_0, d)$ *is defined by* (6.145), *and*

$$V\big(T, x(T)\big) = 2\exp(-T) \quad \textit{if } \big\{t \ge t_0 : \exp(t)V\big(t, x(t)\big) < 2\big\} \neq \emptyset \tag{6.146}$$

Next, we prove the following:

Fact 2 *For the closed-loop system* (1.3) *with* $u = K(t, x)$, *the following property holds for all* $(j, x_0, d) \in Z^+ \times \Re^n \times M_D$:

$$\begin{aligned} &V\big(2j + 2, x(2j + 2, 2j, x_0; d)\big) \\ &\quad \le V(2j, x_0) - \tilde{\rho}\big(V(2j, x_0)\big) + 18\exp(-2j) \end{aligned} \tag{6.147}$$

Proof Obviously, the desired (6.147) holds for $x_0 = 0$. Next, assume that $x_0 \neq 0$. Let $t_{\max} > 2j$ be the maximal existence time of $x(\cdot, 2j, x_0; d)$. We distinguish two cases. The first case is

$$\big\{t \in \big[2j, \min(t_{\max}, 2j + 2)\big) : \exp(t)V\big(t, x(t, 2j, x_0; d)\big) < 2\big\} = \emptyset \tag{6.148}$$

In this case, Fact 1, in conjunction with inequalities (6.110) and (6.112), guarantees that $t_{\max} > 2j + 2$ and that (6.147) holds. The second case is

$$\big\{t \in \big[2j, \min(t_{\max}, 2j + 2)\big) : \exp(t)V\big(t, x(t, 2j, x_0; d)\big) < 2\big\} \neq \emptyset \tag{6.149}$$

Let

$$t_1 := \sup\big\{t \in \big[2j, \min(t_{\max}, 2j + 2)\big) : \exp(t)V\big(t, x(t, 2j, x_0; d)\big) < 2\big\} \tag{6.150}$$

Clearly, from (6.150) we have

$$\sup_{t \to t_1^-} \exp(t) V\big(t, x(t, 2j, x_0; d)\big) \le 2 \tag{6.151}$$

which, by virtue of (6.20), implies $t_{\max} > t_1$. If $t_1 = 2j + 2$, the desired (6.147) follows from (6.151). If $t_1 < 2j + 2$, (6.150) guarantees that $\exp(t)V(t, x(t, 2j, x_0; d)) \ge 2$ for all $t \in [t_1, \min(t_{\max}, 2j + 2))$, and the latter, in conjunction with (6.151), gives

$$\exp(t_1) V\big(t_1, x(t_1, 2j, x_0; d)\big) = 2 \tag{6.152}$$

Using Fact 1 together with (6.110) and (6.152), we get $V(t, x(t, 2j, x_0; d)) \le 18 \exp(-t_1)$ for all $t \in [t_1, \min(t_{\max}, 2j + 2))$. By exploiting (6.20) we conclude that $t_{\max} > 2j + 2$, and therefore the estimate $V(t, x(t, 2j, x_0; d)) \le 18 \exp(-t_1)$ is fulfilled for every $t \in [t_1, 2j + 2]$. The latter implies (6.147), and this completes the proof of Fact 2. □

Finally, we show the following fact.

Fact 3 *The following property holds for the closed-loop system* (1.3) *with* $u = K(t, x)$:

$$V\big(t, x(t, t_0, x_0; d)\big) \le 9V(t_0, x_0) + 18 \exp(-t_0)$$
$$\text{for all } t \in \big[t_0, [t_0] + 2\big], (t_0, x_0, d) \in \Re^+ \times \Re^n \times M_D \tag{6.153}$$

Proof Obviously, (6.153) holds for $x_0 = 0$. Suppose next that $x_0 \ne 0$ and let us on the contrary assume that there exists $\hat{t} \in [t_0, [t_0] + 2]$ with

$$V\big(\hat{t}, x\big(\hat{t}, t_0, x_0; d\big)\big) > 9V(t_0, x_0) + 18 \exp(-t_0) \tag{6.154}$$

We distinguish two cases. First assume that

$$\big\{s \in [t_0, \hat{t}\,] : \exp(s) V\big(s, x(s, t_0, x_0; d)\big) < 2\big\} = \emptyset$$

In this case, (6.110) guarantees that $V(\hat{t}, x(\hat{t}, t_0, x_0; d)) \le 9V(t_0, x_0)$, which contradicts (6.154). Consider the remaining case

$$\big\{s \in [t_0, \hat{t}\,] : \exp(s) V\big(s, x(s, t_0, x_0; d)\big) < 2\big\} \ne \emptyset$$

and let $t_1 := \sup\{s \in [t_0, \hat{t}\,] : \exp(s)V(s, x(s, t_0, x_0; d)) < 2\}$. If $t_1 = \hat{t}$, we would have $V(\hat{t}, x(\hat{t}, t_0, x_0; d)) \le 2 \exp(-t_0)$, which contradicts (6.154). If $t_1 < \hat{t}$, then we would have $\exp(s)V(s, x(s, t_0, x_0; d)) \ge 2$ for all $s \in [t_1, \hat{t}\,]$ and $\exp(t_1)V(t_1, x(t_1, 2j, x_0; d)) = 2$. Therefore (6.110) gives $V(s, \phi(s, t_0, x_0; d)) \le 18 \exp(-t_1)$ for all $s \in [t_1, \hat{t}\,]$, which again contradicts (6.154), and we conclude that (6.153) holds. This completes the proof of Fact 3. □

Inequalities (6.147) and (6.153), in conjunction with Proposition 2.5 in Chap. 2, show that the closed-loop system (1.3) with $u = K(t, x)$ is RGAOS. By invoking definition (6.143)–(6.144) we obtain

$$\big|K(t, x)\big| \le \big|\tilde{k}(t, x)\big| \quad \forall (t, x) \in \Re^+ \times \big(\Re^n \backslash 0\big) \tag{6.155}$$

where $\tilde{k} : \Re^+ \times (\Re^n \backslash \{0\}) \to U$ is the continuous mapping whose existence is guaranteed by Lemma 6.3. Combining definition (6.29) of $\tilde{b}(\cdot,\cdot)$, (6.109), and (6.155), we get, for all $(t,x) \in \Re^+ \times (\Re^n \backslash 0)$,

$$\left|K(t,x)\right| \le \max\left\{b(\tau,y) : (\tau,y) \in \Re^+ \times \Re^n, |\tau - t| + |y - x| \le q(\tau,y)\right\} \quad (6.156)$$

It follows from (6.23) and (6.156) that the following inequality holds for all $(t,x) \in \Re^+ \times \Re^n$:

$$\left|K(t,x)\right| \le a\big(V(t,x)\big) \quad (6.157)$$

Define $\tilde{a}(s) := a(s) + a_1^{-1}(s)$, where $a_1 \in K_\infty$ is defined in (6.20). Taking into account inequalities (6.20) and (6.20), the following inequality holds:

$$\tilde{a}^{-1}\big(\left|K(t,x)\right| + \mu(t)|x|\big) \le V(t,x) \le a_2\big(\beta(t)|x|\big) \quad \forall (t,x) \in \Re^+ \times \Re^n \quad (6.158)$$

where $a_2 \in K_\infty$ and $\mu, \beta \in K^+$are the functions involved in (6.20). Inequalities (6.147), (6.153), (6.158), in conjunction with Proposition 2.5 in Chap. 2, guarantee that the closed-loop system (1.3) with $u = K(t,x)$ and output $\tilde{Y} = K(t,x)$ is RGAOS.

The proof is complete. □

6.6.2 *Control Systems Described by ODEs: The Artstein–Sontag Approach*

This methodology is based on the following idea: given a CLF, design a feedback law so that the time derivative of the CLF along the solutions of the closed-loop system becomes negative definite.

Using this methodology in his pioneering work [2], Artstein studied the above existence problem for affine autonomous control systems without disturbances, $U \subseteq \Re^m$ being a closed convex set and output Y being identically the state of the system, i.e., $H(t,x) \equiv x$ (see also [67]). He showed in [2] that the existence of a time-independent Control Lyapunov Function (CLF) satisfying the "small-control" property is a necessary and sufficient condition for the existence of a continuous stabilizing feedback. Sontag [62] extended the results by presenting an explicit formula of the feedback stabilizer for affine autonomous control systems without disturbances, $U = \Re^m$ and output Y being identically the state of the system. Sontag's formula was exploited recently in [29] for the uniform stabilization of time-varying systems. Freeman and Kokotović [14] extended the idea of the CLF in order to study affine control systems with disturbances, $U \subseteq \Re^m$ being a closed convex set and output Y being identically the state of the system, i.e., $H(t,x) \equiv x$; they introduced the concept of the Robust Control Lyapunov Function (RCLF). In [36] the authors showed that the "small-control" property is not needed for nonuniform in time robust global stabilization of the state ($H(t,x) \equiv x$) of control systems affine in the control with $U = \Re^m$. The result was extended in [35] for the general case of output stability. In all the above works the stabilizing feedback is constructed using a partition of unity methodology or Michael's Theorem (when the continuity of feedback laws suffices).

In this section, we consider control systems of the form (1.3) under the following hypotheses:

(HH2) The vector fields $f : \Re^+ \times D \times \Re^n \times U \to \Re^n$, $H : \Re^+ \times \Re^n \to \Re^k$ are continuous, and for every bounded interval $I \subset \Re^+$ and every compact set $S \subset \Re^n \times U$, there exists $L \geq 0$ such that

$$\begin{aligned}&\big|f(t,d,x,u) - f(t,d,y,v)\big| + \big|H(t,x) - H(t,y)\big| \\ &\quad \leq L|x-y| + L|u-v| \quad \text{for all } (t,d) \in I \times D, (x,u) \in S, (y,v) \in S\end{aligned}$$

Moreover, the set $D \subset \Re^l$ is compact, and $U \subseteq \Re^m$ is a closed convex set.

We next give the definition of the Output Robust Control Lyapunov Function for system (1.3). The definition is in the same spirit with the definition of the notion of Robust Control Lyapunov Function given in [14] for continuous-time finite-dimensional control systems. It should be noticed that the following definition of an ORCLF is different from Definition 6.1. Recall the definition of the Dini derivative $V^0(t,x;v) := \limsup_{h\to 0^+} \frac{V(t+h,x+hv)-V(t,x)}{h}$ for all $(t,x) \in \Re^+ \times \Re^n$, $v \in \Re^n$ (see (2.128) in Chap. 2).

Definition 6.2 We say that (1.3) admits an *Output Robust Control Lyapunov Function* (*ORCLF*) if there exists a locally Lipschitz function $V : \Re^+ \times \Re^n \to \Re^+$ (called the Output Robust Control Lyapunov Function) that satisfies the following properties:

(i) There exist $a_1, a_2 \in K_\infty$ and $\beta, \mu \in K^+$ such that (6.20) holds.
(ii) There exists a function $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ with $\Psi(t,0,0) = 0$ for all $t \geq 0$, a function $q \in \mathcal{E}$, and a C^0 positive definite function $\rho : \Re^+ \to \Re^+$such that for each $u \in U$, the mapping $(t,x) \to \Psi(t,x,u)$ is upper semi-continuous, and the following inequality holds:

$$\inf_{u\in U} \Psi(t,x,u) \leq q(t) \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{6.159}$$

Moreover, for every finite set $\{u_1, u_2, \ldots, u_p\} \subset U$ and for every $\lambda_i \in [0,1]$ $(i = 1, \ldots, p)$ with $\sum_{i=1}^p \lambda_i = 1$, it holds that

$$\begin{aligned}&\sup_{d\in D} V^0\left(t,x; f\left(t,d,x,\sum_{i=1}^p \lambda_i u_i\right)\right) \\ &\quad \leq -\rho\big(V(t,x)\big) + \max\big\{\Psi(t,x,u_i), i = 1,\ldots,p\big\} \quad \forall (t,x) \in \Re^+ \times \Re^n\end{aligned} \tag{6.160}$$

For the case $H(t,x) \equiv x$, we simply call $V : \Re^+ \times \Re^n \to \Re^+$ a State Robust Control Lyapunov Function (SRCLF).

We are now ready to state and prove our main results for the finite-dimensional case (1.3).

Theorem 6.2 *Consider system* (1.3) *under hypotheses* (H1–4) *and* (HH2). *The following statements are equivalent*:

(a) *There exists a* C^∞ *function* $k : \Re^+ \times \Re^n \to U$ *with* $k(t,0)=0$ *for all* $t \geq 0$ *such that the closed-loop system* (1.3) *with* $u = k(t,x)$ *is RGAOS.*
(b) *There exists a* C^0 *function* $k : \Re^+ \times \Re^n \to U$ *with* $f(t,d,x,k(t,x))$ *being locally Lipschitz with respect to* x *and* $f(t,d,0,k(t,0)) = 0$ *for all* $(t,d) \in \Re^+ \times D$ *such that the closed-loop system* (1.3) *with* $u = k(t,x)$ *is RGAOS.*
(c) *System* (1.3) *admits an ORCLF.*

Theorem 6.3 *Consider system* (1.3) *under hypotheses* (H1–4) *and* (HH2). *Assume that system* (1.3) *admits an ORCLF, which satisfies inequality* (6.20) *with* $\beta(t) \equiv 1$ *and inequality* (6.159) *with* $q(t) \equiv 0$ *and that the following hypothesis holds*:

(HH3) *There exist functions* $\eta \in K^+$ *and* $\varphi \in C^v(A;U)$, *where* $v \in \{1,2,\ldots\}$ *and* $A = \bigcup_{t\geq 0}\{t\} \times \{x \in \Re^n : |x| < 4\eta(t)\}$, *with* $\varphi(t,0) = 0$ *for all* $t \geq 0$, *such that*

$$\Psi\big(t,x,\varphi(t,x)\big) \leq 0 \quad \textit{for all } (t,x) \in \Re^+ \times \Re^n \textit{ with } |x| \leq 2\eta(t) \tag{6.161}$$

Then, there exists a mapping $k : \Re^+ \times \Re^n \to U$ *of class* $C^v(\Re^+ \times \Re^n; U)$ *with* $k(t,0) = 0$ *for all* $t \geq 0$ *such that the closed-loop system* (1.3) *with* $u = k(t,x)$ *is URGAOS.*

Finally, if the ORCLF V *and the function* Ψ *involved in property* (ii) *of Definition* 6.2 *are time independent and the following hypothesis holds*:

(HH4) *There exist a constant* $\eta > 0$ *and a function* $\varphi \in C^v(A;U)$ *with* $\varphi(0) = 0$, *where* $v \in \{1,2,\ldots\}$ *and* $A = \{x \in \Re^n : |x| < 4\eta\}$, *such that*

$$\Psi\big(x,\varphi(x)\big) \leq 0 \quad \textit{for all } x \in \Re^n \textit{ with } |x| \leq 2\eta \tag{6.162}$$

then, the mapping k *is time invariant.*

Proof of Theorem 6.2 Implications (a) ⇒ (b) is obvious, and we prove implications (c) ⇒ (a) and (b) ⇒ (c).

(c) ⇒ (a) Suppose that (1.3) admits an ORCLF. Without loss of generality, we may assume that the function $q \in \mathcal{E}$ involved in (6.159) is positive for all $t \geq 0$.

Furthermore, define

$$\Xi(t,x,u) := \Psi(t,x,u) - 8q(t) \quad (t,x,u) \in \Re^+ \times \Re^n \times U \tag{6.163}$$

$$\Xi(t,x,u) := \Xi(0,x,u) \quad (t,x,u) \in (-1,0) \times \Re^n \times U \tag{6.164}$$

The definition of Ξ, given by (6.163), (6.164), guarantees that the function $\Xi : (-1,+\infty) \times \Re^n \times U \to \Re \cup \{+\infty\}$ with $\Xi(t,0,0) = -8q(\max\{0,t\})$ for all $t > -1$ is such that, for each $u \in U$, the mapping $(t,x) \to \Xi(t,x,u)$ is upper semi-continuous. By virtue of (6.159) and the upper semi-continuity of Ξ, it follows that for each $(t,x) \in (-1,+\infty) \times \Re^n$, there exist $u = u(t,x) \in U$ and $\delta = \delta(t,x) \in (0, t+1)$ such that

$$\begin{aligned}&\Xi\big(\tau, y, u(t,x)\big) \leq 0\\ &\quad \forall(\tau,y) \in \big\{(\tau,y) \in (-1,+\infty) \times \Re^n : |\tau - t| + |y - x| < \delta\big\}\end{aligned} \tag{6.165}$$

Using (6.165) and standard partition of unity arguments, we can determine sequences $\{(t_i, x_i) \in (-1,+\infty) \times \Re^n\}_{i=1}^{\infty}$, $\{u_i \in U\}_{i=1}^{\infty}$, and $\{\delta_i\}_{i=1}^{\infty}$ with $\delta_i = \delta(t_i, x_i) \in (0, t_i + 1)$ associated with a sequence of open sets $\{\Omega_i\}_{i=1}^{\infty}$ with

$$\Omega_i \subseteq \big\{(\tau,y) \in (-1,+\infty) \times \Re^n : |\tau - t_i| + |y - x_i| < \delta_i\big\} \tag{6.166}$$

forming a locally finite open covering of $(-1,+\infty) \times \Re^n$ such that

$$\Xi(\tau, y, u_i) \leq 0 \quad \forall(\tau, y) \in \Omega_i \tag{6.167}$$

Also, a family of smooth functions $\{\theta_i\}_{i=1}^{\infty}$ with $\theta_i(t,x) \geq 0$ for all $(t,x) \in (-1,+\infty) \times \Re^n$ can be determined with

$$\operatorname{supp}\theta_i \subseteq \Omega_i \tag{6.168}$$

$$\sum_{i=1}^{\infty} \theta_i(t,x) = 1 \quad \forall(t,x) \in (-1,+\infty) \times \Re^n \tag{6.169}$$

The facts that $\Xi(t,0,0) = -8q(t) < 0$ for all $t \geq 0$ and that the mapping $(t,x) \to \Xi(t,x,0)$ is upper semi-continuous imply that for every $t \geq 0$, there exists $\delta(t) > 0$ such that $\Xi(\tau, y, 0) \leq 0$ for all $(\tau, y) \in \Re^+ \times \Re^n$ with $|\tau - t| + |y| \leq \delta(t)$. Utilizing the compactness of $[0,T]$ for every $T \geq 0$, we conclude that for every $T \geq 0$, there exists $\tilde{\delta}(T) > 0$ such that

$$(\tau, y) \in [0,T] \times \Re^n \quad \text{and} \quad |y| \leq \tilde{\delta}(T) \quad \Rightarrow \quad \Xi(\tau, y, 0) \leq 0 \tag{6.170}$$

Define the following function:

$$\tilde{\eta}(t) := \frac{1}{2}\big([t] + 1 - t\big)\tilde{\delta}\big([t]+1\big) + \frac{1}{2}\big(t - [t]\big)\tilde{\delta}\big([t]+2\big) \quad t \geq 0 \tag{6.171}$$

where $[t]$ denotes the integer part of $t \geq 0$. Notice that by definition (6.171) it follows that $\tilde{\eta}(k) = \frac{1}{2}\tilde{\delta}(k+1)$, $\lim_{t \to (k+1)^-} \tilde{\eta}(t) = \frac{1}{2}\tilde{\delta}(k+2)$ for all $k \in Z^+$, which implies that $\tilde{\eta}$ is continuous. Moreover, definition (6.171) gives

$$0 < \tilde{\eta}(t) \leq \frac{1}{2}\max\big\{\tilde{\delta}\big([t]+1\big); \tilde{\delta}\big([t]+2\big)\big\} \quad \text{for all } t \geq 0,$$

which, in conjunction with (6.170) and the inequality $t \leq [t] + 1 \leq [t] + 2$, implies

$$|x| \leq 2\tilde{\eta}(t) \quad \Rightarrow \quad \Xi(t, x, 0) \leq 0 \tag{6.172}$$

Let $\bar{\eta} : \Re^+ \to (0, +\infty)$ be the positive, continuous, and nonincreasing function defined by $\bar{\eta}(t) := \min_{0 \leq \tau \leq t} \tilde{\eta}(\tau)$. Let $\varphi \in C^{\infty}(\Re; [0,1])$ be a smooth function with $\int_0^1 \varphi(s)\,ds > 0$, $\varphi(s) = 0$ for all $s \leq 0$ and $s \geq 1$. Define $\eta(t) := \int_0^1 \varphi(s)\bar{\eta}(t+s)\,ds$, which is a C^{∞} positive function that satisfies $\eta(t) \leq \tilde{\eta}(t)$ for all $t \geq 0$. Consequently, by virtue of (6.172), we obtain

$$|x| \leq 2\eta(t) \quad \Rightarrow \quad \Xi(t, x, 0) \leq 0 \tag{6.173}$$

Let $h \in C^{\infty}(\Re;[0,1])$ be a smooth nondecreasing function with $h(s)=0$ for all $s \leq 0$ and $h(s)=1$ for all $s \geq 1$. Define

$$k(t,x) := h\left(\frac{|x|^2 - 2\eta^2(t)}{2\eta^2(t)}\right)\sum_{i=1}^{\infty}\theta_i(t,x)u_i \tag{6.174}$$

Clearly, k as defined by (6.174) is a smooth function with $k(t,0)=0$ for all $t \geq 0$. Moreover, since $k(t,x)$ is defined as a (finite) convex combination of $u_i \in U$ and $0 \in U$, we have $k(t,x) \in U$ for all $(t,x) \in \Re^+ \times \Re^n$.

Let $(t,x) \in \Re^+ \times \Re^n$ with $|x| \geq 2\eta(t)$ and define a finite set $J(t,x) = \{j \in \{1,2,\ldots\}; \theta_j(t,x) \neq 0\}$. By virtue of (6.160) and definition (6.174), we get

$$\begin{aligned}
&\sup_{d\in D} V^0\big(t,x; f\big(t,d,x,k(t,x)\big)\big)\\
&\quad = \sup_{d\in D} V^0\left(t,x; f\left(t,d,x,\sum_{j\in J(t,x)}\theta_j(t,x)u_j\right)\right)\\
&\quad \leq -\rho\big(V(t,x)\big) + \max_{j\in J(t,x)}\big\{\Psi(t,x,u_j)\big\}
\end{aligned} \tag{6.175}$$

Notice that for each $j \in J(t,x)$, we obtain from (6.168) that $(t,x) \in \Omega_j$. Consequently, by virtue of (6.167) and definition (6.163), we have that $\Psi(t,x,u_j) \leq 8q(t)$ for all $j \in J(t,x)$. Combining the previous inequality with inequality (6.175), we conclude that the following property holds for all $(t,x,d) \in \Re^+ \times \Re^n \times D$ with $|x| \geq 2\eta(t)$:

$$V^0\big(t,x; f\big(t,d,x,k(t,x)\big)\big) \leq -\rho\big(V(t,x)\big) + 8q(t) \tag{6.176}$$

Let $(t,x) \in \Re^+ \times \Re^n$ with $|x| \leq \sqrt{2}\eta(t)$. Notice that by definition (6.174) we get

$$\sup_{d\in D} V^0\big(t,x; f\big(t,d,x,k(t,x)\big)\big) = \sup_{d\in D} V^0\big(t,x; f(t,d,x,0)\big)$$

By virtue of (6.173), (6.163), and the above inequality, we conclude that (6.176) holds as well for all $(t,x) \in \Re^+ \times \Re^n$ with $|x| \leq \sqrt{2}\eta(t)$. Finally, for the case $(t,x) \in \Re^+ \times \Re^n$ with $\sqrt{2}\eta(t) < |x| < 2\eta(t)$, let $J(t,x) = \{j \in \{1,2,\ldots\}; \theta_j(t,x) \neq 0\}$ and notice that from (6.160) we get

$$\begin{aligned}
&\sup_{d\in D} V^0\big(t,x; f\big(t,d,x,k(t,x)\big)\big)\\
&\quad = \sup_{d\in D} V^0\left(t,x; f\left(t,d,x,h\left(\frac{|x|^2-2\eta^2(t)}{2\eta^2(t)}\right)\sum_{j\in J(t,x)}\theta_j(t,x)u_j\right)\right)\\
&\quad \leq -\rho\big(V(t,x)\big) + \max\big\{\Psi(t,x,0), \Psi(t,x,u_j), j \in J(t,x)\big\}
\end{aligned} \tag{6.177}$$

Taking into account definition (6.163) and inequalities (6.167), (6.168), (6.173), and (6.177), we conclude that (6.176) holds as well for all $(t,x) \in \Re^+ \times \Re^n$ with $\sqrt{2}\eta(t) < |x| < 2\eta(t)$. Consequently, (6.176) holds for all $(t,x) \in \Re^+ \times \Re^n$.

It follows from (6.176) and Theorem 2.4 in Chap. 2 that system (1.3) with $u = k(t,x)$ is RGAOS.

(b) $\Rightarrow$ (c) Since system (1.3) with $u = k(t,x)$ is RGAOS and since $f(t,d,x,k(t,x))$ is Lipschitz with respect to x on each bounded subset of $\Re^+ \times \Re^n \times D$, it follows from Theorem 2.1 in Chap. 2 and Theorem 3.5 in Chap. 3 that there exists a function $V \in C^0(\Re^+ \times \Re^n; \Re^+)$ being locally Lipschitz and functions $a_1, a_2 \in K_\infty$ and $\beta, \mu \in K^+$ such that

$$a_1\big(\big|\big(\mu(t)x, H(t,x)\big)\big|\big) \le V(t,x) \le a_2\big(\beta(t)|x|\big) \quad \forall(t,x) \in \Re^+ \times \Re^n \tag{6.178}$$

$$\sup_{d\in D} V^0\big(t,x; f\big(t,d,x,k(t,x)\big)\big) \le -V(t,x) \quad \forall(t,x) \in \Re^+ \times \Re^n \tag{6.179}$$

We next prove that V is an ORCLF for (1.3). Obviously, property (i) of Definition 6.2 is a consequence of inequality (6.178). Define, for all $(t,x) \in \Re^+ \times \Re^n$,

$$\Psi(t,x,u) := L_V\big(t+|x|\big)L_U\big(t+|x|+2\big|k(t,x)\big|+\big|u-k(t,x)\big|\big)\big|u-k(t,x)\big| \tag{6.180}$$

where $L_U : \Re^+ \to \Re^+$ and $L_V : \Re^+ \to \Re^+$ are continuous nondecreasing functions satisfying

$$\big|f(t,d,x,u) - f(t,d,x,v)\big| \le L_U\big(t+|x|+|u|+|v|\big)|u-v|$$
$$\forall(t,x,d,u,v) \in \Re^+ \times \Re^n \times D \times U \times U \tag{6.181}$$

and

$$\big|V(t,y) - V(t,x)\big| \le L_V\big(t+|x|+|y|\big)|y-x| \quad \forall(t,x) \in \Re^+ \times \Re^n \tag{6.182}$$

Inequality (6.159) with $q(t) \equiv 0$ is an immediate consequence of definition (6.180). Clearly, definition (6.180) implies $\Psi(t,0,0) = 0$ for all $t \ge 0$.

Notice that for every $(t,x) \in \Re^+ \times \Re^n$ and all finite sets $\{u_1, u_2, \ldots, u_p\} \subset U$ and $\lambda_i \in [0,1]$ $(i = 1,\ldots,p)$ with $\sum_{i=1}^p \lambda_i = 1$, definition (6.180), in conjunction with the fact $|\sum_{i=1}^p \lambda_i u_i - k(t,x)| \le \sum_{i=1}^p \lambda_i |u_i - k(t,x)| \le \max_{i=1,\ldots,p} |u_i - k(t,x)|$, implies that

$$\Psi\left(t,x,\sum_{i=1}^p \lambda_i u_i\right) \le \max_{i=1,\ldots,p} \Psi(t,x,u_i) \tag{6.183}$$

By definition (2.128) in Chap. 2 and inequality (6.182) we get, for all $(t,x,v,w) \in \Re^+ \times \Re^n \times \Re^n \times \Re^n$,

$$V^0(t,x;v) \le L_V\big(t+|x|\big)|v-w| + V^0(t,x;w)$$

Combining the above inequality with inequalities (6.179), (6.181) and definition (6.180), we obtain, for all $(t,x,u) \in \Re^+ \times \Re^n \times U$,

$$\begin{aligned}
&\sup_{d\in D} V^0\big(t,x; f(t,d,x,u)\big) \\
&\quad\le \sup_{d\in D} V^0\big(t,x; f\big(t,d,x,k(t,x)\big)\big) \\
&\qquad + L_V\big(t+|x|\big)\sup_{d\in D}\big|f(t,d,x,u) - f\big(t,d,x,k(t,x)\big)\big| \\
&\quad\le -V(t,x) + L_V\big(t+|x|\big)L_U\big(t+|x|+\big|u-k(t,x)\big|+2\big|k(t,x)\big|\big)\big|u-k(t,x)\big| \\
&\quad\le -V(t,x) + \Psi(t,x,u)
\end{aligned}$$

The above inequality, in conjunction with (6.183), implies that inequality (6.160) with $\rho(s) := s$ holds.

The proof is complete. □

Proof of Theorem 6.3 Suppose that (1.3) admits an ORCLF which satisfies (6.159) with $q(t) \equiv 0$. Define

$$\Xi(t,x,u) := \Psi(t,x,u) - \frac{1}{2}\rho\big(V(t,x)\big) \quad (t,x,u) \in \Re^+ \times \Re^n \times U \tag{6.184}$$

$$\Xi(t,x,u) := \Xi(0,x,u) \quad (t,x,u) \in (-1,0) \times \Re^n \times U \tag{6.185}$$

The definition of Ξ, given by (6.184) and (6.185), guarantees that the function $\Xi : (-1,+\infty) \times \Re^n \times U \to \Re \cup \{+\infty\}$ with $\Xi(t,0,0) = 0$ for all $t > -1$ is such that, for each $u \in U$, the mapping $(t,x) \to \Xi(t,x,u)$ is upper semi-continuous. By virtue of (6.159) with $q(t) \equiv 0$ and the upper semi-continuity of Ξ, it follows that for each $(t,x) \in (-1,+\infty) \times (\Re^n \backslash \{0\})$, there exist $u = u(t,x) \in U$ and $\delta = \delta(t,x) \in (0, \min(1, t+1))$ such that

$$\Xi\big(\tau, y, u(t,x)\big) \le 0$$
$$\forall (\tau, y) \in \big\{(\tau, y) \in (-1,+\infty) \times \Re^n : |\tau - t| + |y - x| < \delta\big\} \tag{6.186}$$

Using (6.186) and standard partition of unity arguments, we can determine sequences $\{(t_i, x_i) \in (-1,+\infty) \times (\Re^n \backslash \{0\})\}_{i=1}^{\infty}$, $\{u_i \in U\}_{i=1}^{\infty}$, and $\{\delta_i\}_{i=1}^{\infty}$ with $\delta_i = \delta(t_i, x_i) \in (0, \min(1, t_i + 1))$, associated with a sequence of open sets $\{\Omega_i\}_{i=1}^{\infty}$ with

$$\Omega_i \subseteq \big\{(\tau, y) \in (-1,+\infty) \times \Re^n : |\tau - t_i| + |y - x_i| < \delta_i\big\} \tag{6.187}$$

forming a locally finite open covering of $(-1,+\infty) \times (\Re^n \backslash \{0\})$ and such that

$$\Xi(\tau, y, u_i) \le 0 \quad \forall (\tau, y) \in \Omega_i \tag{6.188}$$

Also, a family of smooth functions $\{\theta_i\}_{i=1}^{\infty}$ with $\theta_i(t,x) \ge 0$ for all $(t,x) \in (-1,+\infty) \times (\Re^n \backslash \{0\})$ can be determined with

$$\operatorname{supp}\theta_i \subseteq \Omega_i \tag{6.189}$$

$$\sum_{i=1}^{\infty} \theta_i(t,x) = 1 \quad \forall (t,x) \in (-1,+\infty) \times \big(\Re^n \backslash \{0\}\big) \tag{6.190}$$

We define

$$\tilde{k}(t,x) := \sum_{i=1}^{\infty} \theta_i(t,x) u_i \quad \text{for } t \ge 0,\ x \ne 0 \tag{6.191}$$

$$\tilde{k}(t,0) := 0 \quad \text{for } t \ge 0 \tag{6.192}$$

$$k(t,x) := \left(1 - h\left(\frac{|x|^2 - 2\eta^2(t)}{2\eta^2(t)}\right)\right)\varphi\big(t, \Pr_{Q(t)}(x)\big) + h\left(\frac{|x|^2 - 2\eta^2(t)}{2\eta^2(t)}\right)\tilde{k}(t,x) \tag{6.193}$$

where $\eta \in K^+$ and $\varphi \in C^v(A;U)$ are the functions involved in hypothesis (HH3), $h \in C^\infty(\Re;[0,1])$ is a smooth nondecreasing function with $h(s)=0$ for all $s \le 0$ and $h(s)=1$ for all $s \ge 1$, and $Q(t)=\{x \in \Re^n : |x| \le 3\eta(t)\}$. Clearly, k as defined by (6.193) is of class $C^v(\Re^+ \times \Re^n;U)$ with $k(t,0)=0$ for all $t \ge 0$.

Next, we show that

$$V^0\big(t,x;f\big(t,d,x,k(t,x)\big)\big) \le -\frac{1}{2}\rho\big(V(t,x)\big) \quad \forall(t,x,d) \in \Re^+ \times \Re^n \times D \tag{6.194}$$

Let $(t,x) \in \Re^+ \times \Re^n$ with $|x| \le \sqrt{2}\eta(t)$. Notice that by definition (6.193) we get

$$\sup_{d \in D} V^0\big(t,x;f\big(t,d,x,k(t,x)\big)\big) = \sup_{d \in D} V^0\big(t,x;f\big(t,d,x,\varphi(t,x)\big)\big)$$

The above inequality, in conjunction with inequalities (6.160), (6.161), implies that inequality (6.194) holds for all $(t,x) \in \Re^+ \times \Re^n$ with $|x| \le \sqrt{2}\eta(t)$. For $(t,x) \in \Re^+ \times (\Re^n \backslash \{0\})$ with $|x| \ge 2\eta(t)$, define $J(t,x)=\{j \in \{1,2,\ldots\};\theta_j(t,x) \ne 0\}$ (a finite set). Notice that by virtue of (6.160) and definitions (6.191), (6.193), we get

$$\begin{aligned}
&\sup_{d \in D} V^0\big(t,x;f\big(t,d,x,k(t,x)\big)\big) \\
&\quad = \sup_{d \in D} V^0\Bigg(t,x;f\Bigg(t,d,x,\sum_{j \in J(t,x)} \theta_j(t,x)u_j\Bigg)\Bigg) \\
&\quad \le -\rho\big(V(t,x)\big) + \max_{j \in J(t,x)} \big\{\Psi(t,x,u_j)\big\}
\end{aligned} \tag{6.195}$$

Notice that for each $j \in J(t,x)$, we obtain from (6.189) that $(t,x) \in \Omega_j$. Consequently, by virtue of (6.188) and definition (6.184), we have that $\Psi(t,x,u_j) \le \frac{1}{2}\rho(V(t,x))$, for all $j \in J(t,x)$. Combining the previous inequality with inequality (6.195), we conclude that (6.194) holds.

In a similar way, we can show that inequality (6.194) holds for all $(t,x) \in \Re^+ \times \Re^n$ with $\sqrt{2}\eta(t) < |x| < 2\eta(t)$.

If the ORCLF V and the function Ψ involved in property (ii) of Definition 6.2 are time-independent, then the partition of unity arguments used above may be repeated on $\Re^n \backslash \{0\}$ instead of $\Re^+ \times (\Re^n \backslash \{0\})$. If in addition hypothesis (HH4) holds, then the constructed feedback is time invariant.

The fact that system (1.3) with $u=k(t,x)$ is URGAOS follows directly from Theorem 2.4 in Chap. 2 and inequality (6.194). The proof is complete. □

6.6.3 *Control Systems Described by ODEs: Remarks and Feedback Design*

The problem with Definition 6.2 of the ORCLF that might arise in practice is the assumption of the knowledge of the function $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ involved in property (ii) of Definition 6.2. Particularly, the following problem arises:

Problem (P) *Consider system* (1.3) *under hypotheses* (H1–4) *and* (HH2). *Is the function* $V : \Re^+ \times \Re^n \to \Re^+$ *involved in the following hypothesis an ORCLF for system* (1.3)*?*

(HH5) The function $V : \Re^+ \times \Re^n \to \Re^+$ is a C^1 function which satisfies property (i) of Definition 6.1, and there exist a function $q \in \mathcal{E}$ and a C^0 positive definite function $\rho : \Re^+ \to \Re^+$ such that the following inequality holds:

$$\inf_{u \in U} \left\{ \frac{\partial V}{\partial t}(t,x) + \sup_{d \in D} \left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,u) \right) \right\} \leq -\rho\big(V(t,x)\big) + q(t) \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{6.196}$$

The proof of implication (b) $\Rightarrow$ (c) of Theorem 6.2 gives insights for the solution to Problem (P): If there exist a continuous function $k : \Re^+ \times \Re^n \to U$ with $k(t,0) = 0$ for all $t \geq 0$, a function $\tilde{q} \in \mathcal{E}$, and a C^0 positive definite function $\tilde{\rho} : \Re^+ \to \Re^+$such that

$$\frac{\partial V}{\partial t}(t,x) + \sup_{d \in D} \left(\frac{\partial V}{\partial x}(t,x) f\big(t,d,x,k(t,x)\big) \right) \leq -\tilde{\rho}\big(V(t,x)\big) + \tilde{q}(t) \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{6.197}$$

then, V is an ORCLF for (1.3). Particularly, the function $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ involved in property (ii) of Definition 6.2 may be defined by

$$\Psi(t,x,u) := \tilde{\rho}\big(V(t,x)\big) + \sup \left\{ \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,d,x,v) : d \in D, v \in U \text{with} \big|v - k(t,x)\big| \leq \big|u - k(t,x)\big| \right\} \tag{6.198}$$

It is easy to check that $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ as defined by (6.198) satisfies inequalities (6.159), (6.160) of Definition 6.2, following exactly the same procedure as in the proof of implication (b) $\Rightarrow$ (c) of Theorem 6.2. Moreover, by virtue of Theorem 6.2, if $V : \Re^+ \times \Re^n \to \Re^+$ is an ORCLF for system (1.3), then the proof of implication (c) $\Rightarrow$ (a) of Theorem 6.2 shows that there exist a continuous function $k : \Re^+ \times \Re^n \to U$ with $k(t,0) = 0$ for all $t \geq 0$, a function $\tilde{q} \in \mathcal{E}$, and a C^0 positive definite function $\tilde{\rho} : \Re^+ \to \Re^+$such that (6.197) holds. Consequently, $V : \Re^+ \times \Re^n \to \Re^+$ is an ORCLF for (1.3) under hypotheses (H1–4) *if and only if* there exist a continuous function $k : \Re^+ \times \Re^n \to U$ with $k(t,0) = 0$ for all $t \geq 0$, a function $\tilde{q} \in \mathcal{E}$, and a C^0 positive definite function $\tilde{\rho} : \Re^+ \to \Re^+$such that (6.197) holds.

The problem with the above solution to Problem (P) is that we can check if $V : \Re^+ \times \Re^n \to \Re^+$ is an ORCLF for (1.3) by constructing a feedback stabilizer for (1.3). On the other hand, our goal in practice is to construct a desired feedback stabilizer based on the mere knowledge of the Lyapunov function $V : \Re^+ \times \Re^n \to \Re^+$ under hypotheses (H1–4), (HH2), and (HH5). Consequently, the above solution to Problem (P) cannot be helpful for feedback construction purposes.

The rest of this paragraph provides sufficient conditions for establishing that $V : \Re^+ \times \Re^n \to \Re^+$ under hypotheses (H1–4), (HH2), and (HH5) is an ORCLF for (1.3).

Indeed, if the mapping $u \to \frac{\partial V}{\partial x}(t,x) f(t,d,x,u)$ is quasi-convex for each fixed $(t,x,d) \in \Re^+ \times \Re^n \times D$, then the mapping $u \to \frac{\partial V}{\partial t}(t,x) + \sup_{d\in D}(\frac{\partial V}{\partial x}(t,x) \times f(t,d,x,u))$ is quasi-convex for each fixed $(t,x) \in \Re^+ \times \Re^n$. Thus, property (ii) of Definition 6.2 is satisfied with $\Psi(t,x,u) := \rho(V(t,x)) + \frac{\partial V}{\partial t}(t,x) + \sup_{d\in D}(\frac{\partial V}{\partial x}(t,x) f(t,d,x,u))$. This is exactly the case arising in affine-in-control systems: for affine-in-control systems, the mapping $u \to \frac{\partial V}{\partial x}(t,x) f(t,d,x,u)$ is convex.

Example 6.6.1 For an autonomous affine-in-control system

$$\dot{x} = f(d,x) + g(d,x)u$$
$$x \in \Re^n, u \in \Re, d \in D \tag{6.199}$$

where $f, g : D \times \Re^n \to \Re^n$ are locally Lipschitz mappings with $f(d,0) = 0$ for all $d \in D$, $D \subset \Re^l$ is a compact set, a SRCLF is a locally Lipschitz, positive definite function $V : \Re^n \to \Re^+$, with $\{x \in \Re^n : V(x) \le a\}$ being a compact set for every $a \ge 0$, which satisfies

$$V^0\big(x; f(x) + g(x)u\big) \le a(x) + b(x)u \quad \forall (x,u) \in \Re^n \times \Re \tag{6.200}$$

for certain locally Lipschitz mappings $a, b : \Re^n \to \Re$ with $a(0) = b(0) = 0$ and such that

$$b(x) = 0 \quad \text{and} \quad x \neq 0 \quad \Rightarrow \quad a(x) < 0 \tag{6.201}$$

$$a(x) + b(x)\varphi(x) < 0 \quad \text{for all } x \in A \backslash \{0\} \tag{6.202}$$

where $\varphi \in C^0(A; \Re)$ is a locally Lipschitz function with $\varphi(0) = 0$, $A = \{x \in \Re^n : |x| < 4\eta\}$, and $\eta > 0$ is a constant.

A locally Lipschitz version of the feedback law proposed by Sontag [53] is defined, for $x \neq 0$, by

$$\tilde{k}(x) := \begin{cases} 0 & \text{if } b(x) = 0 \\ -\frac{a(x)+\sqrt{a^2(x)+b^4(x)}}{b(x)} & \text{if } b(x) \neq 0 \end{cases} \tag{6.203}$$

$$k(x) := \left(1 - h\left(\frac{|x|^2 - 2\eta^2}{2\eta^2}\right)\right)\varphi\big(\mathrm{Pr}_Q(x)\big) + h\left(\frac{|x|^2 - 2\eta^2}{2\eta^2}\right)\tilde{k}(x) \tag{6.204}$$

where $h \in C^0(\Re; [0,1])$ is a locally Lipschitz, nondecreasing function with $h(s) = 0$ for all $s \le 0$ and $h(s) = 1$ for all $s \ge 1$, and $Q = \{x \in \Re^n : |x| \le 3\eta\}$.

Using the arguments in [62], it can be shown that the mapping $k : \Re^n \to \Re$ as defined by (6.203), (6.204) is a locally Lipschitz mapping. In fact, if the functions $a, b : \Re^n \to \Re$, $h \in C^0(\Re; [0,1])$, and $\varphi \in C^0(A; \Re)$ are of class C^v, where $v \in \{1, 2, \ldots\}$, then $k : \Re^n \to \Re$ as defined by (6.203), (6.204) is also a mapping of class C^v.

Inequality (6.200) shows that $V^0(x; f(x)+g(x)k(x)) \leq -P(x)$ for all $x \in \Re^n$, where

$$P(x) := p(x) \quad \text{for all } |x| \geq 2\eta$$
$$P(x) := a(x) + b(x)\varphi(x) \quad \text{for all } |x| \leq \sqrt{2}\eta$$
$$P(x) := \left(1 - h\left(\frac{|x|^2 - 2\eta^2}{2\eta^2}\right)\right)\big(a(x) + b(x)\varphi(x)\big) + h\left(\frac{|x|^2 - 2\eta^2}{2\eta^2}\right)p(x)$$
$$\text{for all } \sqrt{2}\eta < |x| < 2\eta$$

where

$$p(x) := \begin{cases} a(x) & \text{if } b(x) = 0 \\ -\sqrt{a^2(x) + b^4(x)} & \text{if } b(x) \neq 0 \end{cases}$$

By virtue of (6.202), $P : \Re^n \to \Re^+$ is continuous and positive definite. It follows from Proposition 2.2 in Chap. 2 that $V : \Re^n \to \Re^+$ is a SRCLF for (6.199), which satisfies property (ii) of Definition 6.2 with $\Psi(x,u) := P(x) + a(x) + b(x)u$.

The following lemma helps us to give sufficient conditions for a function satisfying hypothesis (HH5) to be an ORCLF.

Lemma 6.4 *For a mapping $f : A \times U \to \Re$, where $U \subseteq \Re^m$ is a closed convex set, define the set-valued map*

$$A \times U \ni (x,u) \to \mathcal{U}(x,u) := \overline{co}\big\{v \in U : f(x,v) \leq f(x,u)\big\} \tag{6.205}$$

and the mapping $\psi : A \times U \to \Re \cup \{+\infty\}$ by

$$\psi(x,u) := \sup\big\{f(x,v) : v \in \mathcal{U}(x,u)\big\} \tag{6.206}$$

Then for every finite set $\{u_1, u_2, \ldots, u_p\} \subset U$ and for every $\lambda_i \in [0,1]$ $(i = 1, \ldots, p)$ with $\sum_{i=1}^p \lambda_i = 1$, it holds that

$$f\left(x, \sum_{i=1}^p \lambda_i u_i\right) \leq \max\big\{\psi(x,u_i), i = 1, \ldots, p\big\} \quad \forall x \in A \tag{6.207}$$

Proof Consider a finite set $\{u_1, u_2, \ldots, u_p\} \subset U$ and $\lambda_i \in [0,1]$ $(i = 1, \ldots, p)$ with $\sum_{i=1}^p \lambda_i = 1$. Let $u \in \{u_1, u_2, \ldots, u_p\}$ be such that $f(x,u) = \max_{i=1,\ldots,p} f(x,u_i)$. It follows from definition (6.205) that $\sum_{i=1}^p \lambda_i u_i \in \mathcal{U}(x,u)$, and consequently, by definition (6.206), we get $f(x, \sum_{i=1}^p \lambda_i u_i) \leq \psi(x,u)$. The previous inequality combined with the fact that $u \in \{u_1, u_2, \ldots, u_p\}$ (which implies $\psi(x,u) \leq \max\{\psi(x,u_i), i = 1, \ldots, p\}$) establishes (6.207). The proof is complete. □

The following lemma is a direct consequence of the previous lemma.

Lemma 6.5 *Let $V : \Re^+ \times \Re^n \to \Re^+$ be a C^1 function which satisfies the following properties*:

(i) *there exist functions* $q \in \mathcal{E}$ *and* $\rho : \Re^+ \to \Re^+$ *being* C^0 *positive definite such that for all* $(t,x) \in \Re^+ \times \Re^n$, *there exists* $u \in U$ *with* $\mathcal{U}(t,x,u) \subseteq \tilde{\mathcal{U}}(t,x) \neq \emptyset$, *where*

$$\mathcal{U}(t,x,u) := \overline{co}\left\{ v \in U : \sup_{d \in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,v)\right) \le \sup_{d \in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,u)\right)\right\}$$

$$\tilde{\mathcal{U}}(t,x) := \left\{ v \in U : \frac{\partial V}{\partial t}(t,x) + \sup_{d \in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,v)\right) \le -\rho\big(V(t,x)\big) + q(t)\right\},$$

(ii) *for each fixed* $u \in U$, *the mapping* $(t,x) \to \sup\{\sup_{d \in D}(\frac{\partial V}{\partial x}(t,x) f(t,d,x,v)) : v \in \mathcal{U}(t,x,u)\}$ *is upper semi-continuous.*

Then, property (ii) *of Definition* 6.2 *holds with* $\Psi(t,x,u) := \rho(V(t,x)) + \frac{\partial V}{\partial t}(t,x) + \sup\{\sup_{d \in D}(\frac{\partial V}{\partial x}(t,x) f(t,d,x,v)) : v \in \mathcal{U}(t,x,u)\}$.

It should be noted that if the mapping $u \to \frac{\partial V}{\partial x}(t,x) f(t,d,x,u)$ is quasi-convex for each fixed $(t,x,d) \in \Re^+ \times \Re^n \times D$, then the set-valued map

$$\mathcal{U}(t,x,u) := \overline{co}\left\{ v \in U : \sup_{d \in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,v)\right) \le \sup_{d \in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,u)\right)\right\}$$

in (i) of Lemma 6.5 satisfies

$$\mathcal{U}(t,x,u) = \left\{ v \in U : \sup_{d \in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,v)\right) \le \sup_{d \in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,u)\right)\right\}$$

Consequently, property (i) of Lemma 6.5 becomes equivalent to the existence of $u \in U$ with $\frac{\partial V}{\partial t}(t,x) + \sup_{d \in D}(\frac{\partial V}{\partial x}(t,x) f(t,d,x,u)) \le -\rho(V(t,x)) + q(t)$.

The following example illustrates the use of Lemma 6.5 for a special class of nonlinear systems.

Example 6.6.2 Consider system (1.3) under hypotheses (H1–3) with $m = 1$, $U = \Re$, and a C^1 function $V : \Re^+ \times \Re^n \to \Re^+$ which satisfies property (i) of Definition 2.6 and

$$\frac{\partial V}{\partial t}(t,x) + \sup_{d \in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,u)\right) = a(t,x)u^2 + b(t,x)u + c(t,x) \tag{6.208}$$

$$\inf_{u \in \Re}\big(a(t,x)u^2 + b(t,x)u + c(t,x)\big) \le -\rho\big(V(t,x)\big) + q(t) \tag{6.209}$$

for all $(t,x,u) \in \Re^+ \times \Re^n \times U$, appropriate continuous mappings $a,b,c : \Re^+ \times \Re^n \to \Re$ with $a(t,0) = b(t,0) = c(t,0) = 0$ for all $t \ge 0$, a function $q \in \mathcal{E}$, and a C^0 positive definite function $\rho : \Re^+ \to \Re^+$. We next prove that $V : \Re^+ \times \Re^n \to \Re^+$ is an ORCLF for (1.3), provided that the following implications hold:

$$a(t,x) < 0 \quad \Rightarrow \quad -\frac{b^2(t,x)}{4a(t,x)} + c(t,x) \le -\rho\big(V(t,x)\big) + q(t) \tag{6.210}$$

$$\begin{aligned}&\text{For every sequence } \big\{(t_i,x_i)\big\}_{i=0}^{\infty} \text{with } a(t_i,x_i) < 0\\ &\text{and } (t_i,x_i) \to (t,x) \text{ with } a(t,x) = 0, \text{ it holds that } \frac{b^2(t_i,x_i)}{|a(t_i,x_i)|} \to 0\end{aligned} \tag{6.211}$$

Following the notation of Lemma 6.5, define

$$\begin{aligned}\mathcal{U}(t,x,u) &= \overline{co}\big\{v \in \Re : a(t,x)v^2 + b(t,x)v \le a(t,x)u^2 + b(t,x)u\big\}\\ \tilde{\mathcal{U}}(t,x) &= \big\{v \in \Re : a(t,x)v^2 + b(t,x)v + c(t,x) \le -\rho\big(V(t,x)\big) + q(t)\big\}\end{aligned}$$

Notice that

1. If $a(t,x) > 0$, then for every $u \in \Re$, the set $\{v \in \Re : a(t,x)v^2 + b(t,x)v \le a(t,x)u^2 + b(t,x)u\}$ is closed and convex, and consequently, there exists $u \in \Re$ with $\mathcal{U}(t,x,u) \subseteq \tilde{\mathcal{U}}(t,x)$ (specifically, by virtue of (6.209), the inclusion $\mathcal{U}(t,x,u) \subseteq \tilde{\mathcal{U}}(t,x)$ holds for $u = -\frac{b(t,x)}{2a(t,x)}$).
2. If $a(t,x) = 0$, then for every $u \in \Re$, the set $\{v \in \Re : b(t,x)v \le b(t,x)u\}$ is closed and convex, and consequently, there exists $u \in \Re$ with $\mathcal{U}(t,x,u) \subseteq \tilde{\mathcal{U}}(t,x)$. Specifically, if $b(t,x) \ne 0$, then the inclusion $\mathcal{U}(t,x,u) \subseteq \tilde{\mathcal{U}}(t,x)$ holds for $u = -\frac{c(t,x)+\rho(V(t,x))}{b(t,x)}$. If $b(t,x) = 0$, then by (6.209) the inclusion $\mathcal{U}(t,x,u) \subseteq \tilde{\mathcal{U}}(t,x)$ holds for every $u \in \Re$.
3. If $a(t,x) < 0$, then, for every $u \ne -\frac{b(t,x)}{2a(t,x)}$, the set $\{v \in \Re : a(t,x)v^2 + b(t,x)v \le a(t,x)u^2 + b(t,x)u\}$ is not convex, and there exist $v_1, v_2 \in \Re$ with $v_1 < v_2$ such that

   $$\big\{v \in \Re : a(t,x)v^2 + b(t,x)v \le a(t,x)u^2 + b(t,x)u\big\} = (-\infty, v_1] \cup [v_2, +\infty)$$

   On the other hand, if $u = -\frac{b(t,x)}{2a(t,x)}$, then it holds that $\{v \in \Re : a(t,x)v^2 + b(t,x)v \le a(t,x)u^2 + b(t,x)u\} = \Re$. Consequently, if $a(t,x) < 0$, then for every $u \in \Re$, it holds that $\mathcal{U}(t,x,u) = \Re$. However, in this case implication (6.210) guarantees that $\tilde{\mathcal{U}}(t,x) = \Re$ and therefore the inclusion $\mathcal{U}(t,x,u) \subseteq \tilde{\mathcal{U}}(t,x)$ holds for every $u \in \Re$.

Thus, property (i) of Lemma 6.5 holds for the function $V : \Re^+ \times \Re^n \to \Re^+$. Since $\Psi(t,x,u) := \rho(V(t,x)) + \frac{\partial V}{\partial t}(t,x) + \sup\{\sup_{d\in D}(\frac{\partial V}{\partial x}(t,x)f(t,d,x,v)) : v \in \mathcal{U}(t,x,u)\}$, by virtue of all the above specifications for the set-valued map $\mathcal{U}(t,x,u)$, we get

$$\Psi(t,x,u) := \rho\big(V(t,x)\big) + \begin{cases} a(t,x)u^2 + b(t,x)u + c(t,x) & \text{if } a(t,x) \ge 0\\ -\frac{b^2(t,x)}{4a(t,x)} + c(t,x) & \text{if } a(t,x) < 0\end{cases} \tag{6.212}$$

Notice that implication (6.211) guarantees that property (ii) of Lemma 6.5 holds and consequently property (ii) of Definition 6.2 holds with Ψ defined by (6.212). It should also noticed that other choices for the mapping $\Psi(t,x,u)$ are possible. For example, the selection

$$\Psi(t,x,u) := \rho\big(V(t,x)\big) + \begin{cases} a(t,x)u^2 + b(t,x)u + c(t,x) & \text{if } a(t,x) \geq 0 \\ b(t,x)u + c(t,x) & \text{if } a(t,x) < 0 \end{cases} \tag{6.213}$$

guarantees that $V : \Re^+ \times \Re^n \to \Re^+$ is an ORCLF for (1.3), provided that (6.208), (6.209), and the following implication hold:

$$a(t,x) < 0 \quad b(t,x) = 0 \quad \Rightarrow \quad c(t,x) \leq -\rho\big(V(t,x)\big) + q(t) \tag{6.214}$$

Notice that if (6.214) holds then property (ii) of Definition 6.2 holds with Ψ defined by (6.213). Moreover, notice that if implication (6.210) holds, then implication (6.214) automatically holds.

The following lemma provides a "patchy" construction by combining the formula provided by Lemma 6.5 and the knowledge of appropriate functions that can be used in certain regions of $\Re^+ \times \Re^n$ as feedback functions.

Lemma 6.6 *Let $V : \Re^+ \times \Re^n \to \Re^+$ be a C^1 function and suppose that there exist sets $\Omega_i \subseteq \Re^+ \times \Re^n$ $(i = 0, \ldots, p)$ with $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=0,\ldots,p} \Omega_i = \Re^+ \times \Re^n$, functions $k_i : \Omega_i \to U$ $(i = 1, \ldots, p)$, a function $q \in \mathcal{E}$, and a C^0 positive definite function $\rho : \Re^+ \to \Re^+$ such that*

1. *for every $(t,x) \in \Omega_0$, there exists $u \in U$ with $\mathcal{U}(t,x,u) \subseteq \tilde{\mathcal{U}}(t,x) \neq \emptyset$, where*

$$\mathcal{U}(t,x,u) := \overline{co}\Bigg\{ v \in U : \sup_{d\in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,v)\right) \leq \sup_{d\in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,u)\right)\Bigg\}$$

$$\tilde{\mathcal{U}}(t,x) := \Bigg\{ v \in U : \frac{\partial V}{\partial t}(t,x) + \sup_{d\in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,v)\right) \leq -\rho(V(t,x)) + q(t)\Bigg\},$$

2. *for all $i = 1, \ldots, p$ and $(t,x) \in \Omega_i$, it holds that*

$$\frac{\partial V}{\partial t}(t,x) + \sup_{d\in D}\left(\frac{\partial V}{\partial x}(t,x) f\big(t,d,x,k_i(t,x)\big)\right) \leq -\rho\big(V(t,x)\big) + q(t)$$

Consider the function $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ *defined by*

$$\Psi(t,x,u) := \rho\big(V(t,x)\big) + \frac{\partial V}{\partial t}(t,x) + \sup\left\{\sup_{d\in D}\left(\frac{\partial V}{\partial x}(t,x) f(t,d,x,v)\right) : v \in \mathcal{U}(t,x,u)\right\}$$
$$\text{for } (t,x) \in \Omega_0 \tag{6.215}$$

$$\Psi(t,x,u) := \rho\big(V(t,x)\big) + \sup\left\{\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,d,x,v) : d \in D, v \in U \text{ with } \big|v - k_i(t,x)\big| \le \big|u - k_i(t,x)\big|\right\}$$
$$\text{for } (t,x) \in \Omega_i, i = 1,\dots,p \tag{6.216}$$

and suppose that $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ *is upper semi-continuous.*

Then property (ii) *of Definition* 6.2 *holds with* $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ *defined by* (6.215), (6.216).

The following example illustrates the efficiency of Lemma 6.6. It shows that the knowledge of appropriate functions that can be used in certain regions of $\Re^+ \times \Re^n$ as feedback functions helps us to obtain less conservative results.

Example 6.6.3 Consider system (1.3) under hypotheses (H1–3) with $m = 1$, $U = \Re$, and a C^1 function $V : \Re^+ \times \Re^n \to \Re^+$ which satisfies property (i) of Definition 6.1 and (6.208), (6.209) for appropriate continuous mappings $a, b, c : \Re^+ \times \Re^n \to \Re$ with $a(t,0) = b(t,0) = c(t,0) = 0$ for all $t \ge 0$, a function $q \in \mathcal{E}$, and a C^0 positive definite function $\rho : \Re^+ \to \Re^+$. We showed in Example 6.6.2 that $V : \Re^+ \times \Re^n \to \Re^+$ is an ORCLF for (1.3), provided that implications (6.210), (6.211) hold. In this example, we show that implication (6.211) only is sufficient for guaranteeing that $V : \Re^+ \times \Re^n \to \Re^+$ is an ORCLF for (1.3). Indeed, let $\Omega_0 := \{(t,x) \in \Re^+ \times \Re^n : a(t,x) \ge 0\}$ and $\Omega_1 := \{(t,x) \in \Re^+ \times \Re^n : a(t,x) < 0\}$. Moreover, define $k_1 : \Omega_1 \to \Re$ as follows: $\forall (t,x) \in \Omega_1$,

$$k_1(t,x) := \frac{\sqrt{b^2(t,x) + 4|a(t,x)c(t,x)| + 4|a(t,x)|\rho(V(t,x))} - b(t,x)}{2a(t,x)} \tag{6.217}$$

The specification of the set-valued map $\mathcal{U}(t,x,u)$ for $(t,x) \in \Omega_0$ has been given in Example 6.6.2. Therefore, the function $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ defined by

$$\Psi(t,x,u) := \rho\big(V(t,x)\big) + a(t,x)u^2 + b(t,x)u + c(t,x) \quad \text{for } (t,x) \in \Omega_0 \tag{6.218}$$

$$\Psi(t,x,u) := \rho\big(V(t,x)\big) + \sup\left\{a(t,x)v^2 + b(t,x)v + c(t,x) : \big|v - k_1(t,x)\big| \le \big|u - k_1(t,x)\big|\right\} \quad \text{for } (t,x) \in \Omega_1 \tag{6.219}$$

Clearly, $\Psi(t,x,u)$ as defined by (6.218) is continuous on the interior of Ω_0. Furthermore, it follows from the continuity of $a(t,x), b(t,x), c(t,x), k_1(t,x)$ on Ω_1 and Theorem 1.4.16 in [7] that $\Psi(t,x,u)$ as defined by (6.219) is upper semi-continuous on Ω_1. The reader should notice that implication (6.211) guarantees that $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ is upper semi-continuous (since $\Psi(t,x,u) \le \rho(V(t,x)) - \frac{b^2(t,x)}{4a(t,x)} + c(t,x)$ for all $(t,x,u) \in \Omega_1 \times \Re$). Consequently, Lemma 6.6 guarantees that property (ii) of Definition 6.2 holds with $\Psi : \Re^+ \times \Re^n \times U \to \Re \cup \{+\infty\}$ as defined by (6.218), (6.219) and that $V : \Re^+ \times \Re^n \to \Re^+$ is an ORCLF for (1.3).

6.6.4 Control Systems Described by RFDEs

In this paragraph we present the extension of the Artstein–Sontag methodology to infinite-dimensional systems described by Retarded Functional Differential Equations (RFDEs). Particularly, we consider control systems of the form (1.10) under hypotheses (S1–5) and the following additional hypothesis:

(SS) The mapping $u \to f(t,d,x,u)$ is Lipschitz on bounded sets, in the sense that there exists a continuous, nondecreasing function $L_U : \Re^+ \to \Re^+$ with the following property:

$$\big|f(t,d,x,u) - f(t,d,x,v)\big| \le L_U\big(t + \|x\|_r + |u| + |v|\big)|u - v|$$
$$\forall (t,x,d,u,v) \in \Re^+ \times C^0\big([-r,0];\Re^n\big) \times D \times U \times U \tag{6.220}$$

Moreover, the set $D \subset \Re^l$ is compact, and $U \subseteq \Re^m$ is a closed convex set.

We next give the definition of the Output Robust Control Lyapunov Functional for system (1.10). The definition is in the same spirit with Definition 6.2 of the notion of ORCLF for finite-dimensional control systems.

Definition 6.3 We say that (1.10) under hypotheses (S1–5) and (SS) admits an *Output Robust Control Lyapunov Functional* (*ORCLF*) if there exists an almost Lipschitz on bounded sets functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, referred to as ORCLF, which satisfies the following properties:

(i) There exist functions $a_1, a_2 \in K_\infty$ and $\beta, \mu \in K^+$ such that the following inequality holds for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$:

$$\max\big\{a_1\big(\big\|H(t,x)\big\|_{\mathcal{Y}}\big), a_1\big(\mu(t)\|x\|_r\big)\big\} \le V(t,x) \le a_2\big(\beta(t)\|x\|_r\big) \tag{6.221}$$

(ii) There exists a function $\Psi : \Re^+ \times \Re^p \times U \to \Re \cup \{+\infty\}$ with $\Psi(t,0,0) = 0$ for all $t \ge 0$ such that for each $u \in U$, the mapping $(t,\varphi) \to \Psi(t,\varphi,u)$ is upper semi-continuous, a function $q \in \mathcal{E}$, a continuous mapping $\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to \Phi(t,x) \in \Re^p$ being completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $\Phi(t,0) = 0$ for all $t \ge 0$, and a C^0 positive definite function $\rho : \Re^+ \to \Re^+$ such that the following inequality holds:

$$\inf_{u \in U} \Psi(t,\varphi,u) \le q(t) \quad \forall t \ge 0,\ \forall \varphi = (\varphi_1,\dots,\varphi_p)' \in \Re^p \tag{6.222}$$

Moreover, for every finite set $\{u_1, u_2, \dots, u_N\} \subset U$ and for all $\lambda_i \in [0,1]$ ($i = 1, \dots, N$) with $\sum_{i=1}^{N} \lambda_i = 1$, it holds that

$$\begin{aligned} &\sup_{d \in D} V^0\left(t, x; f\left(t, d, x, \sum_{i=1}^{N} \lambda_i u_i\right)\right) \\ &\quad \le -\rho\big(V(t,x)\big) \max_{i=1,\dots,N} \big\{\Psi\big(t, \Phi(t,x), u_i\big)\big\} \\ &\quad \forall (t,x) \in \Re^+ \times C^0\big([-r,0]; \Re^n\big) \end{aligned} \tag{6.223}$$

For the case $H(t,x) \equiv x \in C^0([-r,0]; \Re^n)$, we call $V : \Re^+ \times C^0([-r,0]; \Re^n) \to \Re^+$ a State Robust Control Lyapunov Functional (SRCLF).

Remark 6.1 Clearly, in the finite-dimensional case, the continuous mapping $\Phi(t,x) = (\Phi_1(t,x), \dots, \Phi_p(t,x))'$ is replaced by the mapping $\Phi(t,x) := x \in \Re^n$ with $p = n$. The question of the construction of the mapping $\Psi : \Re^+ \times \Re^p \times U \to \Re \cup \{+\infty\}$ can be handled in exactly the same way as shown in the previous section, provided that we can find appropriate continuous mappings $\Phi(t,x) = (\Phi_1(t,x), \dots, \Phi_p(t,x))'$, $G : \Re^+ \times \Re^p \times U \to \Re$ with $\rho(V(t,x)) + \sup_{d \in D} V^0(t, x; f(t,d,x,u)) \le G(t, \Phi(t,x), u)$ for all $(t,x) \in \Re^+ \times C^0([-r,0]; \Re^n)$ and $\inf_{u \in U} G(t, \varphi, u) \le q(t)$ for all $(t, \varphi) \in \Re^+ \times \Re^p$. In this case all constructions of $\Psi : \Re^+ \times \Re^p \times U \to \Re \cup \{+\infty\}$ given in the previous section may be repeated with the quantity $\frac{\partial V}{\partial t}(t,x) + \sup_{d \in D}(\frac{\partial V}{\partial x}(t,x) f(t,d,x,u)) + \rho(V(t,x))$ replaced by the quantity $G(t, \varphi, u)$.

We are now in a position to state and prove our main results for the infinite-dimensional case (1.10).

Theorem 6.4 *Consider system* (1.10) *under hypotheses* (S1–5) *and* (SS). *The following statements are equivalent*:

(a) *There exists a continuous mapping* $k : \Re^+ \times C^0([-r,0]; \Re^n) \ni (t,x) \to k(t,x) \in U$ *being completely locally Lipschitz with respect to* $x \in C^0([-r,0]; \Re^n)$ *with* $k(t,0) = 0$ *for all* $t \ge 0$ *such that the closed-loop system* (1.10) *with* $u = k(t, T_r(t)x)$ *is RGAOS.*
(b) *System* (1.10) *admits an ORCLF.*

Theorem 6.5 *Consider system* (1.10) *under hypotheses* (S1–5) *and* (SS). *The following statements are equivalent*:

(a) *System* (1.10) *admits an ORCLF which satisfies inequalities* (6.221), (6.222) *with* $\beta(t) \equiv 1$ *and* $q(t) \equiv 0$. *Moreover, there exist continuous mappings* $\eta \in K^+$ *and* $A \ni (t, \varphi) \to K(t, \varphi) \in U$, *where* $A = \bigcup_{t \ge 0} \{t\} \times \{\varphi \in \Re^p : |\varphi| < 4\eta(t)\}$, *which is locally Lipschitz with respect to* φ *with* $K(t,0) = 0$ *for all* $t \ge 0$, *such that*

$$\begin{aligned} &\Psi\big(t, \Phi(t,x), K\big(t, \Phi(t,x)\big)\big) \le 0 \\ &\quad \text{for all } (t,x) \in \Re^+ \times C^0\big([-r,0]; \Re^n\big) \text{ with } \big|\Phi(t,x)\big| \le 2\eta(t) \end{aligned} \tag{6.224}$$

where $\Phi = (\Phi_1, \ldots, \Phi_p)' : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^p$ *and* $\Psi : \Re^+ \times \Re^p \times U \to \Re \cup \{+\infty\}$ *are the mappings involved in property* (ii) *of Definition* 6.3.

(b) *There exists a continuous mapping* $k : \Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to k(t,x) \in U$, *completely locally Lipschitz with respect to* $x \in C^0([-r,0];\Re^n)$ *with* $k(t,0) = 0$ *for all* $t \geq 0$, *such that the closed-loop system* (1.10) *with* $u = k(t, T_r(t)x)$ *is URGAOS*.

Remark 6.2 From the proof of Theorem 6.5 it will become apparent that if statement (a) of Theorem 6.5 is strengthened so that the ORCLF V, the mappings $\Phi = (\Phi_1, \ldots, \Phi_p)' : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^p$, Ψ involved in property (ii) of Definition 6.3, and the mapping $K : A \to U$ are time independent, then the continuous mapping k, whose existence is guaranteed by statement (b) of Theorem 6.5, is time invariant. Moreover, the proof of Theorem 6.4 shows that the feedback in statement (a) of Theorem 6.4 is actually a C^∞ function of t and $\Phi(t,x)$; the reader should notice the analogy with Theorem 6.2 (in the finite-dimensional case $\Phi(t,x) := x$ and $p = n$).

Proof of Theorem 6.4 (b) $\Rightarrow$ (a) Suppose that (1.10) admits an ORCLF. Without loss of generality, we may assume that the function $q \in \mathcal{E}$ involved in (6.222) is positive for all $t \geq 0$.

Furthermore, define

$$\Xi(t,\varphi,u) := \Psi(t,\varphi,u) - 8q(t) \quad \text{for } (t,\varphi,u) \in \Re^+ \times \Re^p \times U \tag{6.225}$$

$$\Xi(t,\varphi,u) := \Xi(0,\varphi,u) \quad \text{for } (t,\varphi,u) \in (-1,0) \times \Re^p \times U \tag{6.226}$$

The definition of Ξ, given by (6.225), (6.226), guarantees that the function $\Xi : (-1,+\infty) \times \Re^p \times U \to \Re \cup \{+\infty\}$ with $\Xi(t,0,0) = -8q(\max\{0,t\})$ for all $t > -1$ is such that, for each $u \in U$, the mapping $(t,\varphi) \to \Xi(t,\varphi,u)$ is upper semi-continuous. By virtue of (6.222) and the upper semi-continuity of Ξ, it follows that for each $(t,\varphi) \in (-1,+\infty) \times \Re^p$, there exist $u = u(t,\varphi) \in U$ and $\delta = \delta(t,\varphi) \in (0, t+1)$ such that

$$\begin{aligned} &\Xi\big(\tau, y, u(t,\varphi)\big) \leq 0 \\ &\qquad \forall (\tau,y) \in \big\{(\tau,y) \in (-1,+\infty) \times \Re^p : |\tau - t| + |y - \varphi| < \delta\big\} \end{aligned} \tag{6.227}$$

Using (6.227) and standard partition of unity arguments, we can determine sequences $\{(t_i,\varphi_i) \in (-1,+\infty) \times \Re^p\}_{i=1}^\infty$, $\{u_i \in U\}_{i=1}^\infty$, and $\{\delta_i\}_{i=1}^\infty$ with $\delta_i = \delta(t_i,\varphi_i) \in (0, t_i + 1)$ associated with a sequence of open sets $\{\Omega_i\}_{i=1}^\infty$ with

$$\Omega_i \subseteq \big\{(\tau,y) \in (-1,+\infty) \times \Re^p : |\tau - t_i| + |y - \varphi_i| < \delta_i\big\} \tag{6.228}$$

forming a locally finite open covering of $(-1,+\infty) \times \Re^p$ and such that

$$\Xi(\tau,y,u_i) \leq 0 \quad \forall (\tau,y) \in \Omega_i \tag{6.229}$$

Also, a family of smooth functions $\{\theta_i\}_{i=1}^\infty$ with $\theta_i(t,\varphi) \geq 0$ for all $(t,x) \in (-1,+\infty) \times \Re^p$ can be determined with

$$\operatorname{supp}\theta_i \subseteq \Omega_i \tag{6.230}$$

$$\sum_{i=1}^{\infty} \theta_i(t,\varphi) = 1 \quad \forall (t,\varphi) \in (-1,+\infty) \times \Re^p \tag{6.231}$$

Using exactly the same methodology as in the proof of Theorem 6.2 and the facts that $\Xi(t,0,0) = -8q(t) < 0$ for all $t \geq 0$ and that the mapping $(t,\varphi) \to \Xi(t,\varphi,0)$ is upper semi-continuous, we may establish the existence of a C^∞ positive function $\eta : \Re^+ \to (0,+\infty)$ with the following property:

$$|\varphi| \leq 2\eta(t) \quad \Rightarrow \quad \Xi(t,\varphi,0) \leq 0 \tag{6.232}$$

Let $h \in C^\infty(\Re;[0,1])$ be a smooth nondecreasing function with $h(s) = 0$ for all $s \leq 0$ and $h(s) = 1$ for all $s \geq 1$. We define, for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$,

$$k(t,x) := h\left(\frac{|\Phi(t,x)|^2 - 2\eta^2(t)}{2\eta^2(t)}\right) \sum_{i=1}^{\infty} \theta_i\big(t,\Phi(t,x)\big)u_i \tag{6.233}$$

where $\Phi = (\Phi_1,\ldots,\Phi_p)' : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^p$ is the mapping involved in property (ii) of Definition 6.3. Clearly, k as defined by (6.233) is a mapping satisfying the property that for every bounded $\Omega \subset \Re^+ \times C^0([-r,0];\Re^n)$, it holds that

$$\sup\left\{\frac{|k(t,x)-k(t,y)|}{\|x-y\|_r} : (t,x) \in \Omega, (t,y) \in \Omega, x \neq y\right\} < +\infty$$

with $k(t,0) = 0$ for all $t \geq 0$. Moreover, since $k(t,x)$ is defined as a (finite) convex combination of $u_i \in U$ and $0 \in U$, we have $k(t,x) \in U$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$.

Let $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ with $|\Phi(t,x)| \geq 2\eta(t)$ and define the finite set $J(t,x) = \{j \in \{1,2,\ldots\}; \theta_j(t,\Phi(t,x)) \neq 0\}$. By virtue of (6.223) and definition (6.233), we get

$$\begin{aligned}
&\sup_{d\in D} V^0\big(t,x;f\big(t,d,x,k(t,x)\big)\big) \\
&\quad = \sup_{d\in D} V^0\left(t,x;f\left(t,d,x,\sum_{j\in J(t,x)} \theta_j\big(t,\Phi(t,x)\big)u_j\right)\right) \\
&\quad \leq -\rho\big(V(t,x)\big) + \max_{j\in J(t,x)}\big\{\Psi\big(t,\Phi(t,x),u_j\big)\big\}
\end{aligned} \tag{6.234}$$

Notice that for each $j \in J(t,x)$, we obtain from (6.230) that $(t,\Phi(t,x)) \in \Omega_j$. Consequently, by virtue of (6.229) and definition (6.225), we have that $\Psi(t,\Phi(t,x),u_j) \leq 8q(t)$ for all $j \in J(t,x)$. Combining the previous inequality with inequality (6.234), we conclude that the following property holds for all $(t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D$ with $|\Phi(t,x)| \geq 2\eta(t)$:

$$V^0\big(t,x;f\big(t,d,x,k(t,x)\big)\big) \leq -\rho\big(V(t,x)\big) + 8q(t) \tag{6.235}$$

Let $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ with $|\Phi(t,x)| \leq \sqrt{2}\eta(t)$. Notice that by definition (6.233) we get

$$\sup_{d\in D} V^0\big(t,x;f\big(t,d,x,k(t,x)\big)\big) = \sup_{d\in D} V^0\big(t,x;f(t,d,x,0)\big)$$

By virtue of (6.223), (6.232), (6.225) and the above inequality, we conclude that (6.235) holds as well for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ with $|\Phi(t,x)| \le \sqrt{2}\eta(t)$. Finally, for the case $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ with $\sqrt{2}\eta(t) < |\Phi(t,x)| < 2\eta(t)$, let $J(t,x) = \{j \in \{1,2,\dots\}; \theta_j(t,\Phi(t,x)) \ne 0\}$ and notice that from (6.223) we obtain

$$\begin{aligned} &\sup_{d\in D} V^0\big(t,x;\, f\big(t,d,x,k(t,x)\big)\big) \\ &\quad = \sup_{d\in D} V^0\left(t,x;\, f\left(t,d,x,h\left(\frac{|x|^2-2\eta^2(t)}{2\eta^2(t)}\right)\sum_{j\in J(t,x)}\theta_j(t,x)u_j\right)\right) \\ &\quad \le -\rho\big(V(t,x)\big) + \max\big\{\Psi\big(t,\Phi(t,x),0\big), \Psi\big(t,\Phi(t,x),u_j\big), j \in J(t,x)\big\} \end{aligned} \tag{6.236}$$

Taking into account definition (6.225) and (6.229), (6.230), (6.232), and (6.236), we may conclude that (6.235) holds as well for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ with $\sqrt{2}\eta(t) < |\Phi(t,x)| < 2\eta(t)$. Consequently, (6.235) holds for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$.

It follows from (6.235) and Theorem 2.5 in Chap. 2 that system (1.10) with $u = k(t, T_r(t)x)$ is RGAOS.

(a) $\Rightarrow$ (b) Since system (1.10) with $u = k(t, T_r(t)x)$ is RGAOS, and since for every bounded $\Omega \subset \Re^+ \times C^0([-r,0];\Re^n)$, we have

$$\sup\left\{\frac{|k(t,x)-k(t,y)|}{\|x-y\|_r} : (t,x) \in \Omega, (t,y) \in \Omega, x \ne y\right\} < +\infty,$$

it follows that the closed-loop system (1.10) with $u = k(t, T_r(t)x)$ satisfies hypotheses (S1–5). Moreover, since system (1.10) with $u = k(t, T_r(t)x)$ is RGAOS, it follows from Theorem 2.1 in Chap. 2 that there exists $\mu \in K^+$ such that the system

$$\begin{aligned} &\dot{x}(t) = f\big(t, d(t), T_r(t)x, k\big(t, T_r(t)x\big)\big) \\ &Y(t) = \big\|H(t,x)\big\|_{\mathcal{Y}} + \mu(t)\|x\|_r \\ &x(t) \in \Re^n, Y(t) \in \Re, d(t) \in D \end{aligned} \tag{6.237}$$

satisfies hypotheses (S1–5) and is RGAOS. Notice that system (6.237) is the closed-loop system (1.10) with $u = k(t, T_r(t)x)$ and output defined by $Y(t) = \|H(t,x)\|_{\mathcal{Y}} + \mu(t)\|x\|_r$. It follows from Theorem 3.6 in Chap. 3 that there exist functions $a_1, a_2 \in K_\infty$ and $\beta \in K^+$ and a mapping $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, such that, for all $(t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D$,

$$a_1\big(\big\|H(t,x)\big\|_{\mathcal{Y}} + \mu(t)\|x\|_r\big) \le V(t,x) \le a_2\big(\beta(t)\|x\|_r\big) \tag{6.238}$$

$$V^0\big(t,x;\, f\big(t,d,x,k(t,x)\big)\big) \le -V(t,x) \tag{6.239}$$

We next prove that V is an ORCLF for (1.10). Obviously property (i) of Definition 6.3 is a consequence of inequality (6.238). Define, for all $(t,\varphi) = (t,\varphi_1,\varphi_2)' \in \Re^+ \times \Re \times \Re^m$,

$$\begin{aligned}&\Psi(t,\varphi_1,\varphi_2,u)\\&\quad:=L_V\big(t+|\varphi_1|\big)L_U\big(t+|\varphi_1|+2|\varphi_2|+\big|u-\mathrm{Pr}_U(\varphi_2)\big|\big)\big|u-\mathrm{Pr}_U(\varphi_2)\big|\end{aligned}\tag{6.240}$$

and, for all $(t,x)\in\Re^+\times C^0([-r,0];\Re^n)$,

$$\Phi(t,x):=\begin{bmatrix}\|x\|_r\\k(t,x)\end{bmatrix}\in\Re^{m+1}\tag{6.241}$$

where $L_U:\Re^+\to\Re^+$ is the nondecreasing continuous function involved in (6.220), and $L_V:\Re^+\to\Re^+$ is a continuous, nondecreasing function that satisfies

$$\big|V(t,y)-V(t,x)\big|\le L_V\big(t+\|y\|_r+\|x\|_r\big)\|y-x\|_r$$

for all $(t,x)\in\Re^+\times C^0([-r,0];\Re^n)$. The reader should notice that $\Phi(t,0)=0$ for all $t\ge 0$ and that for every bounded $\Omega\subset\Re^+\times C^0([-r,0];\Re^n)$, it holds that

$$\sup\left\{\frac{|\Phi(t,x)-\Phi(t,y)|}{\|x-y\|_r}:(t,x)\in\Omega,(t,y)\in\Omega,x\ne y\right\}<+\infty.$$

The convexity of the set $U\subseteq\Re^m$ implies that the mapping $\Re^m\ni\varphi_2\to\mathrm{Pr}_U(\varphi_2)$ is continuous and consequently that the mapping $\Re^+\times\Re^{m+1}\ni(t,\varphi_1,\varphi_2)\to\Psi(t,\varphi_1,\varphi_2,u)$ is continuous for each fixed $u\in U$. Notice that for every $\varphi=(t,\varphi_1,\varphi_2)'\in\Re^+\times\Re\times\Re^m$ and every finite set $\{u_1,u_2,\dots,u_N\}\subset U$, $\lambda_i\in[0,1]$ $(i=1,\dots,N)$ with $\sum_{i=1}^N\lambda_i=1$, definition (6.240), in conjunction with the inequalities

$$\left|\sum_{i=1}^N\lambda_iu_i-\mathrm{Pr}_U(\varphi_2)\right|\le\sum_{i=1}^N\lambda_i\big|u_i-\mathrm{Pr}_U(\varphi_2)\big|\le\max_{i=1,\dots,N}\big|u_i-\mathrm{Pr}_U(\varphi_2)\big|$$

implies that

$$\Psi\left(t,\varphi_1,\varphi_2,\sum_{i=1}^N\lambda_iu_i\right)\le\max_{i=1,\dots,N}\Psi(t,\varphi_1,\varphi_2,u_i)\tag{6.242}$$

By virtue of the definition of the Dini derivative $V^0(t,x;v)$ and the properties of the Lyapunov functional, we get, for all $(t,x,v,w)\in\Re^+\times C^0([-r,0];\Re^n)\times\Re^n\times\Re^n$,

$$V^0(t,x;v)\le L_V\big(t+\|x\|_r\big)|v-w|+V^0(t,x;w)\tag{6.243}$$

Combining inequalities (6.220), (6.239), and (6.243), we obtain, for all $(t,x,u)\in\Re^+\times C^0([-r,0];\Re^n)\times U$,

$$\begin{aligned}&\sup_{d\in D}V^0\big(t,x;f(t,d,x,u)\big)\\&\quad\le\sup_{d\in D}V^0\big(t,x;f\big(t,d,x,k(t,x)\big)\big)\\&\qquad+L_V\big(t+\|x\|_r\big)\sup_{d\in D}\big|f(t,d,x,u)-f\big(t,d,x,k(t,x)\big)\big|\\&\quad\le -V(t,x)+L_V\big(t+\|x\|_r\big)L_U\big(t+\|x\|_r+|u|+\big|k(t,x)\big|\big)\big|u-k(t,x)\big|\\&\quad\le -V(t,x)+L_V\big(t+\|x\|_r\big)L_U\big(t+\|x\|_r+\big|u-k(t,x)\big|\\&\qquad+2\big|k(t,x)\big|\big)\big|u-k(t,x)\big|\end{aligned}$$

The above inequality, in conjunction with (6.242) and definitions (6.240), (6.241), implies that inequality (6.223) with $\rho(s) := s$ holds. Moreover, by definition (6.240), for every $(t, \varphi_1, \varphi_2) \in \Re^+ \times \Re \times \Re^m$, there exists $u \in U$ (namely $u = \mathrm{Pr}_U(\varphi_2)$) such that (6.222) holds with $q(t) \equiv 0$. The proof is complete. □

Proof of Theorem 6.5 (a) ⇒ (b) Suppose that (1.10) admits an ORCLF which satisfies (6.222) with $q(t) \equiv 0$. Without loss of generality we may assume that the mapping $\Phi = (\Phi_1, \dots, \Phi_p)' : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re^p$ involved in property (ii) of Definition 6.3 satisfies $\Phi_1(t, x) := V(t, x)$. Define

$$\Xi(t, \varphi, u) := \Psi(t, \varphi, u) - \frac{1}{2}\rho\big(c'\varphi\big) \quad (t, \varphi, u) \in \Re^+ \times \Re^p \times U \tag{6.244}$$

$$\Xi(t, \varphi, u) := \Xi(0, \varphi, u) \quad (t, \varphi, u) \in (-1, 0) \times \Re^p \times U \tag{6.245}$$

$$c = (1, 0, \dots, 0)' \in \Re^p \tag{6.246}$$

The definition of Ξ, given by (6.244)–(6.246), guarantees that the function $\Xi : (-1, +\infty) \times \Re^p \times U \to \Re \cup \{+\infty\}$ with $\Xi(t, 0, 0) = 0$ for all $t > -1$ is such that, for each $u \in U$, the mapping $(t, \varphi) \to \Xi(t, \varphi, u)$ is upper semi-continuous. Let $\Theta := (-1, +\infty) \times \{\varphi \in \Re^p : c'\varphi \neq 0\}$, which is an open set. By virtue of (6.222) with $q(t) \equiv 0$ and the upper semi-continuity of Ξ, it follows that for each $(t, \varphi) \in \Theta$, there exist $u = u(t, \varphi) \in U$ with $\delta = \delta(t, \varphi) \in (0, \min(1, t+1))$, $\delta(t, \varphi) \leq \frac{|c'\varphi|}{2}$, such that

$$\Xi\big(\tau, y, u(t, \varphi)\big) \leq 0 \quad \forall (\tau, y) \in \big\{(\tau, y) \in \Theta : |\tau - t| + |y - \varphi| < \delta\big\} \tag{6.247}$$

Using (6.247) and standard partition of unity arguments, we can determine sequences $\{(t_i, \varphi_i) \in \Theta\}_{i=1}^{\infty}$, $\{u_i \in U\}_{i=1}^{\infty}$, and $\{\delta_i\}_{i=1}^{\infty}$ with $\delta_i = \delta(t_i, \varphi_i) \in (0, \min(1, t_i + 1))$, $\delta_i = \delta(t_i, \varphi_i) \leq \frac{|c'\varphi_i|}{2}$, associated with a sequence of open sets $\{\Omega_i\}_{i=1}^{\infty}$ with

$$\Omega_i \subseteq \big\{(\tau, y) \in \Theta : |\tau - t_i| + |y - \varphi_i| < \delta_i\big\} \tag{6.248}$$

forming a locally finite open covering of Θ and such that

$$\Xi(\tau, y, u_i) \leq 0 \quad \forall (\tau, y) \in \Omega_i \tag{6.249}$$

Also, a family of smooth functions $\{\theta_i\}_{i=1}^{\infty}$ with $\theta_i(t, \varphi) \geq 0$ for all $(t, \varphi) \in \Theta$ can be determined with

$$\operatorname{supp}\theta_i \subseteq \Omega_i \tag{6.250}$$

$$\sum_{i=1}^{\infty} \theta_i(t, \varphi) = 1 \quad \forall (t, \varphi) \in \Theta \tag{6.251}$$

We define, for all $(t, x) \in \Re^+ \times C^0([-r, 0]; \Re^n)$,

$$\begin{aligned} k(t, x) := {} & \left(1 - h\left(\frac{|\Phi(t, x)|^2 - 2\eta^2(t)}{2\eta^2(t)}\right)\right) K\big(t, \mathrm{Pr}_{Q(t)}\big(\Phi(t, x)\big)\big) \\ & + h\left(\frac{|\Phi(t, x)|^2 - 2\eta^2(t)}{2\eta^2(t)}\right)\tilde{k}(t, x) \end{aligned} \tag{6.252}$$

where

$$\tilde{k}(t,x) := \sum_{i=1}^{\infty} \theta_i\big(t, \Phi(t,x)\big)u_i \quad \text{for } t \geq 0, x \neq 0 \tag{6.253}$$

$$\tilde{k}(t,0) := 0 \quad \text{for } t \geq 0 \tag{6.254}$$

where $h \in C^{\infty}(\Re; [0,1])$ is a smooth nondecreasing function with $h(s) = 0$ for all $s \leq 0$ and $h(s) = 1$ for all $s \geq 1$, and $Q(t) = \{\varphi \in \Re^p : |\varphi| \leq 3\eta(t)\}$. It follows from definition (6.252), (6.253), (6.254) and the facts that the continuous mapping Φ is completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ and that the continuous mapping $\varphi \to K(t,\varphi)$ is locally Lipschitz that k is completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $k(t,0) = 0$ for all $t \geq 0$. Moreover, it should be noticed that, if the ORCLF V and the function Ψ involved in property (ii) of Definition 6.3 are time independent, then the partition of unity arguments used above may be repeated on $\Theta := \{\varphi \in \Re^p : c'\varphi \neq 0\}$ instead of $\Theta := (-1,+\infty) \times \{\varphi \in \Re^p : c'\varphi \neq 0\}$. This implies that the constructed feedback is time invariant, provided that the mappings $\Phi = \Phi_1, \ldots, \Phi_p)' : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^p$ and $K : A \to U$ are time independent too.

Exploiting the properties of the mappings $\Xi : (-1,+\infty) \times \Re^p \times U \to \Re \cup \{+\infty\}$ and $\Psi : \Re^+ \times \Re^p \times U \to \Re \cup \{+\infty\}$, inequalities (6.224), (6.249), definitions (6.244), (6.252), and the fact that the mapping $\Phi = (\Phi_1, \ldots, \Phi_p)' : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^p$ involved in property (ii) of Definition 6.3 satisfies $\Phi_1(t,x) := V(t,x)$, we may establish (exactly in the same way as in the proof of Theorem 6.4) the following inequality:

$$V^0\big(t,x; f\big(t,d,x,k(t,x)\big)\big) \leq -\frac{1}{2}\rho\big(V(t,x)\big)$$
$$\forall (t,x,d) \in \Re^+ \times \Re^n \times D \tag{6.255}$$

The fact that system (1.10) with $u = k(t, T_r(t)x)$ is URGAOS follows directly from Theorem 2.5 in Chap. 2 and inequality (6.255).

(b) $\Rightarrow$ (a) Since for every bounded $\Omega \subset \Re^+ \times C^0([-r,0];\Re^n)$,

$$\sup\left\{\frac{|k(t,x)-k(t,y)|}{\|x-y\|_r} : (t,x) \in \Omega,\ (t,y) \in \Omega,\ x \neq y\right\} < +\infty$$

we have that the closed-loop system (1.10) with $u = k(t, T_r(t)x)$ satisfies hypotheses (S1–5). For each $(t_0, x_0, d) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D$, we denote by $x(t,t_0,x_0,d)$ the solution of (1.10) with $u = k(t, T_r(t)x)$ and initial condition $T_r(t_0)x = x_0$ corresponding to $d \in M_D$.

Since system (1.10) with $u = k(t, T_r(t)x)$ is RFC, it follows from Lemma 2.7 in Chap. 2 that there exist $\tilde{\mu}, \beta \in K^+$ and $\tilde{a} \in K_\infty$ such that the following inequality holds for all $(t_0, x_0, d) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D$:

$$\tilde{\mu}(t)\|x\|_r \leq a\big(\beta(t_0)\|x_0\|_r\big) \quad \forall t \geq t_0 \tag{6.256}$$

Using Lemma 3.2 in Chap. 3, we can conclude that there exist $\tilde{\mu}, \tilde{\beta} \in K^+$ and $\tilde{a} \in K_\infty$ such that the following inequality holds for all $(t_0, x_0, d) \in \Re^+ \times$

$C^0([-r,0];\Re^n) \times M_D$:

$$\tilde{\mu}(t)\|x\|_r \le \tilde{\beta}(t_0)\tilde{a}\big(\|x_0\|_r\big) \quad \forall t \ge t_0 \tag{6.257}$$

Without loss of generality, we may assume that $\tilde{\beta} \in K^+$ is nondecreasing.

Since system (1.10) with $u = k(t, T_r(t)x)$ is URGAOS, it follows from (6.257) that the system

$$\begin{aligned} \dot{x}(t) &= f\big(t, d(t), T_r(t)x, k\big(t, T_r(t)x\big)\big) \\ Y(t) &= \big\|H(t,x)\big\|_{\mathcal{Y}} + \mu(t)\|x\|_r \\ &x(t) \in \Re^n, Y(t) \in \Re, d(t) \in D \end{aligned} \tag{6.258}$$

where $\mu(t) := \exp(-t)\frac{\tilde{\mu}(t)}{\tilde{\beta}(t)}$, satisfies hypotheses (S1–5) and is URGAOS. Notice that system (6.258) is the closed-loop system (1.10) with $u = k(t, T_r(t)x)$ and output defined by $Y(t) = \|H(t,x)\|_{\mathcal{Y}} + \mu(t)\|x\|_r$. It follows from Theorem 3.6 in Chap. 3 that there exist functions $a_1, a_2 \in K_\infty$ and a mapping $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, such that

$$a_1\big(\big\|H(t,x)\big\|_{\mathcal{Y}} + \mu(t)\|x\|_r\big) \le V(t,x) \le a_2\big(\|x\|_r\big) \tag{6.259}$$

$$V^0\big(t,x; f\big(t,d,x,k(t,x)\big)\big) \le -V(t,x) \tag{6.260}$$

for all $(t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D$. The rest of the proof is exactly the same as the proof of implication (a) $\Rightarrow$ (b) of Theorem 6.4. The only additional thing that should be noticed is that the mappings $\Phi = (\Phi_1,\dots,\Phi_p)' : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^p$, $\eta \in K^+$, and $A \ni (t,\varphi) \to K(t,\varphi) \in U$ may be selected so that (6.241) holds, $\eta(t) \equiv 1$, and $K(t,\varphi) := \mathrm{Pr}_U(\varphi_2)$ for all $(t,\varphi) = (t,\varphi_1,\varphi_2)' \in \Re^+ \times \Re \times \Re^m$.

The proof is complete. □

6.6.5 Finite-Dimensional Discrete-Time Systems

In this section we consider discrete-time systems of the form (4.56) under hypotheses (L1–5) with $\mathcal{X} = \Re^n$, $\mathcal{Y} = \Re^k$, and $\mathcal{U} = \Re^m$, for which the control set $U \subseteq \mathcal{U} = \Re^m$ is a closed convex set.

We next give the definition of the notion of Output Robust Control Lyapunov Function for discrete-time systems. The definition is in the same spirit with the definition of the notion of Output Robust Control Lyapunov Functions for continuous-time control systems.

Definition 6.4 We say that (4.56) under hypotheses (L1–5) with $\mathcal{X} = \Re^n$, $\mathcal{Y} = \Re^k$, and $\mathcal{U} = \Re^m$, for which the control set $U \subseteq \mathcal{U} = \Re^m$ is a closed convex set, admits an *Output Robust Control Lyapunov Function* (*ORCLF*) if there exists a continuous function $V : Z^+ \times \Re^n \to \Re^+$ (called the Output Robust Control Lyapunov Function) which satisfies the following properties:

(i) There exist $a_1, a_2 \in K_\infty$ and $\beta \in K^+$ such that

$$a_1\big(\big|H(t,x)\big|\big) \le V(t,x) \le a_2\big(\beta(t)|x|\big) \quad \forall (t,x) \in Z^+ \times \Re^n. \tag{6.261}$$

(ii) There exist functions $\rho \in K$ with $\rho(s) \le s$ for all $s \ge 0$, $q \in C^0(Z^+; \Re^+)$ with $\lim_{t\to+\infty} q(t) = 0$, and an upper semi-continuous function $\Psi : Z^+ \times \Re^n \times U \to \Re$ with $\Psi(t, 0, 0) = 0$ for all $t \in Z^+$, which is quasi-convex with respect to $u \in U$, such that the following inequalities hold:

$$\sup_{d\in D} V\big(t+1, f(t,d,x,u)\big) \le \Psi(t,x,u) \quad \forall (t,x,u) \in Z^+ \times \Re^n \times U \tag{6.262}$$

$$\inf_{u\in U} \Psi(t,x,u) \le V(t,x) - \rho\big(V(t,x)\big) + q(t) \quad \forall (t,x) \in Z^+ \times \Re^n \tag{6.263}$$

In the case $H(t,x) \equiv x$, we simply call $V : Z^+ \times \Re^n \to \Re^+$ a State Robust Control Lyapunov Function (SRCLF).

Remark 6.3 If the mapping $u \to V(t+1, f(t,d,x,u))$ is (quasi-)convex for each fixed $(t,x,d) \in Z^+ \times \Re^n \times D$, then the mapping $u \to \sup_{d\in D} V(t+1, f(t,d,x,u))$ is (quasi-)convex for each fixed $(t,x) \in Z^+ \times \Re^n$. It follows that property (ii) of Definition 6.4 is satisfied with $\Psi(t,x,u) := \sup_{d\in D} V(t+1, f(t,d,x,u))$.

The following results show how the existence of an ORCLF is related to the existence of a stabilizing feedback for (4.56).

Proposition 6.1 *Consider system* (4.56) *under hypotheses* (L1–5) *with* $\mathcal{X} = \Re^n$, $\mathcal{Y} = \Re^k$, *and* $\mathcal{U} = \Re^m$, *for which the control set* $U \subseteq \mathcal{U} = \Re^m$ *is a closed convex set. Then the following statements are equivalent*:

(a) *System* (4.56) *admits an ORCLF.*
(b) *There exists* $k \in C^\infty(Z^+ \times \Re^n; U)$ *such that the closed-loop system* (4.56) *with* $u = k(t,x)$ *is RGAOS.*
(c) *There exists* $k \in C^0(Z^+ \times \Re^n; U)$ *such that the closed-loop system* (4.56) *with* $u = k(t,x)$ *is RGAOS.*

Proposition 6.2 *Consider system* (4.56) *under hypotheses* (L1–5) *with* $\mathcal{X} = \Re^n$, $\mathcal{Y} = \Re^k$, *and* $\mathcal{U} = \Re^m$, *for which the control set* $U \subseteq \mathcal{U} = \Re^m$ *is a closed convex set. Then the following statements are equivalent*:

(a) *System* (4.56) *admits an ORCLF, which satisfies* (6.261) *with* $\beta(t) \equiv 1$ *and* (6.263) *with* $q(t) \equiv 0$. *Moreover, there exist mappings* $\eta \in K^+$ *and* $K \in C^\nu(A; U)$, *where* $\nu \in Z^+$ *and* $A = \bigcup_{t=0,1,\dots} \{t\} \times \{x \in \Re^n : |x| < 4\eta(t)\}$, *with* $K(t,0) = 0$ *for all* $t \ge 0$ *and such that*

$$\Psi\big(t,x,K(t,x)\big) \le V(t,x) - \rho\big(V(t,x)\big)$$
$$\textit{for all } (t,x) \in Z^+ \times \Re^n \textit{ with } |x| \le 2\eta(t) \tag{6.264}$$

where $\Psi : \Re^+ \times \Re^n \times U \to \Re$ *is the mapping in property* (ii) *of Definition* 6.4.
(b) *There exist* $k \in C^\nu(Z^+ \times \Re^n; U)$, $\nu \in Z^+$, *such that the closed-loop system* (4.56) *with* $u = k(t,x)$ *is RGAOS.*

From the proof of Proposition 6.2 it will become apparent that if statement (a) of Proposition 6.2 is strengthened so that the ORCLF V, the mappings Ψ involved in property (ii) of Definition 6.4, and the mapping $K : A \to U$ are time independent, then the continuous mapping k, whose existence is guaranteed by statement (b) of Proposition 6.2, is time invariant.

Proof of Proposition 6.1 (a) ⇒ (b) Suppose that (4.56) admits an ORCLF. Without loss of generality, we may assume that the function $q \in C^0(Z^+; \Re^+)$ with $\lim_{t\to+\infty} q(t) = 0$ involved in (6.263) is positive for all $t \in Z^+$. We proceed by noticing some facts.

Fact I *For all* $(t, x_0) \in Z^+ \times \Re^n$, *there exist* $u_0 \in U$ *and a neighborhood* $\mathbf{N}(t, x_0) \subset \Re^n$ *such that*

$$x \in \mathbf{N}(t, x_0) \quad \Rightarrow \quad \Psi(t, x, u_0) \le V(t, x) - \rho\big(V(t, x)\big) + 4q(t) \tag{6.265}$$

Moreover, if $x_0 = 0$, *then we may select* $u_0 = 0$.

Proof of Fact I By virtue of (6.263) and since $q(t) > 0$ for all $t \in Z^+$, it follows that for all $(t, x_0) \in Z^+ \times \Re^n$, there exists $u_0 \in U$ such that

$$\Psi(t, x_0, u_0) \le V(t, x_0) - \rho\big(V(t, x_0)\big) + 2q(t) \tag{6.266}$$

If $x_0 = 0$ (and since $\Psi(t, 0, 0) = 0$ for all $t \in Z^+$), then we may select $u_0 = 0$. Since the mapping $x \to \Psi(t, x, u)$ is upper semi-continuous and the mapping $x \in \Re^n \to V(t, x) - \rho(V(t, x))$ is continuous, there exists a neighborhood $\mathbf{N}(t, x_0) \subset \Re^n$ around x_0 such that, for all $x \in \mathbf{N}(t, x_0)$,

$$\begin{aligned} &\Psi(t, x, u_0) \le \Psi(t, x_0, u_0) + q(t) \\ &V(t, x_0) - \rho\big(V(t, x_0)\big) \le V(t, x) - \rho\big(V(t, x)\big) + q(t) \end{aligned} \tag{6.267}$$

Inequalities (6.266) and (6.267) imply property (6.265).

Fact II *For each fixed* $t \in Z^+$, *there exists a family of open sets* $(\Omega_j^{(t)})_{j\in J(t)}$ *with* $\Omega_j^{(t)} \subset \Re^n\backslash\{0\}$ *for all* $j \in J(t)$, *which is a locally finite open covering of* $\Re^n\backslash\{0\}$, *and a family of points* $(u_j^{(t)})_{j\in J(t)}$ *with* $u_j^{(t)} \in U$ *for all* $j \in J(t)$ *such that*

$$x \in \Omega_j^{(t)} \quad \Rightarrow \quad \Psi\big(t, x, u_j^{(t)}\big) \le V(t, x) - \rho\big(V(t, x)\big) + 4q(t) \tag{6.268}$$

This fact is an immediate consequence of Fact I.

By virtue of Fact II and standard partition of unity arguments, it follows that for each fixed $t \in Z^+$, there exists a family of smooth functions $\theta_0^{(t)} : \Re^n \to [0, 1]$, $\theta_j^{(t)} : \Re^n \to [0, 1]$, with $\theta_j^{(t)}(x) = 0$ if $x \notin \Omega_j^{(t)} \subset \Re^n\backslash\{0\}$ and $\theta_0^{(t)}(x) = 0$ if $x \notin \mathbf{N}(t, 0)$, where $\mathbf{N}(t, 0) \subset \Re^n$ is the neighborhood provided by Fact I for $x_0 = 0$ and $u_0 = 0$, $\theta_0^{(t)}(x) + \sum_{j\in J(t)} \theta_j^{(t)}(x)$ being locally finite and such that $\theta_0^{(t)}(x) + \sum_j \theta_j^{(t)}(x) = 1$ for all $x \in \Re^n$. We define, for each fixed $t \in Z^+$,

$$k(t, x) := \sum_{j\in J(t)} \theta_j^{(t)}(x) u_j^{(t)} \tag{6.269}$$

Clearly, k as defined by (6.269) is a smooth function with $k(t,x)\in U$ (since U is convex and $k(t,x)$ is equal to a convex combination of $u_j^{(t)}\in U$ and $u_0=0\in U$). Notice that $0\notin \Omega_j^{(t)}$ for all $j\in J(t)$, and consequently by definition (6.269) we have $k(t,0)=0$ for all $t\in Z^+$. From the fact that Ψ is quasi-convex with respect to $u\in U$ and from definition (6.269) it also follows that

$$\Psi\big(t,x,k(t,x)\big)=\Psi\left(t,x,\sum_{j\in J'(t,x)}\theta_j^{(t)}(x)u_j^{(t)}\right)\le \max_{j\in J'(t,x)}\big\{\Psi\big(t,x,u_j^{(t)}\big)\big\} \quad (6.270)$$

where $J'(t,x)=\{j\in J(t)\cup\{0\};\theta_j^{(t)}(x)\neq 0\}$ is a finite set. For each $j\in J'(t,x)$, we obtain that $x\in\Omega_j^{(t)}$ or $x\in \mathbf{N}(t,0)$. Consequently, by (6.268) or (6.265) we have that $\Psi(t,x,u_j^{(t)})\le V(t,x)-\rho(V(t,x))+4q(t)$ for all $j\in J'(t,x)$. Combining the previous inequality with inequality (6.262), we conclude that the following property holds for all $(t,x,d)\in Z^+\times\Re^n\times D$:

$$\begin{aligned}V\big(t+1,f\big(t,d,x,k(t,x)\big)\big)&\le\Psi\big(t,x,k(t,x)\big)\\&\le V(t,x)-\rho\big(V(t,x)\big)+4q(t)\end{aligned} \quad (6.271)$$

It follows from (6.271) and Proposition 2.3 in Chap. 2 that system (4.56) with $u=k(t,x)$ is RGAOS.

(b) $\Rightarrow$ (c) Trivial.

(c) $\Rightarrow$ (a) Since system (4.56) with $u=k(t,x)$ is RGAOS and satisfies hypotheses (L1–4), it follows from Proposition 3.1 in Chap. 3 that there exists a continuous $V:Z^+\times\Re^n\to\Re^+$ and functions $a_1,a_2\in K_\infty$ and $\beta\in K^+$ such that inequality (6.261) holds and, for all $(t,x,d)\in Z^+\times\Re^n\times D$,

$$V\big(t+1,f\big(t,d,x,k(t,x)\big)\big)\le\exp(-1)V(t,x) \quad (6.272)$$

We next prove that V is an ORCLF for (4.56). Obviously, property (i) of Definition 6.4 holds. Define

$$\begin{aligned}\Psi(t,x,u):=\sup\big\{&V(t+1,F(t,x,d,v);d\in D,v\in U,\big|v-k(t,x)\big|\\&\le\big|u-k(t,x)\big|\big\}\end{aligned} \quad (6.273)$$

Inequalities (6.262), (6.263) with $\rho(s):=(1-\exp(-1))s$ and $q(t)\equiv 0$ are immediate consequences of inequality (6.272) and definition (6.273). Finally, we prove that the function Ψ as defined by (6.273) is quasi-convex with respect to $u\in U$. Notice that the continuous maps $f:Z^+\times D\times\Re^n\times U\to\Re^n$, $k:Z^+\times\Re^n\to U$, and $V:Z^+\times\Re^n\to\Re^+$ can be continuously extended to $f:\Re\times D\times\Re^n\times U\to\Re^n$, $k:\Re\times\Re^n\to U$, and $V:\Re\times\Re^n\to\Re^+$, respectively. Under hypothesis (L5), it follows from the compactness of $D\subset\Re^l$, the continuity of $f:\Re^+\times D\times\Re^n\times U\to\Re^n$, $k:\Re^+\times\Re^n\to U$, and $V:\Re^+\times\Re^n\to\Re^+$, and Theorem 1.4.16 in [4] that the function Ψ as defined by (6.273) is continuous. Clearly, definition (6.273) implies $\Psi(t,0,0)=0$ for all $t\in Z^+$. Let $(t,x)\in Z^+\times\Re^n$, $u_1,u_2\in U$, and $\lambda\in[0,1]$. Then (6.273) implies

$$\begin{aligned}
&\Psi\big(t,x,\lambda u_1+(1-\lambda)u_2\big)\\
&\quad\le \sup\Big\{V\big(t+1,F(t,x,d,v);d\in D,v\in U\big),\\
&\qquad\qquad \big|v-k(t,x)\big|\le\lambda\big|u_1-k(t,x)\big|+(1-\lambda)\big|u_2-k(t,x)\big|\Big\}\\
&\quad\le \sup\Big\{V\big(t+1,F(t,x,d,v)\big);d\in D,v\in U,\\
&\qquad\qquad |v|\le\max\Big\{\big|u_1-k(t,x)\big|,\big|u_2-k(t,x)\big|\Big\}\Big\}
\end{aligned}$$

If $\max\{|u_1-k(t,x)|,|u_2-k(t,x)|\}=|u_1-k(t,x)|$, then the above inequality implies $\Psi(t,x,\lambda u_1+(1-\lambda)u_2)\le\Psi(t,x,u_1)$.

Similarly, if $\max\{|u_1-k(t,x)|,|u_2-k(t,x)|\}=|u_2-k(t,x)|$, then $\Psi(t,x,\lambda u_1+(1-\lambda)u_2)\le\Psi(t,x,u_2)$. Thus, in any case, it holds that $\Psi(t,x,\lambda u_1+(1-\lambda)u_2)\le\max\{\Psi(t,x,u_1),\Psi(t,x,u_2)\}$. The proof is complete. □

Proof of Proposition 6.2 (a) ⇒ (b) Suppose that (4.56) admits an ORCLF which satisfies (6.261) with $\beta(t)\equiv 1$ and (6.263) with $q(t)\equiv 0$. We proceed by noticing some facts.

Fact I *For all* $(t,x_0)\in Z^+\times(\Re^n\backslash\{0\})$, *there exists* $u_0\in U$ *and a neighborhood* $\mathbf{N}(t,x_0)\subset\Re^n$ *such that*

$$x\in\mathbf{N}(t,x_0)\quad\Rightarrow\quad\Psi(t,x,u_0)\le V(t,x)-\frac{1}{4}\rho\big(V(t,x)\big)\tag{6.274}$$

Proof of Fact I By (6.263) it follows that for all $(t,x_0)\in Z^+\times(\Re^n\backslash\{0\})$, there exists $u_0\in U$ such that

$$\Psi(t,x_0,u_0)\le V(t,x_0)-\frac{3}{4}\rho\big(V(t,x_0)\big)\tag{6.275}$$

Since the mapping $x\to\Psi(t,x,u)$ is upper semi-continuous and the mapping $x\in\Re^n\to V(t,x)-\rho(V(t,x))$ is continuous, there exists a neighborhood $\mathbf{N}(t,x_0)\subset\Re^n$ around x_0 such that, for all $x\in\mathbf{N}(t,x_0)$,

$$\begin{aligned}
\Psi(t,x,u_0)\le\Psi(t,x_0,u_0)+\frac{1}{4}\rho\big(V(t,x_0)\big)\\
V(t,x_0)-\frac{1}{2}\rho\big(V(t,x_0)\big)\le V(t,x)-\frac{1}{4}\rho\big(V(t,x)\big)
\end{aligned}\tag{6.276}$$

Inequalities (6.275) and (6.276) imply property (6.274).

Fact II *For each fixed* $t\in Z^+$, *there exists a family of open sets* $(\Omega_j^{(t)})_{j\in J(t)}$ *with* $\Omega_j^{(t)}\subset\Re^n\backslash\{0\}$ *for all* $j\in J(t)$, *which is a locally finite open covering of* $\Re^n\backslash\{0\}$, *and a family of points* $(u_j^{(t)})_{j\in J(t)}$ *with* $u_j^{(t)}\in U$ *for all* $j\in J(t)$ *such that*

$$x\in\Omega_j^{(t)}\quad\Rightarrow\quad\Psi\big(t,x,u_j^{(t)}\big)\le V(t,x)-\frac{1}{4}\rho\big(V(t,x)\big)\tag{6.277}$$

This fact is an immediate consequence of Fact I.

By virtue of Fact II and standard partition of unity arguments, it follows that for each fixed $t \in Z^+$, there exists a family of smooth functions $\theta_j^{(t)} : \Re^n \to [0,1]$ with $\theta_j^{(t)}(x) = 0$ if $x \notin \Omega_j^{(t)} \subset \Re^n \backslash \{0\}$, $\sum_{j \in J(t)} \theta_j^{(t)}(x)$ being locally finite with $\sum_j \theta_j^{(t)}(x) = 1$ for all $x \in \Re^n \backslash \{0\}$. We define, for each fixed $t \in Z^+$,

$$k(t,x) := \left(1 - h\left(\frac{|x|^2 - 2\eta^2(t)}{2\eta^2(t)}\right)\right) K\big(t, \Pr_{Q(t)}(x)\big) + h\left(\frac{|x|^2 - 2\eta^2(t)}{2\eta^2(t)}\right) \tilde{k}(t,x) \tag{6.278}$$

where

$$\tilde{k}(t,x) := \sum_{j \in J(t)} \theta_j^{(t)}(x) u_j^{(t)} \quad \text{for } t \in Z^+, x \neq 0 \tag{6.279}$$

$$\tilde{k}(t,0) := 0 \quad \text{for } t \geq 0 \tag{6.280}$$

where $h \in C^\infty(\Re; [0,1])$ be a smooth nondecreasing function with $h(s) = 0$ for all $s \leq 0$ and $h(s) = 1$ for all $s \geq 1$, and $Q(t) = \{x \in \Re^n : |x| \leq 3\eta(t)\}$. Clearly, k as defined by (6.278), (6.279), and (6.280) is a function of class $C^\nu(Z^+ \times \Re^n; U)$ with $k(t,x) \in U$ (since U is convex and $k(t,x)$ is equal to a convex combination of $u_j^{(t)} \in U$ and $u_0 = 0 \in U$). It can be shown (by distinguishing cases) that

$$\Psi\big(t, x, k(t,x)\big) \leq V(t,x) - \frac{1}{4}\rho\big(V(t,x)\big)$$

for all $(t,x) \in Z^+ \times \Re^n$. It follows from the above inequality and Proposition 2.3 in Chap. 2 that system (4.56) with $u = k(t,x)$ is URGAOS.

(b) $\Rightarrow$ (a) Since system (4.56) with $u = k(t,x)$ is URGAOS and satisfies hypotheses (L1–4), it follows from Proposition 3.1 in Chap. 3 that there exist a continuous $V : Z^+ \times \Re^n \to \Re^+$ and functions $a_1, a_2 \in K_\infty$ such that inequality (6.261) with $\beta(t) \equiv 1$ holds and

$$\begin{aligned} &V\big(t+1, f\big(t, d, x, k(t,x)\big)\big) \leq \exp(-1) V(t,x) \\ &\forall (t,x,d) \in Z^+ \times \Re^n \times D \end{aligned} \tag{6.281}$$

We next prove that V is an ORCLF for (4.56). Obviously, property (i) of Definition 6.4 holds. The reader can verify that the function Ψ as defined by (6.273) is continuous and satisfies (6.262), (6.263) with $q(t) \equiv 0$. Moreover, the function Ψ as defined by (6.273) is quasi-convex with respect to $u \in U$. The proof is complete. □

6.7 Backstepping

Backstepping is a methodology for constructing control Lyapunov functions or functionals in parallel with the construction of stabilizing feedback laws. Backstepping and its variants are described in detail in [41] and the references therein, and share the common feature with the "adding an integrator" results (see [67]) that it is a very powerful tool for the systematic design of controllers for finite-dimensional nonlinear systems with a lower triangular structure. Adaptive and robust backstepping methods are also available in the past literature (see [14, 26, 41] and numerous references therein).

We will describe in detail the backstepping method for a class of infinite-dimensional nonlinear systems described by RFDEs and also for a class of finite-dimensional discrete-time systems.

6.7.1 Backstepping for Control Systems Described by RFDEs

Here, we will study in detail a class of triangular time-delay nonlinear systems described by RFDEs, i.e.,

$$
\begin{aligned}
&\dot{x}_i(t) = f_i\big(t, d(t), T_r(t)x_1, \ldots, T_r(t)x_i\big) + g_i\big(t, d(t), T_r(t)x_1, \ldots, T_r(t)x_i\big)x_{i+1}(t) \\
&\quad i = 1, \ldots, n-1 \\
&\dot{x}_n(t) = f_n\big(t, d(t), T_r(t)x\big) + g_n\big(t, d(t), T_r(t)x\big)u(t) \\
&\quad x(t) = \big(x_1(t), \ldots, x_n(t)\big) \in \Re^n, d(t) \in D, u(t) \in \Re, t \ge 0
\end{aligned} \tag{6.282}
$$

Lyapunov-based feedback design for various special cases of systems described by RFDEs was used recently in [21, 22, 44, 46, 50, 72], including the input-delayed case. More specifically, the stabilization problem for autonomous and disturbance-free systems of the form (6.282) has been studied in [21, 22, 50, 72].

The following result shows that the construction of a stabilizing feedback law for (6.282) proceeds in parallel with the construction of a State Robust Control Lyapunov Functional. Moreover, sufficient conditions for the existence and design of a stabilizing feedback law $u(t) = k(x(t))$, which is independent of the delay, are given. The result covers the finite-dimensional case too.

Theorem 6.6 *Consider system (6.282), where $r > 0$, $D \subset \Re^l$ is a compact set, the mappings $f_i : \Re^+ \times D \times C^0([-r, 0]; \Re^i) \to \Re$ and $g_i : \Re^+ \times D \times C^0([-r, 0]; \Re^i) \to \Re$ $(i = 1, \ldots, n)$ are continuous with $f_i(t, d, 0) = 0$ for all $(t, d) \in \Re^+ \times D$, and each $g_i : \Re^+ \times D \times C^0([-r, 0]; \Re^i) \to \Re$ $(i = 1, \ldots, n)$ is completely locally Lipschitz with respect to $x \in C^0([-r, 0]; \Re^i)$. Suppose that there exists a nondecreasing function $\varphi \in C^\infty(\Re^+; (0, +\infty))$ such that for every $i = 1, \ldots, n$, it holds that*

$$
\begin{aligned}
&\frac{1}{\varphi(\|x\|_r)} \le g_i(t, d, x) \\
&\quad \le \varphi\big(\|x\|_r\big) \quad \forall (t, x, d) \in \Re^+ \times C^0\big([-r, 0]; \Re^i\big) \times D
\end{aligned} \tag{6.283}
$$

Moreover, suppose that for every $i = 1, \ldots, n$, it holds that

$$
\begin{aligned}
&\sup\left\{ \frac{|f_i(t, d, x) - f_i(t, d, y)|}{\|x - y\|_r} : (t, d) \in \Re^+ \times D, x \in S, y \in S, x \ne y \right\} < +\infty \\
&\quad \text{for every bounded } S \subset C^0\big([-r, 0]; \Re^i\big)
\end{aligned} \tag{6.284}
$$

Then, for every $\sigma > 0$, *there exist functions* $\mu_i \in C^\infty(\Re^i;(0,+\infty))$ *and* $k_i \in C^\infty(\Re^i;\Re)$ $(i = 1,\dots,n)$ *with*

$$k_1(\xi_1) := -\mu_1(\xi_1)\xi_1 \tag{6.285}$$

$$k_j(\xi_1,\dots,\xi_j) := -\mu_j(\xi_1,\dots,\xi_j)\big(\xi_j - k_{j-1}(\xi_1,\dots,\xi_{j-1})\big) \quad j = 2,\dots,n \tag{6.286}$$

such that the functional

$$V(x) := \max_{\theta\in[-r,0]} \exp(2\sigma\theta)\left(x_1^2(\theta) + \sum_{j=2}^{n}\big|x_j(\theta) - k_{j-1}\big(x_1(\theta),\dots,x_{j-1}(\theta)\big)\big|^2\right) \tag{6.287}$$

is a State Robust Control Lyapunov Functional (*SRCLF*) *for* (6.282). *Moreover, the closed-loop system* (6.282) *with* $u(t) = k_n(x(t))$ *is URGAS. More specifically, the inequality* $V^0(x;v) \le -2\sigma V(x)$ *holds for all* $(t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D$ *with* $v = (f_1(t,d,x_1) + g_1(t,d,x_1)x_2(0),\dots,f_n(t,d,x) + g_n(t,d,x)k_n(x(0)))' \in \Re^n$.

Remark 6.4 It is worth noting that the feedback law $u(t) = k_n(x(t))$ is delay-independent. From the proof of Theorem 6.6, the functions $\mu_i \in C^\infty(\Re^i;(0,+\infty))$ $(i = 1,\dots,n)$ will be obtained by an algorithmic procedure similar to the backstepping procedure used for finite-dimensional triangular control systems.

The proof of Theorem 6.6 is based on the following lemma.

Lemma 6.7 *Let* $Q \in C^1(\Re^n;\Re^+)$, $\sigma > 0$, *and consider the following functional*:

$$V : C^0\big([-r,0];\Re^n\big) \to \Re^+ \quad \textit{with } V(x) := \max_{\theta\in[-r,0]} \exp(2\sigma\theta)Q\big(x(\theta)\big) \tag{6.288}$$

The functional $V : C^0([-r,0];\Re^n) \to \Re^+$ *defined by* (6.288) *is Lipschitz on bounded sets of* $C^0([-r,0];\Re^n)$ *and satisfies*

$$V^0(x;v) \le -2\sigma V(x)$$
$$\textit{for all } (x,v) \in C^0\big([-r,0];\Re^n\big) \times \Re^n \textit{ with } Q\big(x(0)\big) < V(x) \tag{6.289}$$

$$V^0(x;v) \le \max\big\{-2\sigma V(x), \nabla Q\big(x(0)\big)v\big\}$$
$$\textit{for all } (x,v) \in C^0\big([-r,0];\Re^n\big) \times \Re^n \textit{ with } Q\big(x(0)\big) = V(x) \tag{6.290}$$

Proof The fact that the functional V as defined by (6.288) is Lipschitz on bounded sets of $C^0([-r,0];\Re^n)$ is a direct consequence of the fact that $Q \in C^1(\Re^n;\Re^+)$ (details are left to the reader).

Clearly, by (6.288) we have

$$
\begin{aligned}
V\big(E_h(x;v)\big) &= \max\Big\{\max_{\theta\in[-r,-h]} \exp(2\sigma\theta) Q\big(x(\theta+h)\big), \\
&\qquad \max_{\theta\in[-h,0]} \exp(2\sigma\theta) Q\big(x(0)+(\theta+h)v\big)\Big\} \\
&= \max\Big\{\exp(-2\sigma h) \max_{s\in[-r+h,0]} \exp(2\sigma s) Q\big(x(s)\big), \\
&\qquad \max_{\theta\in[-h,0]} \exp(2\sigma\theta)\Big[Q\big(x(0)\big)+(\theta+h)\nabla Q\big(x(0)\big)v \\
&\qquad + \int_0^{\theta+h} \big(\nabla Q\big(x(0)+sv\big)-\nabla Q\big(x(0)\big)\big)v\,ds\Big]\Big\} \\
&\le \max\Big\{\exp(-2\sigma h)V(x), \max_{\theta\in[-h,0]} \exp(2\sigma\theta)\big[Q\big(x(0)\big)+(\theta+h)\nabla Q\big(x(0)\big)v\big] \\
&\qquad + h|v| \max_{s\in[0,h]} \big|\nabla Q\big(x(0)+sv\big)-\nabla Q\big(x(0)\big)\big|\Big\}
\end{aligned}
\tag{6.291}
$$

If $Q(x(0)) < V(x)$, then there exists $h > 0$ such that $Q(x(0)) + s\nabla Q(x(0))v \le \frac{1}{2}(V(x) + Q(x(0)))$ for all $s \in [0, h]$. Consequently, in this case it follows from (6.291) that, for sufficiently small $h > 0$,

$$
\begin{aligned}
&h^{-1}\big(V\big(E_h(x;v)\big)-V(x)\big) \\
&\quad \le \max\Big\{\frac{\exp(-2\sigma h)-1}{h}V(x), \exp(-2\sigma h)\frac{1}{2h}\big(Q\big(x(0)\big)-V(x)\big) \\
&\qquad + |v| \max_{s\in[0,h]} \big|\nabla Q\big(x(0)+sv\big)-\nabla Q\big(x(0)\big)\big|\Big\}
\end{aligned}
$$

The above inequality gives (6.289) for the case $Q(x(0)) < V(x)$.

If $Q(x(0)) = V(x)$ and $\nabla Q(x(0))v > -2\sigma V(x)$, then it follows that, for sufficiently small $h > 0$,

$$
\max_{\theta\in[-h,0]} \exp(2\sigma\theta)\big[Q\big(x(0)\big)+(\theta+h)\nabla Q\big(x(0)\big)v\big] = Q\big(x(0)\big)+h\nabla Q\big(x(0)\big)v.
$$

Consequently, from (6.291) we obtain

$$
\begin{aligned}
&h^{-1}\big(V\big(E_h(x;v)\big)-V(x)\big) \\
&\quad \le \max\Big\{\frac{\exp(-2\sigma h)-1}{h}V(x), \\
&\qquad \nabla Q\big(x(0)\big)v + |v| \max_{s\in[0,h]} \big|\nabla Q\big(x(0)+sv\big)-\nabla Q\big(x(0)\big)\big|\Big\}
\end{aligned}
$$

The above inequality gives (6.290) for the case $Q(x(0)) = V(x)$ and $\nabla Q(x(0))v > -2\sigma V(x)$.

If $Q(x(0)) = V(x)$ and $\nabla Q(x(0))v \le -2\sigma V(x)$, then it follows that, for sufficiently small $h > 0$,

$$\max_{\theta\in[-h,0]} \exp(2\sigma\theta)\big[Q\big(x(0)\big) + (\theta + h)\nabla Q\big(x(0)\big)v\big] = \exp(-2\sigma h)Q\big(x(0)\big)$$

Consequently, from (6.291) we obtain

$$\begin{aligned} &h^{-1}\big(V\big(E_h(x; v)\big) - V(x)\big)\\ &\quad\le \frac{\exp(-2\sigma h) - 1}{h} V(x) + |v| \max_{s\in[0,h]} \big|\nabla Q\big(x(0) + sv\big) - \nabla Q\big(x(0)\big)\big| \end{aligned}$$

The above inequality gives (6.290) for the case $Q(x(0)) = V(x)$ and $\nabla Q(x(0))v \le -2\sigma V(x)$.

The proof is complete. □

We are now in a position to provide the proof of Theorem 6.6.

Proof of Theorem 6.6 Inequality (6.284), in conjunction with the fact that $f_i(t, d, 0) = 0$ for all $(t, d) \in \Re^+ \times D$ $(i = 1, \dots, n)$, implies the existence of a nondecreasing function $L \in C^\infty(\Re^+; (0, +\infty))$ such that for every $i = 1, \dots, n$, it holds

$$\big|f_i(t, d, x)\big| \le L\big(\|x\|_r\big)\|x\|_r \quad \forall(t, x, d) \in \Re^+ \times C^0\big([-r, 0]; \Re^i\big) \times D \tag{6.292}$$

Let $\sigma > 0$ be a given number. We next define functions $\mu_i \in C^\infty(\Re^i; (0, +\infty))$, $\gamma_i \in C^\infty(\Re^+; (0, +\infty))$, and $b_i \in C^\infty(\Re^+; (0, +\infty))$ $(i = 1, \dots, n)$ using the following algorithm.

Algorithm Step $i = 1$: Define

$$\mu_1(\xi_1) := \frac{\gamma_1(1 + \xi_1^2) + n\sigma}{b_1(1 + \xi_1^2)} \tag{6.293}$$

where

$$\gamma_1(s) := \exp(\sigma r)L\big(s \exp(\sigma r)\big) + \varphi\big(s \exp(\sigma r)\big) \tag{6.294}$$

$$b_1(s) := \frac{1}{\varphi(s \exp(\sigma r))} \tag{6.295}$$

Step $i \ge 2$: Based on the knowledge of the functions $\mu_j \in C^\infty(\Re^j; (0, +\infty))$ $(j = 1, \dots, i - 1)$ from previous steps, we define the function $\mu_i \in C^\infty(\Re^i; (0, +\infty))$. First define

$$k_0 \equiv 0 \quad k_1(\xi_1) := -\mu_1(\xi_1)\xi_1 \tag{6.296}$$

$$k_j(\xi_1, \dots, \xi_j) := -\mu_j(\xi_1, \dots, \xi_j)\big(\xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1})\big) \quad j = 2, \dots, i - 1 \tag{6.297}$$

$$\begin{aligned} \gamma_j(s) := {}& \exp(\sigma r)L\big(s \exp(\sigma r)B_j\big(s \exp(\sigma r)\big)\big)B_j\big(s \exp(\sigma r)\big)\\ &+ \varphi\big(s \exp(\sigma r)B_j\big(s \exp(\sigma r)\big)\big) \quad j = 1, \dots, i \end{aligned} \tag{6.298}$$

$$b_j(s) := \frac{1}{\varphi(s\exp(\sigma r)B_j(s\exp(\sigma r)))} \quad j = 1, \ldots, i \tag{6.299}$$

where $B_j \in C^\infty(\Re^+; (0, +\infty))$ $(j = 1, \ldots, i)$ are nondecreasing functions that satisfy

$$B_1(s) := 1$$
$$B_j(s) \geq \max\left\{1 + \sum_{l=1}^{j-1} \mu_l(\xi_1, \ldots, \xi_l) : \max_{j=1,\ldots,i} \left|\xi_l - k_{l-1}(\xi_1, \ldots, \xi_{l-1})\right| \leq s\right\}$$
$$\text{for all } s \geq 0 \text{ and } j \geq 2 \tag{6.300}$$

Let $\rho_j \in C^\infty(\Re^+; (0, +\infty))$ $(j = 1, \ldots, i)$ and $\delta_j \in C^\infty(\Re^j; (0, +\infty))$ $(j = 0, \ldots, i-1)$ be functions such that the following inequalities hold:

$$b_j(s') - b_j(s) + s\gamma_j(s) - s'\gamma_j(s') \leq (s - s')\rho_j(s) \quad \forall s \geq s' \geq 0 \tag{6.301}$$

$$\delta_j(\xi_1, \ldots, \xi_j) \geq \left|\nabla k_j(\xi_1, \ldots, \xi_j)\right|\left(1 + \mu_1(\xi_1) + \cdots + \mu_j(\xi_1, \ldots, \xi_j)\right)$$
$$\forall (\xi_1, \ldots, \xi_j) \in \Re^j \tag{6.302}$$

Define

$$\mu_i(\xi_1, \ldots, \xi_i) := b_i^{-1}(p)\left[(n + 1 - i)\sigma + \frac{i-1}{4\sigma}a^2(p, \xi_1, \ldots, \xi_{j-1}) + \gamma_i(p) + c_{i-1}(p)\delta_{i-1}(\xi_1, \ldots, \xi_{j-1})\right] \tag{6.303}$$

where

$$p := \frac{i}{2} + \frac{1}{2}\sum_{j=1}^{i}\left|\xi_j - k_{j-1}(\xi_1, \ldots, \xi_{j-1})\right|^2$$

$$c_0 \equiv 0 \quad c_j(s) := \sum_{k=1}^{j} \gamma_k(s) \quad \text{for } j = 1, \ldots, i \tag{6.304}$$

$$a(s, \xi_1, \ldots, \xi_{i-1}) := c_{i-1}(s)\delta_{i-1}(\xi_1, \ldots, \xi_{i-1}) + c_i(s) + \left(1 + \sum_{j=1}^{i-1}\left(s\mu_j(\xi_1, \ldots, \xi_j) + \delta_{j-1}(\xi_1, \ldots, \xi_{j-1})\right)\right)\sum_{k=1}^{i-1}\rho_k(s) \tag{6.305}$$

It should be noticed that in every step $i \geq 2$ of the above algorithm, we only need to compute the functions $\gamma_i(s)$, $b_i(s)$, $B_i(s)$, $\rho_{i-1}(s)$, $\delta_{i-1}(\xi_1, \ldots, \xi_{i-1})$, and $\mu_i(\xi_1, \ldots, \xi_i)$ (the functions $\gamma_j(s)$, $b_j(s)$, $B_j(s)$, $\rho_{j-1}(s)$, $\delta_{j-1}(\xi_1, \ldots, \xi_{j-1})$, and $\mu_j(\xi_1, \ldots, \xi_j)$ for $j = 1, \ldots, i-1$ have been computed in the previous steps). It can be shown by induction that the following claim holds.

Claim *The following inequality holds for all* $(\xi_1,\ldots,\xi_i)' \in \Re^i$:

$$
\begin{aligned}
&-\sum_{j=1}^{i}\big(\xi_j - k_{j-1}(\xi_1,\ldots,\xi_{j-1})\big)^2 b_j(s)\mu_j(\xi_1,\ldots,\xi_j)\\
&\qquad + s\sum_{j=1}^{i}\big|\xi_j - k_{j-1}(\xi_1,\ldots,\xi_{j-1})\big|\gamma_j(s)\\
&\qquad + s\sum_{j=1}^{i}\big|\xi_j - k_{j-1}(\xi_1,\ldots,\xi_{j-1})\big|c_{j-1}(s)\delta_{j-1}(\xi_1,\ldots,\xi_{j-1})\\
&\quad \le -(n+1-i)\sigma\sum_{j=1}^{i}\big|\xi_j - k_{j-1}(\xi_1,\ldots,\xi_{j-1})\big|^2
\end{aligned}
\tag{6.306}
$$

where

$$
\begin{aligned}
&s = \sum_{j=1}^{i}\big|\xi_j - k_{j-1}(\xi_1,\ldots,\xi_{j-1})\big|\\
&c_0 \equiv 0\\
&c_j(s) := \sum_{k=1}^{j}\gamma_k(s) \quad \textit{for } j = 1,\ldots,i
\end{aligned}
\tag{6.307}
$$

By Lemma 6.7 it follows that the functional V defined by (6.287) satisfies

$$
V^0(x;v) \le -2\sigma V(x) \tag{6.308}
$$

for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times \Re \times D$ with $V(x) > x_1^2(0) + \sum_{j=2}^{n}|x_j(0) - k_{j-1}(x_1(0),\ldots,x_{j-1}(0))|^2$ and $v = (f_1(t,d,x_1) + g_1(t,d,x_1)x_2(0), \ldots, f_n(t,d,x) + g_n(t,d,x)u)' \in \Re^n$, and

$$
V^0(x;v) \le \max\big\{-2\sigma V(x), 2A(t,d,x,u)\big\} \tag{6.309}
$$

for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times \Re \times D$ with $V(x) = x_1^2(0) + \sum_{j=2}^{n}|x_j(0) - k_{j-1}(x_1(0),\ldots,x_{j-1}(0))|^2$ and $v = (f_1(t,d,x_1) + g_1(t,d,x_1)x_2(0), \ldots, f_n(t,d,x) + g_n(t,d,x)u)' \in \Re^n$, where

$$
\begin{aligned}
&A(t,d,x,u)\\
&\quad := \sum_{j=1}^{n-1}\big(x_j(0) - k_{j-1}\big(x_1(0),\ldots,x_{j-1}(0)\big)\big)\big(f_i(t,d,x_1,\ldots,x_i)\\
&\qquad + g_i(t,d,x_1,\ldots,x_i)x_{i+1}(0)\big) - \sum_{j=2}^{n-1}\big(x_j(0) - k_{j-1}\big(x_1(0),\ldots,x_{j-1}(0)\big)\big)
\end{aligned}
$$

$$\times \left(\sum_{l=1}^{j-1} \frac{\partial k_{j-1}}{\partial \xi_l} \big(x_1(0), \dots, x_{j-1}(0)\big) \big(f_l(t, d, x_1, \dots, x_l)\right.$$

$$\left. + g_l(t, d, x_1, \dots, x_l) x_{l+1}(0)\big) \right)$$

$$+ \big(x_n(0) - k_{n-1}\big(x_1(0), \dots, x_{n-1}(0)\big)\big)\big(f_n(t, d, x) + g_n(t, d, x)u\big)$$

$$- \big(x_n(0) - k_{n-1}\big(x_1(0), \dots, x_{n-1}(0)\big)\big) \sum_{l=1}^{n-1} \frac{\partial k_{n-1}}{\partial \xi_l} \big(x_1(0), \dots, x_{n-1}(0)\big)$$

$$\times \big(f_l(t, d, x_1, \dots, x_l) + g_l(t, d, x_1, \dots, x_l) x_{l+1}(0)\big) \tag{6.310}$$

By virtue of the previous claim, it can be shown that for all $(t, x, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times D$ with $u = k_n(x_1(0), \dots, x_n(0))$ and $V(x) = x_1^2(0) + \sum_{j=2}^n |x_j(0) - k_{j-1}(x_1(0), \dots, x_{j-1}(0))|^2$, the following inequalities hold:

$$A(t, d, x, u) \le -\sigma \sum_{j=1}^{n} \big|x_j(0) - k_{j-1}\big(x_1(0), \dots, x_{j-1}(0)\big)\big|^2 \tag{6.311}$$

$$\max_{\theta \in [-r, 0]} \sum_{j=1}^{i} \big|x_j(\theta)\big| \le s B_i(s) \tag{6.312}$$

with $s = \exp(\sigma r) \sum_{j=1}^i |x_j(0) - k_{j-1}(x_1(0), \dots, x_{j-1}(0))|$.

Consequently, by virtue of (6.308), (6.309) and (6.311), we obtain

$$V^0(x; v) \le -2\sigma V(x) \tag{6.313}$$

for all $(t, x, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times D$ with $v = (f_1(t, d, x_1) + g_1(t, d, x_1) x_2(0), \dots, f_n(t, d, x) + g_n(t, d, x) k_n(x(0)))' \in \Re^n$.

Notice that there exist functions $a_1, a_2 \in K_\infty$ such that

$$a_1\big(|\xi|\big) \le \xi_1^2 + \sum_{j=2}^{n} \big|\xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1})\big|^2$$

$$\le a_2\big(|\xi|\big) \quad \forall \xi = (\xi_1, \dots, \xi_n)' \in \Re^n \tag{6.314}$$

Consequently, definition (6.287), in conjunction with (6.314), implies

$$\exp(-2\sigma r) a_1\big(\|x\|_r\big) \le V(x) \le a_2\big(\|x\|_r\big) \quad \forall x \in C^0\big([-r, 0]; \Re^n\big) \tag{6.315}$$

It follows from inequalities (6.313), (6.315) and Theorem 2.5 in Chap. 2 that the closed-loop system (6.282) with $u(t) = k_n(x(t))$ is URGAS.

Finally, we show that V as defined by (6.287) is an SRCLF. Clearly, definition (6.310), in conjunction with (6.311), implies, for all $(t, x, u, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times \Re \times D$ with $V(x) = x_1^2(0) + \sum_{j=2}^n |x_j(0) - k_{j-1}(x_1(0), \dots, x_{j-1}(0))|^2$,

$$
\begin{aligned}
&A(t,d,x,u)\\
&\quad\le -\sigma\left(x_1^2(0)+\sum_{j=2}^{n}\left|x_j(0)-k_{j-1}\big(x_1(0),\dots,x_{j-1}(0)\big)\right|^2\right)\\
&\qquad+\left|x_n(0)-k_{n-1}\big(x_1(0),\dots,x_{n-1}(0)\big)\right|\left|g_n(t,d,x)\right|\left|u-k_n\big(x_1(0),\dots,x_n(0)\big)\right|
\end{aligned}
\tag{6.316}
$$

By virtue of (6.283), (6.298), (6.308), (6.309), (6.312), and (6.316), we obtain

$$
\begin{aligned}
&V^0(x;v)\\
&\quad\le -2\sigma V(x)+2\left|x_n(0)-k_{n-1}\big(x_1(0),\dots,x_{n-1}(0)\big)\right|\left|u-k_n\big(x_1(0),\dots,x_n(0)\big)\right|\\
&\qquad\times\gamma_n\left(\left|x_1(0)\right|+\sum_{j=2}^{n}\left|x_j(0)-k_{j-1}\big(x_1(0),\dots,x_{j-1}(0)\big)\right|\right)
\end{aligned}
\tag{6.317}
$$

for all $(t,x,u,d)\in\Re^+\times C^0([-r,0];\Re^n)\times\Re\times D$ with

$$
v=\big(f_1(t,d,x_1)+g_1(t,d,x_1)x_2(0),\dots,f_n(t,d,x)+g_n(t,d,x)u\big)'\in\Re^n
$$

Define

$$
\Phi(t,x):=x(0)\qquad \rho(w):=2\sigma w,
$$

and

$$
\begin{aligned}
\Psi(t,z,u)&:=2\left|z_n-k_{n-1}(z_1,\dots,z_{n-1})\right|\\
&\qquad\times\gamma_n\left(|z_1|+\sum_{j=2}^{n}\left|z_j-k_{j-1}(z_1,\dots,z_{j-1})\right|\right)\left|u-k_n(z_1,\dots,z_n)\right|.
\end{aligned}
$$

These definitions, in conjunction with inequalities (6.315) and (6.317), guarantee that inequalities (6.221), (6.222), and (6.223) hold (with $q(t)\equiv 0$) for V as defined by (6.287). Consequently, V as defined by (6.287) is an SRCLF. The proof is complete. □

Proof of Claim in the proof of Theorem 6.6 The proof is made by induction. It is straightforward to verify that definitions (6.293), (6.294), and (6.295) guarantee that (6.306) holds for $i=1$. We next assume that inequality (6.306) holds for $i-1$ ($i\ge 2$), i.e.,

$$
\begin{aligned}
&\sum_{j=1}^{i-1}\big(\xi_j-k_{j-1}(\xi_1,\dots,\xi_{j-1})\big)^2 b_j\big(s'\big)\mu_j(\xi_1,\dots,\xi_j)\\
&\qquad+s'\sum_{j=1}^{i-1}\left|\xi_j-k_{j-1}(\xi_1,\dots,\xi_{j-1})\right|\gamma_j\big(s'\big)\\
&\qquad+s'\sum_{j=1}^{i-1}\left|\xi_j-k_{j-1}(\xi_1,\dots,\xi_{j-1})\right|c_{j-1}\big(s'\big)\delta_{j-1}(\xi_1,\dots,\xi_{j-1})\\
&\quad\le-(n+2-i)\sigma\left(\sum_{j=1}^{i-1}\left|\xi_j-k_{j-1}(\xi_1,\dots,\xi_{j-1})\right|^2\right)
\end{aligned}
\tag{6.318}
$$

where

$$
\begin{aligned}
& s' = \sum_{j=1}^{i-1} \left| \xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1}) \right| \\
& c_0 \equiv 0 \\
& c_j(s) := \sum_{k=1}^{j} \gamma_k(s) \quad \text{for } j = 1, \dots, i-1
\end{aligned}
\tag{6.319}
$$

By virtue of inequalities (6.301), (6.318) and definitions (6.307), (6.319), we obtain, for all $(\xi_1, \dots, \xi_i)' \in \Re^i$,

$$
\begin{aligned}
& -\sum_{j=1}^{i} \big(\xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1})\big)^2 b_j(s)\mu_j(\xi_1, \dots, \xi_j) \\
& \quad + s \sum_{j=1}^{i} \left| \xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1}) \right| \gamma_j(s) \\
& \quad + s \sum_{j=1}^{i} \left| \xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1}) \right| c_{j-1}(s)\delta_{j-1}(\xi_1, \dots, \xi_{j-1}) \\
& \le -(n+2-i)\sigma \left(\sum_{j=1}^{i-1} \left| \xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1}) \right|^2 \right) \\
& \quad - \big(\xi_i - k_{i-1}(\xi_1, \dots, \xi_{i-1})\big)^2 \\
& \quad \times \big(b_i(s)\mu_i(\xi_1, \dots, \xi_i) - \gamma_i(s) - c_{i-1}(s)\delta_{i-1}(\xi_1, \dots, \xi_{i-1})\big) \\
& \quad + \left| \xi_i - k_{i-1}(\xi_1, \dots, \xi_{i-1}) \right| a(s, \xi_1, \dots, \xi_{i-1}) \sum_{j=1}^{i-1} \left| \xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1}) \right|
\end{aligned}
\tag{6.320}
$$

where $a(s, \xi_1, \dots, \xi_{i-1})$ is defined by (6.305). Using (6.320) in conjunction with the inequality

$$
\begin{aligned}
& \left| \xi_i - k_{i-1}(\xi_1, \dots, \xi_{i-1}) \right| \sum_{j=1}^{i-1} \left| \xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1}) \right| a_j(s, \xi_1, \dots, \xi_{i-1}) \\
& \quad \le \sigma \sum_{j=1}^{i-1} \left| \xi_j - k_{j-1}(\xi_1, \dots, \xi_{j-1}) \right|^2 \\
& \qquad + \left| \xi_i - k_{i-1}(\xi_1, \dots, \xi_{i-1}) \right|^2 \frac{i-1}{4\sigma} a^2(s, \xi_1, \dots, \xi_{i-1})
\end{aligned}
$$

and the fact that the maps $s \to a(s, \xi_1, \ldots, \xi_{i-1})$, $s \to b_i(s)$ are nondecreasing and nonincreasing, respectively, on $\Re^+$, it is not hard to verify that definition (6.303) guarantees inequality (6.306). Notice that the following inequality is also used:

$$s = \sum_{j=1}^{i} \left|\xi_j - k_{j-1}(\xi_1, \ldots, \xi_{j-1})\right| \le p = \frac{i}{2} + \frac{1}{2} \sum_{j=1}^{i} \left|\xi_j - k_{j-1}(\xi_1, \ldots, \xi_{j-1})\right|^2.$$

The proof is complete. □

Proof of inequalities (6.310) and (6.311) Notice that by virtue of definitions (6.296) and (6.297), for all $i \ge 1$ and $(\xi_1, \ldots, \xi_i)' \in \Re^i$, it holds:

$$\begin{aligned} \sum_{j=1}^{i} |\xi_j| \le \sum_{j=1}^{i} &\left|\xi_j - k_{j-1}(\xi_1, \ldots, \xi_{j-1})\right| \\ &+ \sum_{j=1}^{i} \mu_{j-1}(\xi_1, \ldots, \xi_{j-1}) \left|\xi_{j-1} - k_{j-2}(\xi_1, \ldots, \xi_{j-2})\right| \end{aligned} \tag{6.321}$$

Inequality (6.321) implies

$$\sum_{j=1}^{i} |\xi_j| \le \left(\sum_{j=1}^{i-1} \left(1 + \mu_j(\xi_1, \ldots, \xi_{j-1})\right) \right) \max_{j=1,\ldots,i} \left|\xi_j - k_{j-1}(\xi_1, \ldots, \xi_{j-1})\right| \tag{6.322}$$

Definition (6.287) implies that, for all $x \in C^0([-r, 0]; \Re^n)$ with $V(x) = \sum_{j=1}^{n} |x_j(0) - k_{j-1}(x_1(0), \ldots, x_{j-1}(0))|^2$, we have

$$\begin{aligned} &\max_{\substack{j=1,\ldots,i \\ \theta\in[-r,0]}} \left|x_j(\theta) - k_{j-1}\left(x_1(\theta), \ldots, x_{j-1}(\theta)\right)\right| \\ &\quad \le \exp(\sigma r) \sum_{j=1}^{n} \left|x_j(0) - k_{j-1}\left(x_1(0), \ldots, x_{j-1}(0)\right)\right| \end{aligned} \tag{6.323}$$

Inequality (6.311) is a direct consequence of (6.300), (6.322), and (6.323). Consequently, (6.283) and (6.292), together with (6.311), imply, for all $i \ge 1$ and $(t, x, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times D$ with $V(x) = \sum_{j=1}^{n} |x_j(0) - k_{j-1}(x_1(0), \ldots, x_{j-1}(0))|^2$,

$$\left|f_i(t, d, x_1, \ldots, x_i)\right| \le L\left(s \exp(\sigma r) B_i\left(s \exp(\sigma r)\right)\right) B_i\left(s \exp(\sigma r)\right) s \exp(\sigma r) \tag{6.324}$$

$$\frac{1}{\varphi(s \exp(\sigma r) B_i(s \exp(\sigma r)))} \le g_i(t, d, x_1, \ldots, x_i) \le \varphi\left(s \exp(\sigma r) B_i\left(s \exp(\sigma r)\right)\right) \tag{6.325}$$

where

$$s := \sum_{j=1}^{n} \left|x_j(0) - k_{j-1}\left(x_1(0), \ldots, x_{j-1}(0)\right)\right| \tag{6.326}$$

Using (6.324), (6.325), we obtain, by virtue of definitions (6.298), (6.299),

$$\begin{aligned}
&\left|f_i(t,d,x_1,\dots,x_i)\right| + \left|g_i(t,d,x_1,\dots,x_i)\right|\left|x_{i+1}(0) - k_i\big(x_1(0),\dots,x_i(0)\big)\right| \\
&\quad \le s\gamma_i(s) \quad i = 1,\dots,n-1 && (6.327) \\
&\left|f_n(t,d,x_1,\dots,x_n)\right| \le s\gamma_n(s) && (6.328) \\
&b_i(s) \le g_i(t,d,x_1,\dots,x_i) \le \gamma_i(s) \quad i = 1,\dots,n && (6.329)
\end{aligned}$$

where s is defined by (6.326). Definition (6.310), in conjunction with definitions (6.296) and (6.297), gives, for $u = k_n(x_1(0),\dots,x_n(0))$,

$$\begin{aligned}
&A(t,d,x,u) \\
&\quad \le -\sum_{j=1}^{n} g_i(t,d,x_1,\dots,x_i)\mu_j\big(x_1(0),\dots,x_j(0)\big) \\
&\qquad \times \big(x_j(0) - k_{j-1}\big(x_1(0),\dots,x_{j-1}(0)\big)\big)^2 \\
&\qquad + \sum_{j=1}^{n-1}\left|x_j(0) - k_{j-1}\big(x_1(0),\dots,x_{j-1}(0)\big)\right| \\
&\qquad \times \big|f_j(t,d,x_1,\dots,x_j) + g_j(t,d,x_1,\dots,x_j) \\
&\qquad \times \big(x_{j+1}(0) - k_j\big(x_1(0),\dots,x_j(0)\big)\big)\big| \\
&\qquad + \sum_{j=2}^{n-1}\left|\nabla k_{j-1}\big(x_1(0),\dots,x_{j-1}(0)\big)\right|\left|x_j(0) - k_{j-1}\big(x_1(0),\dots,x_{j-1}(0)\big)\right| \\
&\qquad \times \sum_{l=1}^{j-1}\big|f_l(t,d,x_1,\dots,x_l) + g_l(t,d,x_1,\dots,x_l) \\
&\qquad \times \big(x_{l+1}(0) - k_l\big(x_1(0),\dots,x_l(0)\big)\big)\big| \\
&\qquad + \sum_{j=2}^{n-1}\left|\nabla k_{j-1}\big(x_1(0),\dots,x_{j-1}(0)\big)\right|\left|x_j(0) - k_{j-1}\big(x_1(0),\dots,x_{j-1}(0)\big)\right| \\
&\qquad \times \sum_{l=1}^{j-1}\left|x_l(0) - k_{l-1}\big(x_1(0),\dots,x_{l-1}(0)\big)\right| \\
&\qquad \times \mu_l\big(x_1(0),\dots,x_l(0)\big)\left|g_l(t,d,x_1,\dots,x_l)\right| \\
&\qquad + \left|x_n(0) - k_{n-1}\big(x_1(0),\dots,x_{n-1}(0)\big)\right|\left|f_n(t,d,x)\right| \\
&\qquad + \left|\nabla k_{n-1}\big(x_1(0),\dots,x_{n-1}(0)\big)\right|\left|x_n(0) - k_{n-1}\big(x_1(0),\dots,x_{n-1}(0)\big)\right| \\
&\qquad \times \sum_{l=1}^{n-1}\big|f_l(t,d,x_1,\dots,x_l) + g_l(t,d,x_1,\dots,x_l) \\
&\qquad \times \big(x_{l+1}(0) - k_l\big(x_1(0),\dots,x_l(0)\big)\big)\big| \\
&\qquad + \left|\nabla k_{n-1}\big(x_1(0),\dots,x_{n-1}(0)\big)\right|\left|x_n(0) - k_{n-1}\big(x_1(0),\dots,x_{n-1}(0)\big)\right|
\end{aligned}$$

$$\times \sum_{l=1}^{n-1} \left| x_l(0) - k_{l-1}\big(x_1(0), \ldots, x_{l-1}(0)\big) \right|$$
$$\times \mu_l\big(x_1(0), \ldots, x_l(0)\big) \left| g_l(t, d, x_1, \ldots, x_l) \right| \tag{6.330}$$

Combining (6.330) with (6.326), (6.327), (6.328), and (6.329), we obtain, for $u = k_n(x_1(0), \ldots, x_n(0))$,

$$\begin{aligned} & A(t,d,x,u) \\ & \leq -\sum_{j=1}^{n} b_j(s)\mu_j\big(x_1(0), \ldots, x_j(0)\big)\big(x_j(0) - k_{j-1}\big(x_1(0), \ldots, x_{j-1}(0)\big)\big)^2 \\ & \quad + s \sum_{j=2}^{n} \left| x_j(0) - k_{j-1}\big(x_1(0), \ldots, x_{j-1}(0)\big) \right| \gamma_j(s) \\ & \quad + s \sum_{j=2}^{n} \left| \nabla k_{j-1}\big(x_1(0), \ldots, x_{j-1}(0)\big) \right| \left| x_j(0) - k_{j-1}\big(x_1(0), \ldots, x_{j-1}(0)\big) \right| \\ & \quad \times \sum_{l=1}^{j-1} \big(1 + \mu_l\big(x_1(0), \ldots, x_l(0)\big)\big) \gamma_l(s) \end{aligned} \tag{6.331}$$

Inequality (6.331), together with inequality (6.302), implies, for $u = k_n(x_1(0), \ldots, x_n(0))$,

$$\begin{aligned} & A(t,d,x,u) \\ & \leq -\sum_{j=1}^{n} b_j(s)\mu_j\big(x_1(0), \ldots, x_j(0)\big)\big(x_j(0) - k_{j-1}\big(x_1(0), \ldots, x_{j-1}(0)\big)\big)^2 \\ & \quad + s \sum_{j=1}^{n} \left| x_j(0) - k_{j-1}\big(x_1(0), \ldots, x_{j-1}(0)\big) \right| \gamma_j(s) \\ & \quad + s \sum_{j=1}^{n} \delta_{j-1}\big(x_1(0), \ldots, x_{j-1}(0)\big) \left| x_j(0) - k_{j-1}\big(x_1(0), \ldots, x_{j-1}(0)\big) \right| \\ & \quad \times \sum_{l=1}^{j-1} \gamma_l(s) \end{aligned} \tag{6.332}$$

Clearly, inequality (6.332), in conjunction with (6.306) for $i = n$, shows that (6.310) holds. The proof is complete. □

The following example shows how the backstepping methodology can be used for the stabilization of uncertain systems.

Example 6.7.1 Consider the control system

$$\begin{aligned}
\dot{x}_1(t) &= d_1(t)\int_{t-r}^{t} x_1^2(\theta)\,d\theta + x_2(t)\\
\dot{x}_2(t) &= d_2(t)\big\|T_r(t)x_2\big\|_r + u(t)\\
&\big(x_1(t),x_2(t)\big)\in\Re^2,\ \big(d_1(t),d_2(t)\big)\in[-1,1]^2,\ u(t)\in\Re
\end{aligned}\tag{6.333}$$

Clearly, system (6.333) is a control system described by RFDEs, which satisfies the hypotheses of Theorem 6.6. More specifically, inequality (6.283) holds with $\varphi \equiv 1$. In order to design a delay free stabilizing feedback for (6.333), we follow the algorithm in the proof of Theorem 6.6. Notice that inequality (6.292) holds with $L(w) = 1 + rw$. Let $\sigma > 0$ be given.

Step $i = 1$: We define

$$\mu_1(\xi_1) := \exp(\sigma r)\big(1 + r\big(1+\xi_1^2\big)\exp(\sigma r)\big) + 1 + 2\sigma \tag{6.334}$$

$$\gamma_1(s) := \exp(\sigma r)\big(1 + rs\exp(\sigma r)\big) + 1 \qquad b_1(s) \equiv 1 \tag{6.335}$$

Step $i = 2$: We define

$$k_1(\xi_1) := -\big(\exp(\sigma r)\big(1 + r\big(1+\xi_1^2\big)\exp(\sigma r)\big) + 1 + 2\sigma\big)\xi_1 \tag{6.336}$$

$$\begin{aligned}
\delta_1(\xi_1) := &\big(\exp(\sigma r)\big(1 + r\big(1+3\xi_1^2\big)\exp(\sigma r)\big) + 1 + 2\sigma\big)\\
&\times\big(\exp(\sigma r)\big(1 + r\big(1+\xi_1^2\big)\exp(\sigma r)\big) + 2 + 2\sigma\big)
\end{aligned}\tag{6.337}$$

$$B_2(s) := \exp(\sigma r)\big(1 + r\big(1+s^2\big)\exp(\sigma r)\big) + 2 + 2\sigma \tag{6.338}$$

$$\begin{aligned}
\gamma_2(s) &:= 1 + \exp(\sigma r)\big(1 + sr\exp(\sigma r)B_2\big(\exp(\sigma r)s\big)\big)B_2\big(\exp(\sigma r)s\big)\\
b_2(s) &\equiv 1
\end{aligned}\tag{6.339}$$

$$\rho_1(s) := \exp(\sigma r)\big(1 + 2rs\exp(\sigma r)\big) + 1 \tag{6.340}$$

$$a(s,\xi_1) := \gamma_1(s)\delta_1(\xi_1) + \gamma_1(s) + \gamma_2(s) + \big(1 + s\mu_1(\xi_1)\big)\rho_1(s) \tag{6.341}$$

and

$$\mu_2(\xi_1,\xi_2) := \sigma + \frac{1}{4\sigma}a^2(p,\xi_1) + \gamma_2(p) + \gamma_1(p)\delta_1(\xi_1) \tag{6.342}$$

where

$$p := 1 + \frac{1}{2}\xi_1^2 + \frac{1}{2}\big|\xi_2 + \big(\exp(\sigma r)\big(1 + r\big(1+\xi_1^2\big)\exp(\sigma r)\big) + 1 + 2\sigma\big)\xi_1\big|^2 \tag{6.343}$$

The stabilizing feedback law is given by

$$\begin{aligned}
u(t) = -\mu_2\big(x_1(t),x_2(t)\big)\big(&x_2(t)\\
&+ \big(\exp(\sigma r)\big(1 + r\big(1+x_1^2(t)\big)\exp(\sigma r)\big) + 1 + 2\sigma\big)x_1(t)\big)
\end{aligned}\tag{6.344}$$

By virtue of Theorem 6.6, the functional

$$\begin{aligned}
V(x) := \max_{\theta\in[-r,0]} &\exp(2\sigma\theta)\big(x_1^2(\theta)\\
&+ \big|x_2(\theta) + \big(\exp(\sigma r)\big(1 + r\big(1+x_1^2(\theta)\big)\exp(\sigma r)\big) + 1 + 2\sigma\big)x_1(\theta)\big|^2\big)
\end{aligned}\tag{6.345}$$

is a SRCLF, and system (6.333) with (6.344) is URGAS.

6.7.2 Backstepping for Finite-Dimensional Discrete-Time Systems

In this section we consider triangular discrete-time single input systems (i.e., $u \in \Re$) of the form (4.56) under hypotheses (L1–5) with $\mathcal{X} = \Re^n$, $\mathcal{Y} = \Re^k$, and $U = \mathcal{U} = \Re$, for which we suppose that there exist continuous functions $F_i : Z^+ \times \Re^{i+1} \to \Re$ $(i = 1, \ldots, n)$ with $f_i(t, 0) = 0$ for all $t \geq 0$ such that, for all $(t, x, u) \in Z^+ \times \Re^n \times \Re$,

$$f(t, 0, x, u) = \big(F_1(t, x_1, x_2), F_2(t, x_1, x_2, x_3), \ldots, F_n(t, x_1, \ldots, x_n, u)\big)' \tag{6.346}$$

The following theorem is the main result of this section and provides sufficient conditions for the robust stabilization of system (4.56).

Theorem 6.7 *Consider system* (4.56) *under hypotheses* (L1–5) *with* $\mathcal{X} = \Re^n$, $\mathcal{Y} = \Re^k$, $U = \mathcal{U} = \Re$, *and* $D \subset \Re^l$ *being compact with* $0 \in D$ *and suppose that there exist continuous functions* $F_i : Z^+ \times \Re^{i+1} \to \Re$ $(i = 1, \ldots, n)$ *with* $F_i(t, 0) = 0$ *for all* $t \in Z^+$ *such that* (6.346) *holds*. *Furthermore*, *suppose that there exist continuous functions* $k_i : Z^+ \times \Re^i \to \Re$ $(i = 1, \ldots, n)$ *with* $k_i(t, 0) = 0$ *for all* $t \in Z^+$ *such that the following identities hold for all* $(t, x) \in Z^+ \times \Re^n$:

$$F_1\big(t, x_1, k_1(t, x_1)\big) = 0 \tag{6.347}$$

$$\begin{aligned} &F_i\big(t, x_1, \ldots, x_i, k_i(t, x_1, \ldots, x_i)\big) \\ &\quad = k_{i-1}\big(t + 1, F_1(t, x_1, x_2), \ldots, F_{i-1}(t, x_1, \ldots, x_i)\big) \\ &\quad \textit{for } i = 2, \ldots, n \end{aligned} \tag{6.348}$$

Consider the following vector fields defined on $\Re^+ \times \Re^n$:

$$\tilde{F}(t, x) := f\big(t, 0, x, k_n(t, x)\big) \tag{6.349}$$

$$\mathrm{F}^{(0)}(t, x) := x \tag{6.350}$$

$$\mathrm{F}^{(i)}(t, x) := \tilde{F}\big(t + i - 1, \mathrm{F}^{(i-1)}(t, x)\big) \quad \textit{for } i \geq 1 \tag{6.351}$$

Let $p \in C^0(\Re^n; \Re^+)$ *be a positive definite function with* $p(0) = 0$ *that satisfies* $p(x) \geq K|x|$ *for all* $x \in \Re^n$ *for certain constant* $K > 0$, *and let* $\gamma > 1$ *and* $\lambda \in (0, 1)$ *be constants*. *Let* $D(\gamma, \lambda) \subseteq D$ *the set of all* $d \in D$ *that satisfy the following property*: $\forall (t, x) \in Z^+ \times \Re^n$,

$$\sum_{i=0}^{n-1} \gamma^i p\Big(\mathrm{F}^{(i)}\big(t + 1, f\big(t, d, x, k_n(t, x)\big)\big)\Big) \leq \lambda \sum_{i=0}^{n-1} \gamma^i p\big(\mathrm{F}^{(i)}(t, x)\big) \tag{6.352}$$

Then the following statements hold:

(i) *For every pair* $\gamma > 1$, $\lambda \in (0, 1)$ *with* $\lambda\gamma \geq 1$, *the set* $D(\gamma, \lambda) \subseteq D$ *is a nonempty compact set with* $0 \in D(\gamma, \lambda)$.

(ii) *For every pair* $\gamma > 1$, $\lambda \in (0, 1)$ *with* $\lambda\gamma \geq 1$, *the closed-loop system* (4.56) *with* $u(t) = k_n(t, x(t))$ *and* $d(t) \in D(\gamma, \lambda)$ *is RGAS*.

The main idea that lies behind the proof of Theorem 6.7 is to construct a continuous feedback that guarantees the so-called "dead-beat property of order n" (see [61]) for the nominal system (4.56) with $d = 0$. Then by making use of a Control Lyapunov Function for the nominal closed-loop system, we establish RGAS for disturbances that belong to an appropriate set, namely, the set $D(\gamma, \lambda) \subseteq D$. The proof of Theorem 6.7 is based on the following lemma, which is similar to Theorem 3.2 in [61].

Lemma 6.8 (Finite-time stabilization and explicit construction of CLFs for triangular single-input systems) *Consider the single-input discrete-time system*

$$x(t+1) = F\big(t, x(t), u(t)\big)$$
$$u(t) \in \Re, x(t) := (x_1(t), \dots, x_n(t)) \in \Re^n,\ t \in Z^+ \tag{6.353}$$

where

$$F(t, x, u) = \big(F_1(t, x_1, x_2), F_2(t, x_1, x_2, x_3), \dots, F_n(t, x_1, \dots, x_n, u)\big)' \tag{6.354}$$

for certain continuous mappings $F_i : Z^+ \times \Re^{i+1} \to \Re$ ($i = 1, \dots, n$) with $F_i(t, 0) = 0$ for all $t \in Z^+$. Suppose that there exist continuous functions $k_i : Z^+ \times \Re^i \to \Re$ ($i = 1, \dots, n$) with $k_i(t, 0) = 0$ for all $t \in Z^+$ such that the identities (6.347), (6.348) hold for all $(t, x) \in Z^+ \times \Re^n$. Then zero is RGAS for the closed-loop system (6.353) with $u(t) = k_n(t, x(t))$, namely, the system

$$x(t+1) = \tilde{F}\big(t, x(t)\big) \tag{6.355}$$

where

$$\tilde{F}(t, x) := F\big(t, x, k_n(t, x)\big) \tag{6.356}$$

Moreover, the closed-loop system (6.353) with $u(t) = k_n(t, x(t))$ (system (6.355)), has the dead-beat property of order n, i.e., for every $(t_0, x_0) \in Z^+ \times \Re^n$, the unique solution $x(t)$ of the closed-loop system (6.353) with $u(t) = k_n(t, x(t))$ and initial condition $x(t_0) = x_0$ satisfies

$$x(t) = 0 \quad \textit{for all } t \geq t_0 + n \tag{6.357}$$

Furthermore, let $p \in C^0(\Re^n; \Re^+)$ be a positive definite function with $p(0) = 0$ that satisfies $p(x) \geq K|x|$ for all $x \in \Re^n$ for certain constant $K > 0$ and the vector fields $(t, x) \in Z^+ \times \Re^n \to \mathrm{F}^{(i)}(t, x) \in \Re^n$ defined by (6.350), (6.351) with $\tilde{F}$ defined by (6.356). Then, for every $\gamma > 1$, the continuous function $V_\gamma : Z^+ \times \Re^n \to \Re^+$ defined by

$$V_\gamma(t, x) := \sum_{i=0}^{n-1} \gamma^i p\big(\mathrm{F}^{(i)}(t, x)\big) \tag{6.358}$$

is an SRCLF for (6.353). Particularly, for every $\gamma > 1$, there exist functions $a_2 \in K_\infty$ and $\beta \in K^+$ such that

$$K|x| \le V_\gamma(t,x) \le a_2\big(\beta(t)|x|\big) \quad \forall (t,x) \in Z^+ \times \Re^n \tag{6.359}$$

$$V_\gamma\big(t+1, F(t,x,u)\big) \le \Psi_\gamma(t,x,u) \quad \forall (t,x,u) \in Z^+ \times \Re^n \times \Re \tag{6.360}$$

$$\inf_{u\in\Re} \Psi_\gamma(t,x,u) \le \Psi_\gamma\big(t,x,k_n(t,x)\big) \le \frac{1}{\gamma} V_\gamma(t,x) \quad \forall (t,x) \in Z^+ \times \Re^n \tag{6.361}$$

where the function $\Psi_\gamma : \Re^+ \times \Re^n \times \Re \to \Re^+$ *is quasi-convex with respect to* $u \in \Re$ *with* $\Psi_\gamma(t,0,0) = 0$ *for all* $t \in Z^+$ *and is defined by*

$$\Psi_\gamma(t,x,u) := \sup\big\{V_\gamma(t+1, F\big(t,x,k_n(t,x)+v\big); |v| \le \big|u - k_n(t,x)\big|\big\} \tag{6.362}$$

Proof By the definition of the vector fields $\mathrm{F}^{(i)}(t,x)$, we notice that for every $(t_0,x_0) \in Z^+ \times \Re^n$, the solution of (6.355) with initial condition $x(t_0) = x_0$ satisfies $x(i+t_0) = \mathrm{F}^{(i)}(t_0,x_0)$ for all $i \ge 0$. In order to prove that system (6.355) satisfies the dead-beat property of order n, it suffices to show that $\mathrm{F}^{(n)}(t,x) \equiv 0$ for all $(t,x) \in Z^+ \times \Re^n$. Using induction arguments, it is established that

$$\mathrm{F}^{(i)}\big(t+1, \tilde{F}(t,x)\big) = \mathrm{F}^{(i+1)}(t,x) \quad \text{and} \quad \mathrm{F}^{(i)}(t,0) := 0 \quad \text{for all } i \ge 0 \tag{6.363}$$

The proof of the above relations is easy and is left to the reader. Define the sets $S^{(i)}(t) \subseteq \Re^n$ for $t \in Z^+$ by the following formulas:

$$S^{(0)}(t) := \Re^n \tag{6.364}$$

$$S^{(i)}(t) := \big\{x \in S^{(i-1)}(t) : x_{n-i+1} = k_{n-i}(t,x_1,\dots,x_{n-i})\big\} \quad \text{for } 1 \le i \le n-1 \tag{6.365}$$

$$S^{(i)}(t) := \{0\} \quad \text{for } i \ge n \tag{6.366}$$

where $k_i : \Re^+ \times \Re^i \to \Re (i = 1,\dots,n)$ are the functions involved in (6.347), (6.348). Notice that the definitions of the sets $S^{(i)}(t)$ above imply that, for $1 \le i \le n-1$,

$$S^{(i)}(t) := \big\{x \in \Re^n : x_n = k_{n-1}(t,x_1,\dots,x_{n-1}),\dots, x_{n-i+1} = k_{n-i}(t,x_1,\dots,x_{n-i})\big\} \tag{6.367}$$

Next, we make the following claim.

Claim $\mathrm{F}^{(i)}(t,x) \in S^{(i)}(t+i)$ *for all* $i \ge 0$.

Clearly, definitions (6.350) and (6.364) imply that the above claim is true for $i = 0$. In order to prove the above claim, by virtue of definition (6.351), it suffices to prove the implication that $\tilde{F}(t+i,x) \in S^{(i+1)}(t+i+1)$ if $x \in S^{(i)}(t+i)$.

Since $\tilde{F}(t,0) = 0$ for all $t \ge 0$, it follows that the above implication is true for $i \ge n$. For the case $0 \le i \le n-1$, the above implication is an immediate consequence of (6.363) and properties (6.347), (6.348).

Notice that the previous claim and definition (6.366) imply that

$$\mathrm{F}^{(n)}(t,x) \equiv 0 \quad \text{for all } (t,x) \in Z^+ \times \Re^n \tag{6.368}$$

Thus, system (6.355) satisfies the dead-beat property of order n.

Let $\gamma > 1$ and consider the function $V_\gamma : \Re^+ \times \Re^n \to \Re^+$ defined by (6.358). Since all vector fields $\mathrm{F}^{(i)}(t,x)$ $(i \geq 0)$ defined by (6.350), (6.351) and p are continuous on $\Re^+ \times \Re^n$ with $\mathrm{F}^{(i)}(t,0) = 0$ and $p(0) = 0$ for all $t \geq 0$, by virtue of Lemma 2.3 in Chap. 2, there exist $a_2 \in K_\infty$ and $\beta \in K^+$ such that the right-hand side inequality (6.359) holds. The left-hand side inequality (6.359) is an immediate consequence of definitions (6.350), (6.358) and the fact that $p(x) \geq K|x|$ for all $x \in \Re^n$. Inequality (6.360) is an immediate consequence of definition (6.362). We next prove inequality (6.361). Notice that by virtue of definition (6.358) and property (6.363), we have, for all $(t,x) \in Z^+ \times \Re^n$,

$$
\begin{aligned}
V_\gamma\bigl(t+1, \tilde{F}(t,x)\bigr) &= \sum_{i=0}^{n-1} \gamma^i p\bigl(\mathrm{F}^{(i)}\bigl(t+1, \tilde{F}(t,x)\bigr)\bigr) \\
&= \sum_{i=0}^{n-1} \gamma^i p\bigl(\mathrm{F}^{(i+1)}(t,x)\bigr) = \sum_{i=1}^{n} \gamma^{i-1} p\bigl(\mathrm{F}^{(i)}(t,x)\bigr)
\end{aligned}
$$

Consequently, from (6.368), definitions (6.356), (6.362), and the above equality it follows that

$$
\begin{aligned}
\Psi_\gamma\bigl(t,x,k_n(t,x)\bigr) &= V_\gamma\bigl(t+1, \tilde{F}(t,x)\bigr) \\
&= \frac{1}{\gamma}\sum_{i=1}^{n-1} \gamma^i p\bigl(\mathrm{F}^{(i)}(t,x)\bigr) \leq \frac{1}{\gamma}\sum_{i=0}^{n-1} \gamma^i p\bigl(\mathrm{F}^{(i)}(t,x)\bigr) = \frac{1}{\gamma} V_\gamma(t,x)
\end{aligned}
$$

We conclude that inequality (6.361) holds. The proof of the fact that the function $\Psi_\gamma(t,x,u)$ is quasi-convex with respect to $u \in \Re$ is identical to the proof of implication (c) $\Rightarrow$ (a) of Proposition 6.1 and is omitted. It follows from Proposition 2.3 in Chap. 2 that the closed-loop system (6.353) with $u(t) = k_n(t,x(t))$ is RGAS. The proof is complete. □

Proof of Theorem 6.7 Notice that for the case $d = 0$, it follows from (6.346) that system (4.56) has the triangular structure (6.354). Thus Lemma 6.8 holds, and (6.352) for $d = 0$ and $1/\gamma \leq \lambda < 1$ is a consequence of inequality (6.361). This proves that $D(\gamma,\lambda) \subseteq D$ is a nonempty set with $0 \in D(\gamma,\lambda)$. The compactness of $D(\gamma,\lambda) \subseteq D$ follows from the compactness of D and continuity of all mappings involved in (6.352) with respect to d. We next prove statement (ii). Let the Lyapunov function $V_\gamma : Z^+ \times \Re^n \to \Re^+$ be defined by (6.358) and notice that inequality (6.352), in conjunction with definition (6.358), implies

$$
\begin{gathered}
V_\gamma\bigl(t+1, f\bigl(t,d,x,k_n(t,x)\bigr)\bigr) \leq \lambda V_\gamma(t,x) \\
\forall (t,x,d) \in Z^+ \times \Re^n \times D(\gamma,\lambda)
\end{gathered}
\tag{6.369}
$$

From (6.359) and (6.369) in conjunction with Proposition 2.3 in Chap. 2 it follows that the closed-loop system (4.56) with $u(t) = k_n(t,x(t))$ and $d(t) \in D(\gamma,\lambda)$ is RGAS. The proof is complete. □

Example 6.7.2 Consider the nonlinear planar autonomous system

$$\begin{aligned} x_1(t+1) &= \big(1+d(t)\big)\big|x_1(t)\big| - x_2^2(t) \\ x_2(t+1) &= x_2(t) + u(t) \\ &\big(x_1(t), x_2(t)\big) \in \Re^2, u(t) \in \Re, d(t) \in D := [-1,1], t \in Z^+ \end{aligned} \tag{6.370}$$

Notice that the dynamics of system (6.370) satisfy (6.356) with $f_1(t, x_1, x_2) := |x_1| - x_2^2$ and $f_2(t, x_1, x_2, u) := x_2 + u$. Moreover, (6.347) and (6.348) are satisfied for $k_1(t, x_1) := |x_1|^{\frac{1}{2}}$ and $k_2(t, x_1, x_2) := -x_2 + ||x_1| - x_2^2|^{\frac{1}{2}}$. Consequently, the vector fields $\mathrm{F}^{(i)}(t, x)$ are defined by (6.350), (6.351):

$$\mathrm{F}^{(0)}(t,x) := (x_1, x_2)' \qquad \mathrm{F}^{(1)}(t,x) := \big(|x_1| - x_2^2, \big||x_1| - x_2^2\big|^{\frac{1}{2}}\big)'$$

We select $p(x_1, x_2) := |x_1| + |x_2|$. Thus, inequality (6.352) is equivalent to the following inequality:

$$\begin{aligned} &\big|(1+d)|x_1| - x_2^2\big| + \big||x_1| - x_2^2\big|^{\frac{1}{2}} \\ &\quad + \gamma\big|\big|(1+d)|x_1| - x_2^2\big| - \big||x_1| - x_2^2\big|\big| \\ &\quad + \gamma\big|\big|(1+d)|x_1| - x_2^2\big| - \big||x_1| - x_2^2\big|\big|^{\frac{1}{2}} \\ &\leq \lambda|x_1| + \lambda|x_2| + \gamma\lambda\big||x_1| - x_2^2\big| + \gamma\lambda\big||x_1| - x_2^2\big|^{\frac{1}{2}} \end{aligned} \tag{6.371}$$

Clearly, inequality (6.371) is satisfied for all $(x_1, x_2) \in \Re^2$ if $\lambda\gamma > 1$ and $|d| \leq \min\{\frac{\lambda}{1+\gamma}, \frac{\lambda^2}{\gamma^2}, \frac{(\gamma\lambda-1)^2}{\gamma^2}\}$. Thus, we conclude that the closed-loop system (6.370) with $u(t) = -x_2(t) + ||x_1(t)| - x_2^2(t)|^{\frac{1}{2}}$ and $d(t) \in D(\gamma, \lambda)$ for $\gamma = \exp(c) + 1$ and $\lambda = \exp(-c)$, i.e., for $d(t) \in \{d \in [-1,1] : |d| \leq \frac{1}{\exp(2c)(\exp(c)+1)^2}\}$, is RGAS. Notice that larger values for the constant $c > 0$ (or equivalently, larger values for λ) give smaller values for the radius of the disturbance set $D(\gamma, \lambda)$.

6.8 Small-Gain Method

The purpose of this section is to briefly introduce the small-gain method arising from modern nonlinear control design for dynamical systems with strong nonlinearities and intricate structures. Like previously presented synthesis methods, the small-gain control method is also an efficient tool for the explicit design of nonlinear controllers. However, the small-gain method distinguishes itself from other constructive feedback design approaches in that it is motivated by robust nonlinear control design with dynamic uncertainties. When modeling a complex engineering system, those unmodeled dynamics such as friction in mechanical systems are often referred to as *dynamic uncertainty*. The state variables of the unmodeled dynamics are not perfectly measurable and therefore pose serious technical challenges for nonlinear control design. Other control problems also motivate us to consider the presence of dynamic uncertainty. For example, even when the complete mathematical model of a physical system is known, the controller design problem may be intractable either because the exact model has an intricate structure or because there is a large number of the states (measurable or not). To address these challenges, we purposefully decompose the model into a simplified and tractable control subsystem interacted with another subsystem which we will consider as the dynamic uncertainty. The advantage of this trick in nonlinear control design will be seen below, but it is important to stress that the small-gain method shares a design philosophy radically different from other constructive design approaches. Nonetheless, when combined with other methods such as backstepping, the small-gain methodology turns out to be powerful for robust adaptive output-feedback control design [23] and decentralized nonlinear control design [24, 27]. The reader should consult the literature for other extensions to control design problems for discrete-time systems, time-delay systems, and coupled differential-difference equations (see the review article [32] and references therein), to name only a few.

6.8.1 What are Small-Gain Design Techniques?

The nonlinear small-gain control method consists of two crucial results:

- Gain assignment by feedback.
- Nonlinear small-gain theorem for the stability analysis of the closed-loop system.

Before describing the problem of gain assignment by feedback, let us recall the nonlinear, ISS small-gain theorem for an interconnection of two ISS subsystems

$$\dot{x}_1 = f_1(x_1, x_2, u_1) \tag{6.372}$$

$$\dot{x}_2 = f_2(x_1, x_2, u_2) \tag{6.373}$$

where $x_i \in \Re^{n_i}$ and $u_i \in \Re^{m_i}$ for $i = 1, 2$. Assume that both f_1 and f_2 are locally Lipschitz and vanish at the origin. See Fig. 1.2 for an illustration, where $y_1 = x_1$ and $y_2 = x_2$.

It is shown in [28] that the interconnection of two ISS systems remains to be ISS, provided that the loop gain is less than the identity. More precisely:

Theorem 6.8 *Assume that each subsystem is ISS in the sense of Sontag, that is, there exist $\beta_i \in KL$ and $\gamma_i, \gamma_i^u \in K$ with $i = 1, 2$ such that the solutions of each x_i-subsystem satisfy*

$$\left|x_1(t)\right| \le \beta_1\left(\left|x_1(0)\right|, t\right) + \gamma_1\left(\|x_2\|\right) + \gamma_1^u\left(\|u_1\|\right) \tag{6.374}$$

$$\left|x_2(t)\right| \le \beta_2\left(\left|x_2(0)\right|, t\right) + \gamma_2\left(\|x_1\|\right) + \gamma_2^u\left(\|u_2\|\right) \tag{6.375}$$

where $\|\cdot\|$ denotes the L_∞-norm. If there exist functions $\rho_i \in K_\infty$, $i = 1, 2$, such that one of the following two equivalent small-gain conditions holds:

$$(Id + \rho_1) \circ \gamma_1 \circ (Id + \rho_2) \circ \gamma_2 \le Id, \tag{6.376}$$

$$(Id + \rho_2) \circ \gamma_2 \circ (Id + \rho_1) \circ \gamma_1 \le Id \tag{6.377}$$

with Id standing for the identity mapping, then the total interconnected (x_1, x_2)*-system is ISS, that is, the solutions* $(x_1(t), x_2(t))$ *satisfy*:

$$\left|\left(x_1(t), x_2(t)\right)'\right| \le \beta\left(\left|\left(x_1(0), x_2(0)\right)'\right|, t\right) + \gamma\left(\left\|(u_1, u_2)'\right\|\right) \tag{6.378}$$

for some functions $\beta \in KL$ *and* $\gamma \in K$.

Remark 6.5 It is worth noting that a close-form expression for the (not necessarily unique) gain function γ can be obtained on the basis of the functions γ_i, γ_i^u, and ρ_i, $i = 1, 2$. Also, as shown in [28], Theorem 6.8 can be extended to more general interconnected systems in the sense that one subsystem or both subsystems are input-to-state practically stable (ISpS), input-to-output stable (IOS), or even input-to-output practically stable (IOpS). The term "practical" is used to take into account the case when the unforced subsystems do not have an equilibrium (e.g., when non-vanishing disturbances occur).

At first sight, ISS may look a restrictive stability requirement for nonlinear systems. Surprisingly, Sontag [63] has shown that any nonlinear finite-dimensional system of the form $\dot{x} = f(x, u)$ can be made ISS by feedback (or feedback transformation) if and only if it is stabilizable. A refinement of this important result shows that, for some special class of nonlinear systems, we may even assign somehow an arbitrary gain for the derived ISS system. This result is referred to as *gain assignment by feedback* and plays a crucial role in nonlinear control design based on small-gain theorems. To illustrate this point, let us consider an interconnected system composed of an ISS subsystem with given gain $\gamma_1 \in K_\infty$ (which may not be small) and a controlled dynamical system which may not be ISS, as depicted in Fig. 6.1.

The objective of the gain assignment result is twofold: first, apply a pre-feedback law à la Sontag [63] to render the non-ISS subsystem ISS. Then, redesign this con-

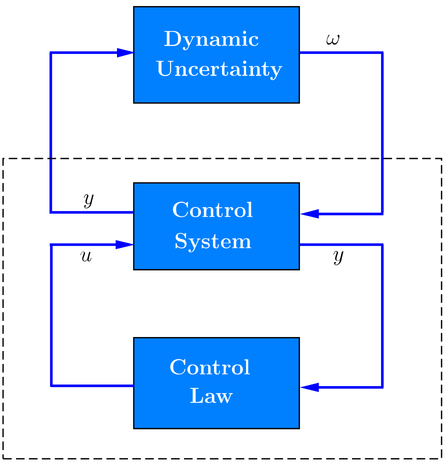

Fig. 6.1 Control of interconnected systems via partial-state feedback

trol law to make the interconnection gain so small that the small-gain condition in (6.376) or (6.377) holds. For example, such a gain to be assigned to the non-ISS system can be $\varepsilon\gamma_1^{-1}$ with $0 < \varepsilon < 1$.

In the next subsection, we describe this strategy in greater details. It should be mentioned that it is a control design tool based on *partial-state* feedback.

6.8.2 Gain Assignment via Feedback

We begin with the robust stabilization of the following dynamically perturbed nonlinear system:

$$\dot{z} = q(z, y) \tag{6.379}$$

$$\dot{y} = u + \omega(z, y) \tag{6.380}$$

where $z \in \Re^{n_0}$ is the (unmeasured) state of the dynamic uncertainty, $y \in \Re$ is the (measured) state of the controlled system, and u is the control input. Assume that the z-system (6.379) is ISS with a given gain $\gamma_1 \in K_\infty$. It is also assumed that q and ω are locally Lipschitz and vanish at the origin and that the nonlinear perturbation $\omega(z, y)$ satisfies the following property:

$$\left|\omega(z, y)\right| \le \phi_1\big(|z|\big) + \phi_2\big(|y|\big) \tag{6.381}$$

for two smooth positive semidefinite functions ϕ_1 and ϕ_2. Without loss of generality, we may assume that ϕ_1 is of class K_∞ and $\phi_2(|y|) = |y|\phi_3(|y|)$ with a smooth function ϕ_3.

To design a globally stabilizing control law $u = \alpha(y)$ for system (6.379)–(6.380), we can apply the small-gain method as follows. First, pick a function $\gamma_2 \in K_\infty$ that, together with γ_1, satisfies the small-gain condition in (6.376) or (6.377). For simplicity, pick $\gamma_2(s) = \varepsilon\gamma_1^{-1}(s)$ for a constant ε in $(0, 1)$. This way, the small-gain condition in (6.376) or (6.377) holds with linear ρ_1 and ρ_2. Then, we design a partial-state feedback law $u = \alpha(y)$ that makes the y-subsystem ISS with gain γ_2. After this is done, a direct application of Small-Gain Theorem 6.8 yields the global stability of the closed-loop system.

The following gain assignment result shows when γ_2 can be assigned as the ISS-gain for the y-subsystem by feedback.

Theorem 6.9 *There always exists a continuous feedback law $u = \alpha(y)$ such that the closed-loop y-system is ISS with gain γ_2 when z is considered as the input. Moreover, if the following condition holds*:

$$\phi_1 \circ \gamma_2^{-1}(s) \quad \textit{is linearly bounded near the origin} \tag{6.382}$$

then, the feedback law $u = \alpha(y)$ can be made smooth.

Proof Consider the Lyapunov function candidate $V = \frac{1}{2}y^2$. By direct computation, its time derivative satisfies

$$\begin{aligned}\dot{V} &= y\big(u + \omega(z, y)\big) \\ &\le y\big(u + y\phi_3\big(|y|\big)\big) + |y|\phi_1\big(|z|\big)\end{aligned} \tag{6.383}$$

Take the control law

$$u = -y\phi_3\big(|y|\big) - k\phi_1 \circ \gamma_2^{-1}\big(|y|\big)\,\mathrm{sgn}(y) \tag{6.384}$$

for some constant $k > 1$.

From (6.383) and (6.384) it follows that

$$\dot{V} \le -k|y|\phi_1 \circ \gamma_2^{-1}\big(|y|\big) + |y|\phi_1\big(|z|\big) \tag{6.385}$$

which implies directly that the closed-loop y-system is ISS with gain γ_2.

Clearly, the control law in (6.384) is continuous and can be made smooth under assumption (6.382). The proof is complete. □

We borrow an example from the seminal work of Byrnes and Isidori [7] to illustrate our result. Consider the control system

$$\begin{aligned}\dot{z} &= -z^5 + y^2 \\ \dot{y} &= u + z^2\end{aligned} \tag{6.386}$$

Clearly, (6.386) is a special member of the class of dynamically perturbed nonlinear systems (6.379)–(6.380). A direct application of Theorem 6.9 yields a continuous partial-state feedback of the form

$$u = -k|y|^{3/5}y^{1/5} \tag{6.387}$$

for any constant $k > 1$ that globally asymptotically stabilizes the system (6.386). As it can be directly checked, assumption (6.382) fails to hold. Thus, the control law (6.387) cannot be made smooth. In fact, Byrnes and Isidori have shown by means of the center manifold theory that there is no continuously differentiable control law of the form $u = \alpha(y)$ that even locally asymptotically stabilizes the system (6.386).

To establish a gain assignment result for higher-dimensional systems, consider a chain of integrators subject to external inputs denoted as $\omega := (\omega_1, \dots, \omega_n)'$:

$$\begin{aligned}\dot{x}_1 &= x_2 + \omega_1 \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \omega_{n-1} \\ \dot{x}_n &= u + \omega_n\end{aligned} \tag{6.388}$$

The following gain assignment result is a special case of [28, Theorem 2.2].

Theorem 6.10 *For any function $\gamma \in K_\infty$, if γ^{-1} is linearly bounded near the origin, then there exists a feedback law of the form $u = \alpha(x)$ such that the closed-loop system (6.388) is ISS with respect to the input ω and, in particular, IOS with the assigned gain γ when $y = x_1$ is taken as the output.*

With Theorem 6.10 at hand, the global stabilization problem by partial-state feedback is easily solvable, via the proposed small-gain method, for

$$\begin{aligned}
\dot{z} &= q(z, x_1) \\
\dot{x}_1 &= x_2 + \omega_1(z, x_1) \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \omega_{n-1}(z, x_1) \\
\dot{x}_n &= u + \omega_n(z, x_1)
\end{aligned} \tag{6.389}$$

where z is the unmeasured state of the dynamic uncertainty, x is the measured state of the controlled system, and u is the control input.

6.9 Observers and Dynamic Feedback

Consider the following discrete-time system:

$$\begin{aligned}
x(t+1) &= f\big(t, d(t), x(t), u(t)\big) \\
&\quad x(t) \in \mathcal{X}, d(t) \in D, u(t) \in U, t \in Z^+
\end{aligned} \tag{6.390}$$

$$Y(t) = H\big(t, x(t)\big) \quad Y(t) \in \mathcal{Y} \tag{6.391}$$

under hypotheses (L1–5). We consider the existence of a dynamic time-varying output feedback law $w(t+1) = g(t, y(t), w(t))$, $u(t) = k(t, y(t), w(t))$, $w(t) \in W$ (Dynamic ROFS problem), where $y(t)$ denotes the measured output given by

$$y(t) = h\big(t, x(t)\big) \quad y(t) \in \mathcal{Y}' \tag{6.392}$$

and $\mathcal{Y}'$, W are normed linear spaces. We will further assume:

(L6) For the output map $h \in CU(Z^+ \times \mathcal{X}; \mathcal{Y}')$ involved in (6.392) with $h(t, 0) = 0$ for all $t \in Z^+$, there exists a set $S \subseteq \mathcal{Y}'$ such that $S = h(t, \mathcal{X})$ for all $t \in Z^+$.

We next give the notion of robust complete observability for discrete-time systems. The definition given here directly extends the corresponding notions given in [15, 66], concerning autonomous continuous-time systems.

Definition 6.5 Consider the system (6.390) and let $(d_i, u_i) \in D \times U$, $i = 0, 1, \dots,$ and define recursively the following family of continuous mappings:

$$\begin{aligned}
&F_0(t, x) = x \quad F_1\big(t, x, d^{(1)}, u^{(1)}\big) = f(t, d_0, x, u_0) \\
&F_i\big(t, x, d^{(i)}, u^{(i)}\big) := f\big(t + i - 1, d_{i-1}, F_{i-1}\big(t, x, d^{(i-1)}, u^{(i-1)}\big), u_{i-1}\big) \quad i \geq 2 \\
&y_0(t, x) = h(t, x) \quad y_i\big(t, x, d^{(i)}, u^{(i)}\big) := h\big(t + i, F_i\big(t, x, d^{(i)}, u^{(i)}\big)\big) \quad i \geq 1
\end{aligned}$$

where $d^{(i)} := (d_0, \dots, d_{i-1})$, $u^{(i)} := (u_0, \dots, u_{i-1})$ for $i \geq 1$. For an integer $p \geq 1$, define the continuous mapping for all $(t, x, d^{(p)}, u^{(p)}) \in \Re^+ \times \mathcal{X} \times D^p \times U^p$ by

$$y^{(p)}\big(t, x, d^{(p)}, u^{(p)}\big) := \big(y_0(t, x), \dots, y_{p-1}\big(t, x, d^{(p-1)}, u^{(p-1)}\big)\big)$$

We say that a continuous function $k \in CU(Z^+ \times \mathcal{X}; W)$, where W is a normed linear space, is *robustly completely observable from the output* $y = h(t, x)$ *with respect to* (6.390) if there exist an integer $p \geq 1$ and a continuous function (called the reconstruction map) $\Psi \in CU(Z^+ \times S \times (\mathcal{Y}')^p \times \mathcal{U}^p; W)$ such that, for all $(t, x, d^{(p)}, u^{(p)}) \in Z^+ \times \mathcal{X} \times D^p \times U^p$, it holds that

$$k\big(t + p, F_p\big(t, x, d^{(p)}, u^{(p)}\big)\big) \\ = \Psi\big(t + p, y_p\big(t, x, d^{(p)}, u^{(p)}\big), y^{(p)}\big(t, x, d^{(p)}, u^{(p)}\big), u^{(p)}\big) \tag{6.393}$$

We say that system (6.390) is *robustly completely observable from the output* $y = h(t, x)$ if the identity function $k(t, x) = x$ is completely observable.

Remark 6.6

(a) Notice that for every input $(d, u) \in M_D \times M_U$ and for every $(t_0, x_0) \in Z^+ \times \mathcal{X}$, the unique solution $x(t)$ of (6.390) corresponding to (d, u) and initiated from x_0 at time t_0 satisfies the following relation: for all $t \geq t_0 + p$,

$$k\big(t, x(t)\big) \\ = \Psi\big(t, y(t), y(t - p), y(t - p + 1), \dots, y(t - 1), u(t - p), \dots, u(t - 1)\big)$$

Following the terminology in [64], if system (6.390) is robustly completely observable from the output $y = h(t, x)$, then every control $(d, u) \in M_D \times M_U$ final-state distinguishes between any two events in time $p \in Z^+$.

(b) Notice that every continuous function of the measured output $k(t, x) = \theta(t, h(t, x))$, where $\theta : Z^+ \times S \to W$ is a continuous function with the property

"for every pair of bounded sets $I \subset Z^+$, $A \subseteq S$ and for every $\varepsilon > 0$, the set $\theta(I \times A)$ is bounded, and there exists $\delta > 0$ such that $\|\theta(t, y) - \theta(t, y_0)\|_W < \varepsilon$ for all $t \in I$ and $y, y_0 \in A$ with $\|y - y_0\|_{\mathcal{Y}'} < \delta$,"

is robustly completely observable from the measured output.

(c) Notice that since $0 \in \mathcal{X}$ is an equilibrium point for (6.390) and $h(t, 0) = 0$ for all $t \geq 0$, by setting $x = 0$ and $u^{(p)} = 0$ in (6.393), we obtain

$$k(t, 0) = \Psi(t, 0, 0) \quad \forall t \geq p.$$

Without loss of generality, we may assume that the reconstruction map Ψ is continuously extended to $Z^+ \times S \times (\mathcal{Y}')^p \times \mathcal{U}^p$ so that the above equality holds for all $t \in Z^+$.

The following proposition provides sufficient conditions for the solvability of the ROFS problem for (6.390), (6.391), (6.392).

Proposition 6.3 *Consider the ROFS problem for* (6.390), (6.391) *with measured output given by* (6.392) *under hypotheses* (L1–6). *Suppose that*:

(i) *There exists a continuous function* $k \in CU(Z^+ \times \mathcal{X}; U)$ *with* $f(t, d, 0, k(t, 0)) = 0$ *for all* $(t, d) \in Z^+ \times D$ *such that the closed-loop system* (6.390), (6.391) *with* $u(t) = k(t, x(t))$ *is RGAOS.*

(ii) *The feedback function* $k \in CU(Z^+ \times \mathcal{X}; U)$ *is robustly completely observable from the output* $y = h(t, x)$ *with respect to* (6.390).

Then the continuous dynamic ROFS problem for (6.390), (6.391), (6.392) *with measured output* $y = h(t, x)$ *and stabilized output* $Y = H(t, x)$ *is globally solvable, i.e., there exist a normed linear space* W *and* continuous *functions* $k \in CU(Z^+ \times S \times W; U)$ *and* $g \in CU(Z^+ \times S \times W; W)$ *with* $f(t, d, 0, k(t, 0, 0)) = 0$ *and* $g(t, 0, 0) = 0$ *for all* $(t, d) \in Z^+ \times D$ *such that the following system with state space* $\mathcal{X} \times W$ *is RGAOS*:

$$\begin{aligned} x(t+1) &= f\big(t, d(t), x(t), k\big(t, h\big(t, x(t)\big), w(t)\big)\big) \\ w(t+1) &= g\big(t, h\big(t, x(t)\big), w(t)\big) \\ Y(t) &= H\big(t, x(t)\big) \end{aligned} \tag{6.394}$$

Proof Since $k \in CU(Z^+ \times \mathcal{X}; U)$ is robustly completely observable from the output $y = h(t, x)$ with respect to (6.390), there exist an integer $p \geq 1$ and a reconstruction map $\Psi \in CU(Z^+ \times S \times (\mathcal{Y}')^p \times \mathcal{U}^p; \mathcal{U})$ such that for all $(t, x, d^{(p)}, u^{(p)}) \in Z^+ \times \mathcal{X} \times D^p \times U^p$ (6.393), holds. Consider the following system:

$$\begin{aligned} w_1(t+1) &= y(t) \\ w_2(t+1) &= w_1(t) \\ &\vdots \\ w_p(t+1) &= w_{p-1}(t) \\ w_{p+1}(t+1) &= u(t) \\ w_{p+2}(t+1) &= w_{p+1}(t) \\ &\vdots \\ w_{2p}(t+1) &= w_{2p-1}(t) \\ u(t) &= \Psi\big(t, y(t), P\big(w(t)\big)\big) \\ w_i(t) &\in \mathcal{Y}' \quad i = 1, \ldots, p \\ w_i(t) &\in \mathcal{U} \quad i = p+1, \ldots, 2p \\ w(t) := \big(w_1(t), \ldots, w_{2p}(t)\big) &\in W := (\mathcal{Y}')^p \times \mathcal{U}^p \quad t \in Z^+ \end{aligned} \tag{6.395}$$

where

$$P(w) := (w_p, w_{p-1}, \ldots, w_1, w_{2p}, w_{2p-2}, \ldots, w_{p+1}) \tag{6.396}$$

Clearly, for every $(t_0, x_0, w_0, d) \in Z^+ \times \mathcal{X} \times W \times M_D$, the solution of (6.390), (6.391), (6.392) with (6.395) and initial condition $(x(t_0), w(t_0)) = (x_0, w_0)$ corresponding to input $d \in M_D$ satisfies, for all $t \geq t_0 + p$,

$$\begin{aligned} w_i(t) &= y(t-i) \quad i = 1, \ldots, p \\ w_{p+i}(t) &= u(t-i) \quad i = 1, \ldots, p \end{aligned} \tag{6.397}$$

It follows from (6.397) and Remark 6.6(a) that

$$u(t) = k\big(t, x(t)\big) = \Psi\big(t, y(t), P\big(w(t)\big)\big) \quad \forall t \ge t_0 + p \tag{6.398}$$

Equality (6.398) shows that the implemented control law $u(t) = \Psi(t, y(t), P(w(t)))$ coincides with the control action given by the state feedback law $u(t) = k(t, x(t))$ after p time units. By virtue of Theorem 2.1 in Chap. 2 and since the closed-loop system (6.390), (6.391) with $u(t) = k(t, x(t))$ is RGAOS, there exist functions $\sigma \in KL$ and $\beta \in K^+$ such that for all $d \in M_D$ and $(t_0, x_0) \in Z^+ \times \mathcal{X}$, the unique solution $x(t)$ of (6.390), (6.391) with $u(t) = k(t, x(t))$ initiated from $x_0 \in \mathcal{X}$ at time $t_0 \in Z^+$ and corresponding to $d \in M_D$ satisfies (2.8). It follows from (6.398) that for all $d \in M_D$ all $(t_0, x_0, w_0) \in Z^+ \times \mathcal{X} \times W$, the unique solution $(x(t), w(t))$ of (6.390), (6.391), (6.392) with (6.395) and initial condition $(x(t_0), w(t_0)) = (x_0, w_0)$ corresponding to input $d \in M_D$ satisfies

$$\begin{aligned} &\big\|H\big(t, x(t)\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|x(t)\big\|_{\mathcal{X}} \\ &\quad \le \sigma\big(\beta(t_0+p)\big\|x(t_0+p)\big\|_{\mathcal{X}}, t - t_0 - p\big) \quad \forall t \ge t_0 + p \end{aligned} \tag{6.399}$$

Notice that by virtue of Remark 6.6(c) and since $f(t, d, 0, k(t, 0)) = 0$ for all $(t, d) \in Z^+ \times D$, we may conclude that $0 \in \mathcal{X} \times W$ is an equilibrium point for system (6.390), (6.391), (6.392) with (6.395). Moreover, by hypotheses (L1–6) it follows that system (6.390), (6.391), (6.392) with (6.395) satisfies hypotheses (L1–5) and consequently, by virtue of Lemmas 1.3, 1.4 in Chap. 1, and Lemma 2.7 in Chap. 2, there exist functions $\mu \in K^+$ and $a \in K_\infty$ such that for all $d \in M_D$ and $(t_0, x_0, w_0) \in Z^+ \times \mathcal{X} \times W$, the unique solution $x(t)$ of (6.390), (6.391), (6.392) with (6.395) initiated from $(x(t_0), w(t_0)) = (x_0, w_0)$ at time $t_0 \ge 0$ and corresponding to $d \in M_D$ satisfies

$$\big\|x(t)\big\|_{\mathcal{X}} + \big\|w(t)\big\|_{W} \le \mu(t) a\big(\|x_0\|_{\mathcal{X}} + \|w_0\|_W\big) \quad \forall t \ge t_0 \tag{6.400}$$

Combining estimates (6.399) and (6.400), we conclude that the closed-loop system (6.390), (6.391), (6.392) with (6.395) satisfies the Robust Output Attractivity Property (Property P3 of Definition 2.2 in Chap. 2). By Lemma 2.1 the closed-loop system (6.390), (6.391), (6.392) with (6.395) is RGAOS. The proof is complete. □

An immediate consequence of Proposition 6.3 is the following proposition, which provides a necessary and sufficient condition for the solvability of the dynamic ROFS problem for (6.390), (6.391), (6.392).

Proposition 6.4 (Separation principle) *Consider the ROFS problem for* (6.390), (6.391) *with measured output given by* (6.392) *under hypotheses* (L1–6). *The following statements are equivalent*:

(a) *There exist a normed linear space W' and continuous functions $k \in CU(Z^+ \times \mathcal{X} \times W'; U)$ and $g \in CU(Z^+ \times \mathcal{X} \times W'; W')$ with $f(t, d, 0, k(t, 0, 0)) = 0$ and*

$g(t,0,0)=0$ for all $(t,d)\in Z^+\times D$ such that the following system with state space $\mathcal{X}\times W'$ is RGAOS:

$$\begin{aligned} x(t+1) &= f\big(t,d(t),x(t),k\big(t,x(t),w'(t)\big)\big) \\ w'(t+1) &= g\big(t,x(t),w'(t)\big) \\ Y(t) &= H\big(t,x(t)\big) \end{aligned} \tag{6.401}$$

Moreover, the functions $k\in CU(Z^+\times\mathcal{X}\times W';U)$ and $g\in CU(Z^+\times\mathcal{X}\times W';W')$ are robustly completely observable from the output $y'=(h(t,x),w')$ with respect to the system:

$$\begin{aligned} &x(t+1)=f\big(t,d(t),x(t),u_1(t)\big) \\ &w'(t+1)=u_2(t) \\ &\quad\big(x(t),w'(t)\big)\in\mathcal{X}\times W', u(t)=\big(u_1(t),u_2(t)\big)\in U\times W',\ d(t)\in D, t\in Z^+. \end{aligned} \tag{6.402}$$

(b) *The continuous dynamic ROFS problem for* (6.390), (6.391), (6.392) *with measured output $y=h(t,x)$ and stabilized output $Y=H(t,x)$ is globally solvable*, *i.e*., *there exist a normed linear space W and* continuous *functions $k\in CU(Z^+\times S\times W;U)$ and $g\in CU(Z^+\times S\times W;W)$ with $f(t,d,0,k(t,0,0))=0$ and $g(t,0,0)=0$ for all $(t,d)\in Z^+\times D$ such that system* (6.394) *with state space $\mathcal{X}\times W$ is RGAOS.*

Implication (a) $\Rightarrow$ (b) of Proposition 6.4 is an immediate application of Proposition 6.3 to the control system (6.402) with input (u_1,u_2). Implication (b) $\Rightarrow$ (a) of Proposition 6.4 is an immediate consequence of Remark 6.6(b). We remark that since the component $x(t)$ of the solution of (6.402) does not depend on the input u_2, the requirement that the functions $k\in CU(Z^+\times\mathcal{X}\times W';U)$ and $g\in CU(Z^+\times\mathcal{X}\times W';W')$ are robustly completely observable from the output $y'=(h(t,x),w')$ with respect to the system (6.402) implies the requirement that, for every $w'\in W'$, the functions $(t,x)\in Z^+\times\mathcal{X}\to k(t,x,w')\in U$ and $(t,x)\in Z^+\times\mathcal{X}\to g(t,x,w')\in W'$ are robustly completely observable from the output $y=h(t,x)$ with respect to the system (6.390), (6.391), (6.392).

Example 6.9.1 Consider the ROFS problem for the system

$$\begin{aligned} &x_1(t+1)=x_2(t) \\ &x_2(t+1)=x_2^2(t)+u(t) \\ &x_3(t+1)=d(t)x_3(t)+\exp(t)x_2(t) \\ &Y(t)=x(t) \\ &\quad x:=(x_1,x_2,x_3)\in\Re^3, u(t)\in\Re, t\in Z^+, d(t)\in[-r,r] \end{aligned} \tag{6.403}$$

where $r\in[0,1)$, with measured output $y=x_1$. First notice that the feedback function $k(t,x):=-x_2^2$ stabilizes system (6.403). We prove this claim by considering the Lyapunov function $V(t,x) := |x_1| + 3\exp(t)|x_2| + |x_3|$, which clearly satisfies the following inequalities:

$$|Y| = |x| \le V(t,x) \le 5\exp(t)|x| \quad \forall (t,x) \in Z^+ \times \Re^3 \tag{6.404}$$

$$\begin{aligned} & V\big(t+1, x_2, x_2^2 + k(t,x), dx_3 + \exp(t)x_2\big) \\ & \quad \le \big(1+\exp(t)\big)|x_2| + r|x_3| \\ & \quad \le \max\left\{\frac{2}{3}, r\right\} V(t,x) \quad \forall (t,x,d) \in Z^+ \times \Re^3 \times [-r,r] \end{aligned} \tag{6.405}$$

and since $\max\{\frac{2}{3}, r\} < 1$, by virtue of Proposition 2.3 in Chap. 2, we conclude that the closed-loop system (6.403) with $u(t) = k(t, x(t))$ is RGAOS. Moreover, the feedback function $k(t,x) := -x_2^2$ is robustly completely observable from the output $y = x_1$. Particularly, we define the continuous mappings (following the notation of Definition 6.5)

$$F_0(t,x) = x,\; F_1\big(t,x,d^{(1)},u^{(1)}\big) = \begin{pmatrix} x_2 \\ x_2^2 + u_0 \\ d_0 x_3 + \exp(t)x_2 \end{pmatrix}$$

$$y_0(t,x) = x_1,\; y_1\big(t,x,d^{(1)},u^{(1)}\big) := x_2$$

Clearly, we have

$$k\big(t+1, F_1\big(t,x,d^{(1)},u^{(1)}\big)\big) = \Psi(t+1, y_1, y_0, u_0) := -\big(y_1^2 + u_0\big)^2$$

Consequently, the closed-loop system (6.403) with

$$\begin{aligned} w(t+1) &= u(t) \\ u(t) &= -\big(y^2(t) + w(t)\big)^2 \\ w(t) &\in \Re,\; t \in Z^+ \end{aligned}$$

is RGAOS.

6.10 Bibliographical and Historical Notes

1. A vast number of papers and books is devoted to the stabilization of linear systems described by ODEs. The reader can use the reference list included in [64], where many papers and books on linear system theory are cited. For time-delay linear systems, one can consult [51].
2. The goal of solving the Robust Output Feedback Stabilization problem for nonlinear systems is highly nontrivial. The issue of robustness to modeling and actuator errors and measurement noise makes the problem harder. There are classes of nonlinear systems described by ODEs for which stabilizing feedback laws can be designed which also take into account actuator errors and measurement noise, e.g., triangular globally Lipschitz systems. For infinite-dimensional systems, the Robust Output Feedback Stabilization problem is currently under intensive research.

3. Sampled-data globally stabilizing feedback laws for homogeneous systems were proposed in [16].
4. Clearly, a necessary condition for the existence of a stabilizing state feedback law is asymptotic controllability (see [64]). However, there are systems which are asymptotically controllable and cannot be stabilized by a continuous state feedback law, e.g., Artstein's cycles (see, e.g., [5, 64]). The existence of a Control Lyapunov Function under the assumption of asymptotic controllability is studied in [1, 59, 64] (see also references therein). The existence of feedback stabilizers under the assumption under the assumption of asymptotic controllability is studied in [9, 16, 65].
5. The design of stabilizing feedback laws by means of the Artstein–Sontag approach for nonaffine control systems described by ODEs was recently studied in [33, 47]. Recently, the Artstein–Sontag approach was extended in [39], so that it can be applied under weaker hypotheses.
6. A comparison between the Artstein–Sontag approach and the Coron–Rosier approach is provided next.

   - the Artstein–Sontag methodology allows the use of Lipschitz RCLFs, while the Coron–Rosier methodology requires continuously differentiable RCLFs; this is exactly the reason why the Artstein–Sontag approach can be extended to infinite-dimensional systems while the Coron–Rosier approach faces important difficulties with infinite-dimensional systems,
   - the Artstein–Sontag approach can provide explicit formulas for stabilizing feedback laws (see [36, 62]), while the Coron–Rosier approach leads mostly to existential results,
   - the Artstein–Sontag methodology can provide time-independent feedback laws while the Coron–Rosier methodology always gives time-varying feedback laws,
   - the Coron–Rosier approach does not require convexity assumptions; on the other hand, the Artstein–Sontag approach requires that the control set is convex and demands additional properties for the RCLF.

7. Backstepping, also called "adding an integrator" by European authors initially, is a powerful method to recursively design adaptive and nonlinear controllers for systems with a cascade structure or taking the strict-feedback form. As stated in [41], the origin of this idea can be traced back at least to the year 1989 when several related schemes were presented by different authors. See the excellent monograph [41] for the details. Also see [11, 13, 19, 26, 30, 34, 40, 44, 60, 69] for other recent extensions. Special results, which guarantee additional properties for the constructed feedback (e.g., linearity), are given in [45, 68, 70]. The backstepping-based result provided in Sect. 6.7 for systems described by RFDEs appeared in [33], and the backstepping-based result in Sect. 6.7 for discrete-time systems appeared in [34].
8. Another method of proving stability that can be used for feedback design is the use of Matrosov functions (see [43]). However, this approach is not studied here.

9. In many cases, the existence of a control Lyapunov function can lead to practical and/or semiglobal stabilization: see [48, 49, 66].
10. The small-gain method for nonlinear feedback design was introduced in [28] for important classes of dynamically perturbed nonlinear systems. It was quickly generalized to nonlinear systems with (linear) input unmodeled dynamics [42] and (nonlinear) input dynamic uncertainties [25]. Other interesting extensions for nonlinear decentralized systems, time-delay systems, and coupled differential-difference equations can be found in [24, 32] and references therein.
11. The Output Feedback stabilization problem and the corresponding observer design problems for continuous-time systems have been studied by many authors (see [6, 14, 35, 53–58, 71]). The separation principle holds for linear systems described by ODEs. Recent results in [3, 8] have provided the extension of the separation principle for nonlinear systems described by ODEs. The output feedback stabilization results for discrete-time systems contained in the present work first appeared in [31].

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Chapter 7
Applications

7.1 Introduction

This chapter aimed to demonstrate the wide applicability of the analysis and synthesis tools presented in previous chapters. While there are many good control engineering applications we may choose, a tough decision must be made for want of space. We depart from practical application examples commonly adopted by control engineers and researchers, and will study three nonconventional, but challenging and important, case studies from three different areas: mathematical biology, numerical analysis, and mathematical economics. These emerging applications are driving forces behind our search for new tools for nonlinear systems and control. The chapter objective is to present novel solutions to the feedback stabilization problem by means of the tools presented in previous chapters.

In the following, we describe in details these interesting control problems:

- The stabilization of a delayed chemostat model (Sect. 7.2). The application shows how the small-gain results of Chap. 5 can be applied for the robust stabilization of uncertain systems with delays. Moreover, the application illustrates the potential of the use of novel small-gain results to difficult problems in Mathematical Biology and process control engineering.
- The stabilization of numerical schemes for the numerical solution of systems described by ODEs (Sect. 7.3). This application shows how feedback stabilization methodologies can be used for the solution of classical problems in areas of applied mathematics different from mathematical control theory. A well-known problem in numerical analysis is converted to an abstract feedback stabilization problem and is solved by applying modern nonlinear feedback design methodologies.
- The stabilization of the price of a commodity by manipulation of buffer stocks (Sect. 7.4). The application shows the efficiency of the proposed feedback stabilization methodologies for the solution of problems in mathematical economics. The application also shows how government policies may be driven by nonlinear mathematical control theory (hence the term "cybernetics" was coined by the

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mathematician N. Wiener from the greek word $\kappa\upsilon\beta\epsilon\rho\nu\tilde{\omega}$ = govern, to refer to control theory and related areas).

It is clear that the list of potential applications of modern nonlinear control theory in general, and the presented stability and stabilization results in particular, is long. We would partially achieve our objective of writing this monograph if the reader were inspired by the contents of the present chapter and were ready to tackle even more challenging stabilization problems.

7.2 Stabilization of a Delayed Chemostat Model

The small-gain results presented in this book can allow a transient period during which the solutions do not satisfy the IOS inequalities (Theorems 5.1 and 5.2 in Chap. 5). Here, we show how the obtained small-gain results can be used for the feedback stabilization of uncertain chemostat models (see also [4] for applications of small-gain results to biological systems).

More specifically, we consider the robust global feedback stabilization problem for the chemostat model with delays (1.13), where $b \geq 0$ is the cell mortality rate, $r \geq 0$ is the maximum delay, and $K(s) > 0$ is a possibly variable yield coefficient. The functions $\mu : [0, s_{\text{in}}] \to [0, \mu_{\max}]$ with $\mu(0) = 0$ and $\mu(s) > 0$ for all $s > 0$ and $K : [0, s_{\text{in}}] \to (0, +\infty)$ are assumed to be locally Lipschitz functions. It should be noted that the case of variable yield coefficients has been studied recently (see [34, 35]) and has been proposed for the justification of experimental results. The reader should notice that chemostat models with time delays were considered in [32, 33]. By assuming the existence of a nontrivial equilibrium point for (1.13), i.e., the existence of $(X^*, s^*, D^*) \in (0, +\infty) \times (0, s_{\text{in}}) \times (0, +\infty)$ such that (1.14) holds, and using the change of coordinates and input transformation (1.15) we obtain the equivalent transformed system (1.16).

The stabilization problem for the equilibrium point $(X^*, s^*, D^*) \in (0, +\infty) \times (0, s_{\text{in}}) \times (0, +\infty)$ is crucial: in [30] it is shown that the equilibrium point is unstable even if $\mu : (0, s_{\text{in}}) \to (0, \mu_{\max}]$ is strictly monotone (e.g., the Monod specific growth rate). Moreover, as remarked in [30], the chemostat model (1.11) under (1.12) allows the expression of the effect of the time difference between consumption of nutrient and growth of the cells (see the discussion on pp. 238–240 in [30]).

Here, we solve the feedback stabilization problem for the chemostat by providing a *delay-free* feedback which achieves global stabilization (see Theorem 7.1 below). The proof of the theorem relies on the small-gain results of the book. No knowledge of the maximum delay $r \geq 0$ is assumed. The stabilization is "global": here the adjective "global" refers only to "admissible states," i.e., for all initial conditions $(X_0, s_0(\cdot)) \in (0, +\infty) \times C^0([-r, 0]; (0, s_{\text{in}}))$.

In order to solve the robust feedback stabilization problem for the chemostat model (1.16), we will assume that

(H) There exists $s_p \in (0, s^*)$ such that $\mu(s) > b$ for all $s \in [s_p, s_{\text{in}}]$.

Hypothesis (H) is automatically satisfied for the case of a monotone specific growth rate. Hypothesis (H) can be satisfied for nonmonotone specific growth rates (e.g., Haldane or generalized Haldane growth expressions). By using the trajectory-based small-gain Theorem 5.2, we can prove the following theorem.

Theorem 7.1 *Let $a > 0$ be a constant that satisfies*

$$\min_{s_p \le s \le s_{\text{in}}} \mu(s) - b > aD^* \frac{s^*}{s_{\text{in}}} \tag{7.1}$$

Then, the locally Lipschitz delay-free feedback law given by

$$D(t) = \frac{K(s(t))\mu(s(t))X(t) + aD^*(s^* - \min(s(t), s^*))}{s_{\text{in}} - \min(s(t), s^*)} \tag{7.2}$$

achieves the robust global stabilization of the equilibrium point $(X^*, s(\cdot)) \in (0, +\infty) \times C^0([-r, 0]; (0, s_{\text{in}}))$ *with* $s(\theta) = s^*$ *for all* $\theta \in [-r, 0]$, *for the uncertain chemostat model* (1.13) *under hypothesis* (H).

In the new coordinates (see (1.15)), the feedback law (7.2) takes the form

$$\begin{aligned} u(t) = \ln\Big(& g\big(x_2(t)\big)\exp\big(x_1(t)\big)\min\big(G + \exp\big(x_2(t)\big), G + 1\big) \\ & + \frac{a}{G+1}\max\big(1 - \exp\big(x_2(t)\big), 0\big)\Big) \end{aligned} \tag{7.3}$$

The feedback law (7.3), or (7.2), is a delay-free feedback, which achieves global stabilization of $0 \in \Re \times C^0([-r, 0]; \Re)$ for system (1.16) no matter how large the delay is. Furthermore, no knowledge of the maximum delay $r \ge 0$ is needed for the implementation of (7.3). As a result, the proof of Theorem 7.1 is equivalent to the proof of robust global asymptotic stability of the equilibrium point $0 \in \Re \times C^0([-r, 0]; \Re)$ for system (1.16).

Before we give the proof of Theorem 7.1, it is important to understand the intuition that leads to the construction of the feedback law (7.3) and the ideas behind the proof of Theorem 7.1. To explain the procedure, we follow the following arguments:

1. For the stabilization of the equilibrium point $0 \in \Re \times C^0([-r, 0]; \Re)$, we first start with the stabilization of the subsystem

$$\begin{aligned} \dot{x}_2(t) = {} & D^*\big(G\exp\big(-x_2(t)\big) + 1\big) \\ & \times \Big(\exp\big(u(t)\big) - \big(G + \exp\big(x_2(t)\big)\big)g\big(x_2(t)\big)\exp\big(x_1(t)\big)\Big) \end{aligned}$$

   with x_1 as the disturbance input and u as the control input. Any feedback law which satisfies $u(t) = \ln(g(x_2(t))\exp(x_1(t))(G + 1))$ for $x_2(t) \ge 0$ and $u(t) > \ln(g(x_2(t))\exp(x_1(t))(G + \exp(x_2(t))))$ for $x_2(t) < 0$ achieves ISS stabilization of the subsystem with x_1 as the input.
2. In order to prove URGAS for the composite system by means of small-gain arguments, one has to show the ISS property of the x_1-subsystem $\dot{x}_1(t) =$

$\min_{t-r\leq\tau\leq t}\tilde{\mu}(x_2(\tau)) + d(t)(\max_{t-r\leq\tau\leq t}\tilde{\mu}(x_2(\tau)) - \min_{t-r\leq\tau\leq t}\tilde{\mu}(x_2(\tau))) - D^*\exp(u(t)) - b$ with x_2 as the input. Notice that the feedback selection from the previous step gives $\dot{x}_1(t) = \min_{t-r\leq\tau\leq t}\tilde{\mu}(x_2(\tau)) + d(t)(\max_{t-r\leq\tau\leq t}\tilde{\mu}(x_2(\tau)) - \min_{t-r\leq\tau\leq t}\tilde{\mu}(x_2(\tau))) - D^*g(x_2(t))\exp(x_1(t))(G+1) - b$ for $x_2(t)\geq 0$ and $\dot{x}_1(t) < \min_{t-r\leq\tau\leq t}\tilde{\mu}(x_2(\tau)) - D^*g(x_2(t))\exp(x_1(t))(G+\exp(x_2(t))) - b$ for $x_2(t) < 0$. The estimation of the derivative $\dot{x}_1(t)$ shows that the ISS inequality for the x_1-subsystem does not hold unless we have $\min_{t-r\leq\tau\leq t}\tilde{\mu}(x_2(\tau)) > b$ for all t sufficiently large. By virtue of hypothesis (H), there exists $y < 0$ such that the ISS inequality for the the x_1-subsystem holds if $\min_{t-r\leq\tau\leq t} x_2(\tau) \geq y$ for all t sufficiently large.

3. The feedback law $u(t) > \ln(g(x_2(t))\exp(x_1(t))(G+\exp(x_2(t))))$ for $x_2(t) < 0$ is selected such that the inequality $\min_{t-r\leq\tau\leq t} x_2(\tau) \geq y$ holds for all initial conditions after a transient period. Since the ISS inequalities will hold only after this transient period, the trajectory-based small-gain result Theorem 5.2 must be used for the proof of URGAS of the closed-loop system.

Proof of Theorem 7.1 Consider the solution $(x_1(t), x_2(t)) \in \Re^2$ of (1.16) with (7.3) with arbitrary initial condition $x_1(0)\in\Re$, $T_r(0)x_2 \in C^0([-r,0];\Re)$ and corresponding to arbitrary input $d \in M_D$. The following equations hold for system (1.16) with (7.3):

$$\begin{aligned} \dot{x}_2(t) &= aD^*\frac{G\exp(-x_2(t))+1}{G+1}\big(1-\exp\big(x_2(t)\big)\big) \quad \text{if } x_2(t)\leq 0 \\ \dot{x}_2(t) &= D^*g\big(x_2(t)\big)\big(G\exp\big(-x_2(t)\big)+1\big)\exp\big(x_1(t)\big) \\ &\quad\times\big(1-\exp\big(x_2(t)\big)\big) \quad \text{if } x_2(t)>0 \end{aligned} \tag{7.4}$$

Equations (7.4) imply that the function $V(t) = x_2^2(t)$ is nonincreasing, and consequently, we obtain

$$\big|x_2(t)\big| \leq \big\|T_r(0)x_2\big\|_r \quad \text{for all } t\in[0,t_{\max}) \tag{7.5}$$

Using the fact that $\mu : (0, s_{\text{in}}) \to (0, \mu_{\max}]$ and definition (1.17) of $\tilde{\mu}$, we get that $\tilde{\mu}(x_2) \leq \mu_{\max}$ for all $x_2\in\Re$. This implies the following differential inequality:

$$\dot{x}_1(t) \leq 2\mu_{\max} - b$$

which by direct integration yields the estimate

$$x_1(t) \leq x_1(0) + (2\mu_{\max} - b)t \quad \text{for all } t\in[0,t_{\max}) \tag{7.6}$$

Define $\kappa(s) := (G+1)D^*\max_{|y|\leq s} g(y)$. Inequalities (7.5) and (7.6) imply that the following differential inequality holds:

$$\dot{x}_1(t) \geq -b - \frac{aD^*}{G+1} - \kappa\big(\big\|T_r(0)x_2\big\|_r\big)\exp\big(x_1(0) + (2\mu_{\max} - b)t\big)$$

which by direct integration yields the following estimate for all $t\in[0,t_{\max})$:

$$x_1(t) \geq x_1(0) - \left(b + \frac{aD^*}{G+1}\right)t - \kappa\left(\left\| T_r(0)x_2 \right\|_r\right) \exp\left(x_1(0)\right) \frac{\exp((2\mu_{\max} - b)t) - 1}{2\mu_{\max} - b} \tag{7.7}$$

Inequalities (7.5), (7.6), and (7.7) and a standard contradiction argument show that system (1.16) with (7.3) is forward complete, i.e., $t_{\max} = +\infty$. Therefore, inequalities (7.5), (7.6), and (7.7) hold for all $t \geq 0$, and since system (1.16) with (7.3) is autonomous, it follows that system (1.16) with (7.3) is Robustly Forward Complete.

Define $W(t) = x_2^2(t)$ if $x_2(t) \leq 0$ and $W(t) = 0$ if otherwise. Using (7.4), we obtain the existence of a positive definite function $\rho \in C^0(\Re^+; \Re^+)$ such that

$$\dot{W}(t) \leq -\rho\left(W(t)\right) \quad \text{for all } t \geq 0 \tag{7.8}$$

Lemma 2.13 in Chap. 2 implies the existence of $\sigma \in KL$ such that, for all $x_1(0) \in \Re$, $T_r(0)x_2 \in C^0([-r, 0]; \Re)$, and $d \in M_D$, it holds that

$$W(t) \leq \sigma\left(W(0), t\right) \quad \text{for all } t \geq 0 \tag{7.9}$$

Inequality (7.9), in conjunction with Theorem 3.1 in Chap. 3, shows the existence of $a \in K_\infty$ such that for all $x_1(0) \in \Re$, $T_r(0)x_2 \in C^0([-r, 0]; \Re)$, and $d \in M_D$, there exists $\xi \geq r$ with $\xi \leq r + a(|x_2(0)|)$ satisfying

$$x_2(t-r) \geq -\frac{c}{2} \quad \text{for all } t \geq \xi \tag{7.10}$$

where $c := \ln(\frac{s_{\text{in}} - s_p}{s_p G}) > 0$, and $s_p < s^*$ is the constant involved in hypothesis (H). Define $S := \Re \times C^0([-r, 0]; [-c/2, +\infty))$. Inequality (7.10) shows that $(x_1(t), T_r(t)x_2) \in S$ for all $t \geq \xi$ and that inequality (5.16) holds for appropriate $a \in K_\infty$ and $c(t) \equiv 1$.

Notice that for $(x_1(t), T_r(t)x_2) \in S$, the functionals

$$V_1(t) = \max_{\theta \in [-r,0]} \exp(2\sigma\theta)\left|z(t+\theta)\right|^2 \qquad V_2 = \left|x_1(t)\right|^2 \tag{7.11}$$

where $\sigma > 0$ and

$$x_2(t) = c\left(\exp\left(z(t)\right) - 1\right) \tag{7.12}$$

are well defined. Moreover, by considering the differential equations

$$\dot{z}(t) = aD^* \frac{G\exp(c(1 - \exp(z(t)))) + 1}{c(G+1)} \exp\left(-z(t)\right)\left(1 - \exp\left(c\left(\exp\left(z(t)\right) - 1\right)\right)\right)$$
$$\text{if } z(t) \leq 0$$
$$\dot{z}(t) = c^{-1}D^* g\left(c\left(\exp\left(z(t)\right) - 1\right)\right)\left(G\exp\left(c\left(1 - \exp\left(z(t)\right)\right)\right) + 1\right)\exp\left(x_1(t) - z(t)\right)$$
$$\times \left(1 - \exp\left(c\left(\exp\left(z(t)\right) - 1\right)\right)\right) \quad \text{if } z(t) > 0,$$

we conclude from Lemma 2.14 in Chap. 2 and Lemma 6.7 in Chap. 6 that, for every $\gamma_{1,2} \in K_\infty$, there exists $\sigma_1 \in KL$ such that

$$V_1(t) \leq \max\left\{\sigma_1\left(V_1(t_0), t - t_0\right), \sup_{t_0 \leq \tau \leq t} \gamma_{1,2}\left(V_2(\tau)\right)\right\} \quad \text{for all } t \geq t_0 \geq 0 \tag{7.13}$$

Finally, using hypothesis (H) and definitions (7.11), we guarantee that there exists a positive definite function $\rho \in C^0(\Re^+;\Re^+)$ such that the following holds for every $\varepsilon > 0$:

"Whenever

$$(1+\varepsilon)\ln\left(\frac{(G+\exp(c(\exp(\sqrt{V_1(t)})-1)))D^* \max_{|z|\le\sqrt{V_1(t)}} g(c(\exp(z)-1))}{\min_{|z|\le\exp(\sigma r)\sqrt{V_1(t)}} \tilde{\mu}(c(\exp(z)-1)) - b - \frac{a}{G+1}D^*(1-\exp(c(\exp(-\sqrt{V_1(t)})-1)))}\right) \le \left|x_1(t)\right|,$$

$$(1+\varepsilon)\ln\left(\frac{\max_{|z|\le\exp(\sigma r)\sqrt{V_1(t)}} \tilde{\mu}(c(\exp(z)-1)) - b}{(G+\exp(c(\exp(-\sqrt{V_1(t)})-1)))D^* \min_{|z|\le\sqrt{V_1(t)}} g(c(\exp(z)-1))}\right) \le \left|x_1(t)\right|$$

then $2x_1(t)\dot{x}_1(t) \le -\rho(x_1^2(t))$."

Therefore, Lemma 2.14 in Chap. 2 implies that there exists $\sigma_2 \in KL$ such that

$$V_2(t) \le \max\left\{\sigma_2\big(V_2(t_0), t-t_0\big), \sup_{t_0\le\tau\le t} \gamma_{2,1}\big(V_1(\tau)\big)\right\} \quad \text{for all } t \ge t_0 \ge 0 \tag{7.14}$$

where

$$\begin{aligned} &\gamma_{2,1}(s) := (1+\varepsilon)^2(\ln(\max\{g_1(s), g_2(s)\}))^2 \\ &g_1(s) \\ &\quad := \frac{(G+\exp(c(\exp(\sqrt{s})-1)))D^* \max_{|z|\le\sqrt{s}} g(c(\exp(z)-1))}{\min_{|z|\le\exp(\sigma r)\sqrt{s}} \tilde{\mu}(c(\exp(z)-1)) - b - \frac{aD^*}{G+1}(1-\exp(c(\exp(-\sqrt{s})-1)))} \\ &g_2(s) := \frac{\max_{|z|\le\exp(\sigma r)\sqrt{s}} \tilde{\mu}(c(\exp(z)-1)) - b}{(G+\exp(c(\exp(-\sqrt{s})-1)))D^* \min_{|z|\le\sqrt{s}} g(c(\exp(z)-1))} \end{aligned} \tag{7.15}$$

Inequalities (7.5), (7.6), (7.7), (7.13), and (7.14) guarantee that inequalities (5.10), (5.11), (5.13), (5.14), (5.15), and (5.17) hold for appropriate $\sigma \in KL$, $\nu \in K^+$, and $a \in K_\infty$ with $c(t) \equiv 1$, $p \equiv 0$, $\gamma_{1,2}(s) := \gamma_{2,1}(\frac{s}{2})$, $\gamma_{1,1}(s) = \gamma_{2,2}(s) \equiv 0$, $L := V_1 + V_2$, and $H(t, x_1, x_2) := \sqrt{x_1^2 + \|x_2\|_r^2}$. Finally, notice that the MAX-preserving mapping $\Gamma : \Re_+^2 \to \Re_+^2$ with $\Gamma_i(x) = \max_{j=1,2} \gamma_{i,j}(x_j)$ $(i = 1, 2)$ satisfies the cyclic small-gain conditions.

By virtue of Theorem 5.2, we conclude that the autonomous system (1.16) with (7.3) is URGAS. □

7.3 Applications to Numerical Analysis

It is well known that step size control can enhance the performance of a numerical scheme when applied to a system of Ordinary Differential Equations (ODEs). In fact the use of the word "control" suggests that methods and techniques used in Mathematical Control Theory can be (in principle) used in order to achieve certain objectives for the numerical solution of systems of ODEs. For example, in

[10] the authors use a "Proportional-Integral" technique which is similar to the "Proportional-Integral" controller used in Linear Control Theory in order to keep the local discretization error within certain bounds (see also [8, 9, 12]). Theoretical results on the behavior of adaptive time-stepping methods have been presented in [22, 24], and the control theoretic notion of input-to-state stability (ISS) has been successfully used in [6, 7] in order to explain the behavior of attractors under discretization.

In this section, we consider the problem of selecting the step size for numerical schemes so that the numerical solution presents the same qualitative behavior as the original system of ODEs. It is well known that any consistent and stable numerical scheme for ODEs inherits the asymptotic stability of the original equation in a practical sense, even for more general attractors than equilibria; see, for instance, [6, 7] and [31] (Chap. 7). Practical asymptotic stability means that the system exhibits an asymptotically stable set close to the original attractor, i.e., in our case a small neighborhood around the equilibrium point, which shrinks down to the attractor as the time step h tends to 0. In contrast to these results, here we investigate the case in which the numerical approximation is asymptotically stable in the usual sense, i.e., not only practically.

We focus on nonlinear systems for which an equilibrium point is the global attractor. First, we show how the problem of appropriate step size selection can be converted to a rigorous abstract feedback stabilization problem for a particular hybrid system; see also [16]—the reader should notice that the standard stability analysis of numerical schemes uses the study of a discrete-time system, e.g., [11, 13, 15, 23, 31], not a hybrid system. Therefore, we are in a position to use all methods of feedback design for nonlinear systems. Secondly, we apply small-gain methods for the solution of the problem for a class of nonlinear systems.

7.3.1 Conversion to a Feedback Stabilization Problem

Consider the autonomous dynamical system

$$\dot{z}(t) = f\big(z(t)\big) \quad z(t) \in \Re^n \tag{7.16}$$

where $f : \Re^n \to \Re^n$ is a locally Lipschitz vector field with $f(0) = 0$. For all $z_0 \in \Re^n$ and $t \geq 0$, the solution of (7.16) with initial condition $z(0) = z_0$ will be denoted by $z(t, 0, z_0)$, or $z(t, z_0)$ or simply $z(t)$ if there is no confusion from the context. The numerical approximation of system (7.16) will be the following system with variable sampling partition:

$$\begin{aligned} \dot{x}(t) &= F\big(h_i, x(\tau_i)\big) \quad t \in [\tau_i, \tau_{i+1}) \\ \tau_0 &= 0,\ \tau_{i+1} = \tau_i + h_i \\ h_i &= \varphi\big(x(\tau_i)\big)\exp\big(-u(\tau_i)\big) \\ x(t) &\in \Re^n,\ u(t) \in [0, +\infty) \end{aligned} \tag{7.17}$$

where $\varphi \in C^0(\Re^n;(0,r])$, $r>0$ is a constant, and $F:\bigcup_{x\in\Re^n}([0,\varphi(x)]\times\{x\})\to\Re^n$ is a (not necessarily continuous) vector field with $F(h,0)=0$ for all $h\in[0,\varphi(0)]$ and $\lim_{h\to 0^+}F(h,z)=f(z)$ for all $z\in\Re^n$. Modelling numerical approximations by hybrid systems instead of the usual discrete-time system enables us to simultaneously investigate the qualitative behavior for all time steps $h\in(0,\varphi(x)]$ simultaneously.

We assume that there exists a continuous nondecreasing function $M:\Re^+\to\Re^+$ such that

$$\left|F(h,x)\right|\le|x|M\big(|x|\big)\quad \text{for all } x\in\Re^n \text{and } h\in\left[0,\varphi(x)\right] \tag{7.18}$$

It should be noticed that the hybrid system (7.17) under hypothesis (7.18) is an autonomous system with variable sampling partition. Some remarks are needed in order to justify the name "numerical approximation of system (7.16)" for the hybrid system (7.17):

1. Notice that the condition $\lim_{h\to 0^+}F(h,z)=f(z)$ is a consistency condition for the numerical scheme applied to (7.16).
2. The sequence $\{h_i\}_0^\infty$ is the sequence of step sizes used in order to obtain the numerical solution. Notice that for the case $\varphi(x)\equiv r$, constant inputs $u(t)\equiv u\ge 0$ will produce constant step sizes with $h_i\equiv r\exp(-u)$. Moreover, notice that variable step sizes can be represented easily by selecting in an appropriate way the input $u:\Re^+\to\Re^+$.
3. The constant $r>0$ is the maximum allowable step size.
4. The function $\varphi\in C^0(\Re^n;(0,r])$ determines the maximum allowable step size $\varphi(x(\tau_i))$ for each $x(\tau_i)\in\Re^n$. This is important for implicit numerical schemes as shown below.

All consistent s-stage Runge–Kutta methods can be represented by the hybrid system (7.17). More specifically, let $x_0\in\Re^n$ and consider a consistent s-stage Runge–Kutta method for (7.16),

$$Y_i=x_0+h\sum_{j=1}^{s}a_{ij}f(Y_j)\quad i=1,\dots,s \tag{7.19}$$

$$x=x_0+h\sum_{i=1}^{s}b_if(Y_i) \tag{7.20}$$

with $\sum_{i=1}^{s}b_i=1$. If the scheme is explicit, i.e., if $a_{ij}=0$ for $j\ge i$, then there always exists a unique solution to (7.19). If the scheme is implicit, then in order to be able to guarantee that (7.19) admit a unique solution, it may be necessary to restrict the step size to $h\in[0,\varphi(x_0)]$ for some maximal step size $\varphi(x_0)$ depending on the state $x_0\in\Re^n$.

A suitable choice for $\varphi(x)$ may be obtained in the following way. Let $\gamma:\Re^+\to\Re^+$ be a continuous nondecreasing function with $|f(x)|\le|x|\gamma(|x|)$ for all $x\in\Re^n$

(such a function always exists since $f : \Re^n \to \Re^n$ is a locally Lipschitz vector field with $f(0) = 0$). Let $L_\lambda : \Re^n \to (0, +\infty)$ be a continuous function satisfying

$$L_\lambda(x_0) \geq \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in B_\lambda(x_0), x \neq y\right\}$$

for all $x_0 \in (\Re^n \backslash \{0\})$, with $B_\lambda(x_0) := \{x \in \Re^n : |x - x_0| \leq \lambda|x_0|\}$, $\lambda \in (0, 1)$. The continuous function $\varphi(x) := \frac{\lambda}{|A|(L_\lambda(x)+\gamma(|x|))}$, where $|A| := \max_{i=1,\dots,s} \sum_j^s |a_{ij}|$, guarantees that for all $x_0 \in \Re^n$ and $h \in [0, \varphi(x_0)]$ equations (7.19) have a unique solution satisfying $Y_i \in B_\lambda(x_0)$, $i = 1, \dots, s$. Note however that this bound may be conservative. For instance, if we apply the implicit Euler scheme ($s = 1$, $a_{11} = b_1 = 1$) to an asymptotically stable linear ODE of the form $\dot{x} = Jx$ with a Hurwitz matrix $J \in \Re^{n\times n}$, then (7.19) becomes

$$Y_1 = x_0 + hJY_1 \quad \Leftrightarrow \quad (I - hJ)Y_1 = x_0$$

which always has a unique solution because all eigenvalues of $-J$ and thus of $I - hJ$ have positive real parts for all $h \geq 0$; hence, $I - hJ$ is invertible for all $h \geq 0$.

We define

$$F(h, x_0) := h^{-1}(x - x_0) = \sum_{i=1}^{s} b_i f(Y_i) \tag{7.21}$$

A moment's thought reveals that for every locally bounded $u : \Re^+ \to \Re^+$ and for every $x_0 \in \Re^n$, the solution of (7.17) with (7.21) coincides at each τ_i, $i \geq 0$, with the numerical solution of (7.16) with $x(0) = x_0$ obtained by using the Runge–Kutta numerical scheme (7.19), (7.20) and using the discretization step sizes $h_i = \varphi(x(\tau_i))\exp(-u(\tau_i))$, $i \geq 0$. The reader should notice that other ways (besides (7.21)) of defining the vector field $F : \bigcup_{x\in\Re^n}([0, \varphi(x)] \times \{x\}) \to \Re^n$ may be possible; here we have selected the simplest way of obtaining a piecewise linear numerical solution.

Moreover, the reader should notice that appropriate step size restriction can always guarantee that (7.18) holds for $F : \bigcup_{x\in\Re^n}([0, \varphi(x)] \times \{x\}) \to \Re^n$ as defined by (7.21). For example, if $\varphi(x) := \frac{\lambda}{|A|(L_\lambda(x)+\gamma(|x|))}$ is the step size restriction described above, then $F : \bigcup_{x\in\Re^n}([0, \varphi(x)] \times \{x\}) \to \Re^n$ as defined by (7.21) satisfies $|F(h, x)| \leq |x|[1 + r(1 + \lambda)(\sum_{i=1}^s |b_i|)\gamma((1 + \lambda)|x|)]$ for all $x \in \Re^n$ and $h \in [0, \varphi(x)]$. Thus, (7.18) holds with $M(y) := 1 + r(1 + \lambda)(\sum_{i=1}^s |b_i|)\gamma((1 + \lambda)y)$.

Assume next that $0 \in \Re^n$ is Uniformly Globally Asymptotically Stable (UGAS) for (7.16). Our goal is to be able to produce numerical solutions using the numerical approximation (7.17) which have the correct qualitative behavior. More specifically, we would like to be in a position to know a continuous function $\varphi : \Re^n \to (0, r]$ such that the numerical solution produced by (7.17) has a correct qualitative behavior (e.g., $\lim_{t\to+\infty} x(t) = 0$). However, we would like to be able to guarantee that a correct behavior for the numerical solution can be obtained by using arbitrary discretization step sizes smaller than $\varphi(x(\tau_i))$ (i.e., if we obtain the correct qualitative behavior using the discretization step sizes $h_i = \varphi(x(\tau_i))$, $i \geq 0$, we would like to obtain a correct qualitative behavior using the smaller discretization step sizes $h_i = \varphi(x(\tau_i))\exp(-u(\tau_i))$, $i \geq 0$, where $u : \Re^+ \to \Re^+$ is an arbitrary locally bounded function). This is equivalent to requiring that $0 \in \Re^n$ is Uniformly Robustly Globally Asymptotically Stable (URGAS) for (7.17).

The reader should notice that continuity for the function $\varphi : \Re^n \to (0, r]$ is essential: without assuming continuity, it may happen that $\liminf_{x \to 0} \varphi(x) = 0$, and this would require discretization step sizes of vanishing magnitude as $t \to +\infty$. Moreover, since we want to be able to determine a continuous function $\varphi : \Re^n \to (0, r]$ which "stabilizes" the hybrid system (7.17), we are essentially studying a feedback stabilization problem for the hybrid system (7.17). Hence, we are in a position to pose the problem rigorously. We consider the following feedback stabilization problems:

(P1) *Existence Problem* Is there a continuous function $\varphi : \Re^n \to (0, r]$ such that $0 \in \Re^n$ is URGAS for system (7.17)?
(P2) *Design Problem* Construct, if possible, a continuous function $\varphi : \Re^n \to (0, r]$ such that $0 \in \Re^n$ is URGAS for system (7.17).

Here, we are looking for asymptotic stability with respect to the equilibrium in question. This is a strong property which cannot in general be deduced from practical stability (guaranteed by general results on attractors in [6, 7] and [31]). In [31] (Chap. 5), several results for our problem focusing on specific classes of ODEs are derived using classical numerical stability concepts like A-stability, B-stability, and the like.

It should be emphasized that standard step size control algorithms do not solve problem (P1). The following example illustrates this point.

Example 7.3.1 Consider the linear planar system

$$\dot{x}_1 = -0.005x_1 + x_2 \qquad \dot{x}_2 = -x_1 - 0.005x_2 \tag{7.22}$$

Following the suggestions in [12] (pp. 167–169) based on the estimation of the local discretization error, we have used the formula

$$h_{\text{new}} = h \min\left\{P, 0.8\sqrt{\frac{1}{\text{err}}}\right\} \tag{7.23}$$

where

$$\text{err} = \sqrt{\frac{1}{2}\left(\frac{x_{1,\text{EULER}} - x_{1,\text{HEUN}}}{\text{sc}_1}\right)^2 + \frac{1}{2}\left(\frac{x_{2,\text{EULER}} - x_{2,\text{HEUN}}}{\text{sc}_2}\right)^2} \tag{7.24}$$

$$\text{sc}_i = \text{Atol} + \text{Rtol} \max\left\{|x_i|, |x_{i,\text{HEUN}}|\right\} \quad i = 1, 2 \tag{7.25}$$

Atol > 0 is the tolerance for absolute errors, Rtol > 0 is the tolerance for relative errors, $P \geq 1$ is a constant factor which determines the magnitude of a (possible) increase of the step size, $x_{i,\text{EULER}}$ $(i = 1, 2)$ are the approximations of the components of the solution by the explicit Euler scheme, and $x_{i,\text{HEUN}}$ $(i = 1, 2)$ are the approximations of the components of the solution by Heun's 2nd-order scheme.

Figure 7.1 shows the logarithm of the value of the squared Euclidean norm for the numerical solution obtained by Heun's 2nd-order scheme with Atol = Rtol = 10^{-2},

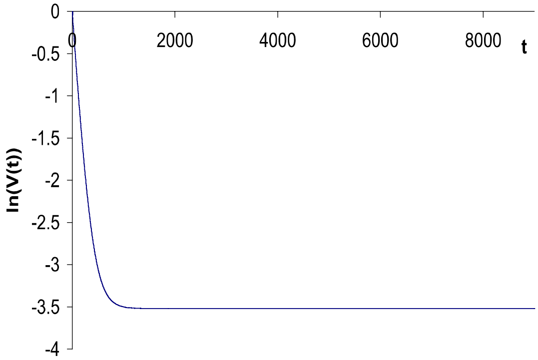

Fig. 7.1 The value of the logarithm of the squared Euclidean norm $V(t) = |x(t)|^2$ for the numerical solution of (7.22) with $\text{Atol} = \text{Rtol} = 10^{-2}$, $P = 2$

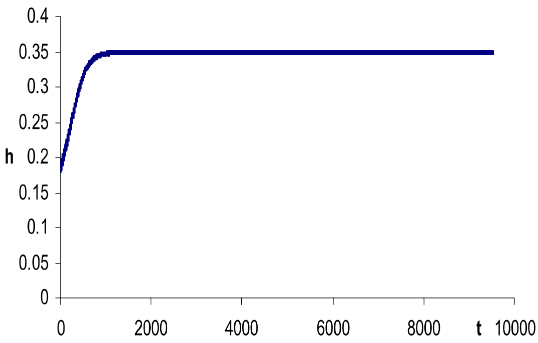

Fig. 7.2 The value of the applied step size for the numerical solution of (7.22), with $\text{Atol} = \text{Rtol} = 10^{-2}$, $P = 2$

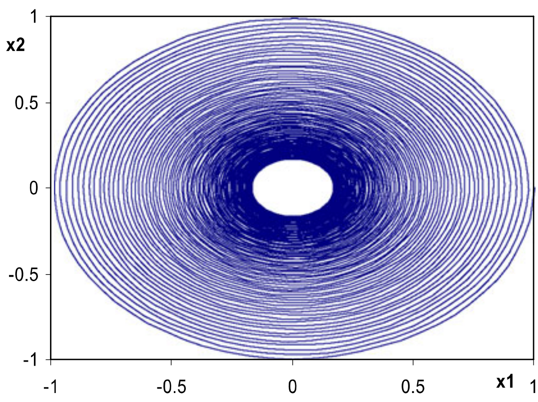

Fig. 7.3 The phase portrait for the numerical solution of (7.22) with $\text{Atol} = \text{Rtol} = 10^{-2}$, $P = 2$

$P = 2$, and initial condition $(x_1, x_2) = (1, 0)$: the numerical solution exhibits an asymptotically stable limit cycle of radius $r = 0.17195$, which is clearly shown in the phase portrait in Fig. 7.3. Indeed, Fig.7.2 shows that the step size tends to take values inside the interval $[0.347, 0.351]$ for large times. The limit cycle shrinks to the origin as $\text{Atol}, \text{Rtol} \to 0$, but exists for all $\text{Atol}, \text{Rtol} > 0$. Figure 7.4 shows the logarithm of the value of the squared Euclidean norm for the numerical solution obtained by Heun's 2nd-order scheme with $\text{Atol} = \text{Rtol} = 10^{-3}$, $P = 2$, and initial

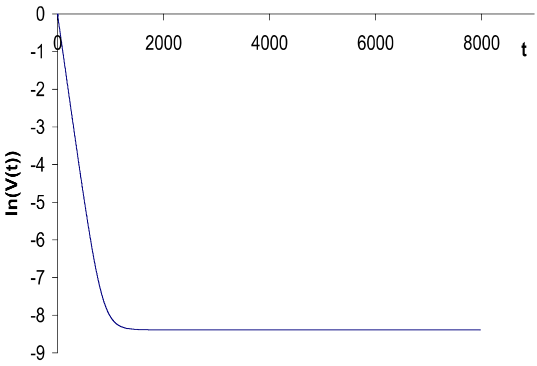

Fig. 7.4 The value of the logarithm of the squared Euclidean norm $V(t) = |x(t)|^2$ for the numerical solution of (7.22) with Atol = Rtol = 10^{-3}, $P = 2$

condition $(x_1, x_2) = (1, 0)$: the numerical solution exhibits an asymptotically stable limit cycle of radius $r = 0.015$. It is clear that (standard) step size control based on the local discretization error does not give the desired qualitative behavior.

7.3.2 Small-Gain Methods

Consider the following system described by ODEs:

$$\dot{z} = f_0(z) \tag{7.26}$$

$$\begin{aligned} \dot{x}_1 &= -a_1(x_1)x_1 + f_1(z) \\ \dot{x}_i &= -a_i(x_i)x_i + f_i(z, x_1, \ldots, x_{i-1}) \quad i = 2, \ldots, n \end{aligned} \tag{7.27}$$

where $z \in \Re^m$, $x = (x_1, \ldots, x_n)' \in \Re^n$, $f_0 : \Re^m \to \Re^m$, $f_1 : \Re^m \to \Re$, $f_i : \Re^m \times \Re^{i-1} \to \Re$ $(i = 2, \ldots, n)$, and $a_i : \Re \to \Re$ $(i = 1, \ldots, n)$ are locally Lipschitz mappings with $f_0(0) = 0$, $f_1(0) = \cdots = f_n(0, 0, \ldots, 0) = 0$. We assume that there exist constants $L_i > 0$ $(i = 1, \ldots, n)$ such that

$$a_i(y) \geq L_i \quad \forall y \in \Re \tag{7.28}$$

We also assume that $0 \in \Re^m$ is UGAS for (7.26). Under the previous assumptions, using the fact that system (7.26), (7.27) has a structure of systems in cascade, we can prove by induction that for every $j = 1, \ldots, n$, $0 \in \Re^m \times \Re^j$ is UGAS for system (7.26) with

$$\begin{aligned} \dot{x}_1 &= -a_1(x_1)x_1 + f_1(z) \\ \dot{x}_i &= -a_i(x_i)x_i + f_i(z, x_1, \ldots, x_{i-1}) \quad i = 2, \ldots, j \end{aligned} \tag{7.29}$$

The proof is based on the fact that for every $x_{i,0} \in \Re$ and for every measurable $u : \Re^+ \to \Re$, the solution of $\dot{x}_i = -a_i(x_i)x_i + u$ with initial condition $x_i(0) = x_{i,0}$ satisfies the following estimate, which is obtained from the variations of constants formula and (7.28):

$$\left|x_i(t)\right| \leq \exp\left(-\frac{L_i}{2}t\right)|x_{i,0}| + \frac{1}{L_i} \sup_{0 \leq s \leq t} \left|u(s)\right| \quad \forall t \geq 0 \tag{7.30}$$

Consequently, the solution of $\dot{x}_i = -a_i(x_i)x_i + f_i(z, x_1, \dots, x_{i-1})$ satisfies $|x_i(t)| \le \exp(-\frac{L_i}{2}t)|x_{i,0}| + \frac{1}{L_i}\sup_{0\le s\le t}|f_i(z(s), x_1(s), \dots, x_{i-1}(s))|$. Using the functions $V_1(z,x) = |z|$, $V_{i+1}(z,x) = |x_i|$ $(i = 1, \dots, n)$ and the set $S(t) := \Re^m \times \Re^n$, one can show that hypotheses (SG3), (SG4), and (SG5) hold for system (7.26), (7.27) with gains $\gamma_{i,j}(s)$ $(i, j = 1, \dots, n+1)$ satisfying $\gamma_{i,j}(s) \equiv 0$ for $j \ge i$. Therefore, the cyclic small-gain conditions hold, and Theorem 5.2 guarantees UGAS for system (7.26), (7.27).

Suppose that a stable numerical scheme is available for (7.26), i.e., there exist $\varphi \in C^0(\Re^m; (0, r])$, $r > 0$, and $F_0 : \bigcup_{z\in\Re^m}([0, \varphi(z)] \times \{z\}) \to \Re^m$ with $F_0(h, 0) = 0$ for all $h \in [0, \varphi(0)]$ and $\lim_{h\to 0^+} F_0(h, z) = f_0(z)$ for all $z \in \Re^m$ such that $0 \in \Re^m$ is URGAS for the hybrid system

$$\begin{aligned} &\dot{z}(t) = F_0(h_i, z(\tau_i)) \quad t \in [\tau_i, \tau_{i+1}) \\ &\tau_0 = 0,\ \tau_{i+1} = \tau_i + h_i \\ &h_i = \varphi\big(z(\tau_i)\big)\exp\big(-u(\tau_i)\big) \\ &x(t) \in \Re^n,\ u(t) \in [0, +\infty) \end{aligned} \tag{7.31}$$

We propose the following first-order numerical scheme for (7.27):

$$\begin{aligned} &x_1(t+h) = x_1(t) - ha_1\big(x_1(t)\big)x_1(t+h) + hf_1\big(z(t)\big) \\ &x_i(t+h) = x_i(t) - ha_i\big(x_i(t)\big)x_i(t+h) + hf_i\big(z(t), x_1(t), \dots, x_{i-1}(t)\big) \\ &\quad i = 2, \dots, n \end{aligned} \tag{7.32}$$

The above scheme is a partitioned scheme which treats differently the state x_i and the states $z, x_1, \dots, x_{i-1}$ for each differential equation. The resulting hybrid system is system (7.31) with

$$\begin{aligned} \dot{x}_1(t) &= \frac{-a_1(x_1(\tau_i))}{1 + h_i a_1(x_1(\tau_i))}x_1(\tau_i) + \frac{1}{1 + h_i a_1(x_1(\tau_i))}f_1\big(z(\tau_i)\big) \\ \dot{x}_j(t) &= \frac{-a_j(x_j(\tau_i))}{1 + h_i a_j(x_j(\tau_i))}x_j(\tau_i) \\ &\quad + \frac{1}{1 + h_i a_j(x_j(\tau_i))}f_j\big(z(\tau_i), x_1(\tau_i), \dots, x_{j-1}(\tau_i)\big) \\ j &= 2, \dots, n \end{aligned} \tag{7.33}$$

We have

Theorem 7.2 $0 \in \Re^m \times \Re^n$ *is URGAS for system* (7.31), (7.33).

The proof of the above theorem is based on the following technical lemma.

Lemma 7.1 *Let* $a : \Re \to \Re$ *be a continuous function with* $L = \inf_{y\in\Re} a(y) > 0$, *and let a constant* $r > 0$. *Then for every sequence* $\{h_i\}_0^\infty$ *with* $h_i \in (0, r]$ *for all* $i \ge 0$, *for every locally bounded function* $v : \Re^+ \to \Re$, *and for every* $x_0 \in \Re$, *the solution of*

$$\dot{x}(t) = \frac{-a(x(\tau_i))}{1 + h_i a(x(\tau_i))} x(\tau_i) + \frac{1}{1 + h_i a(x(\tau_i))} v(\tau_i) \quad t \in [\tau_i, \tau_{i+1})$$
$$\tau_{i+1} = \tau_i + h_i, h_i \in (0, r], x(t) \in \Re \tag{7.34}$$

with initial condition $x(0) = x_0 \in \Re$, $\tau_0 = 0$ *satisfies the following estimate*:

$$\left|x(t)\right| \le \exp(\sigma r)|x_0| \exp(-\sigma t) + \frac{1}{\sigma L} \sup_{0 \le s \le t} \left|v(s)\right| \quad \forall t \in \left[0, \sup_{i \ge 0} \tau_i\right) \tag{7.35}$$

where $\sigma > 0$ *is any constant such that* $\frac{1}{1+s} \le \exp(-\sigma s)$ *for all* $s \in [0, rL]$, *i.e.*, $\sigma \le \frac{\ln(1+rL)}{rL}$.

Proof Notice that for every $i \ge 0$, it holds that

$$x(\tau_{i+1}) = x_0 \prod_{j=0}^{i} \big(1 + h_j a\big(x(\tau_j)\big)\big)^{-1} + \sum_{j=0}^{i} \left[h_j v(\tau_j) \left(\prod_{k=j}^{i} \big(1 + h_k a\big(x(\tau_k)\big)\big)^{-1} \right) \right] \tag{7.36}$$

and using the definition $L = \inf_{y \in \Re} a(y) > 0$, we obtain the following bound from (7.36):

$$\left|x(\tau_{i+1})\right| \le |x_0| \prod_{j=0}^{i} (1 + h_j L)^{-1} + \max_{j=0,\dots,i} \left|v(\tau_j)\right| \sum_{j=0}^{i} \left[h_j \left(\prod_{k=j}^{i} (1 + h_k L)^{-1} \right) \right] \tag{7.37}$$

Let $\sigma > 0$ such that $\frac{1}{1+s} \le \exp(-\sigma s)$ for all $s \in [0, rL]$. It holds

$$\prod_{j=0}^{i} (1 + h_j L)^{-1} \le \prod_{j=0}^{i} \exp(-\sigma L h_j) = \exp(-\sigma L \tau_{i+1})$$

and

$$\begin{aligned}
\sum_{j=0}^{i} \left[h_j \left(\prod_{k=j}^{i} (1 + h_k L)^{-1} \right) \right] &\le \sum_{j=0}^{i} \left[h_j \left(\prod_{k=j}^{i} \exp(-\sigma L h_k) \right) \right] \\
&= \sum_{j=0}^{i} \big[h_j \exp\big(-\sigma L(\tau_{i+1} - \tau_j)\big) \big] \\
&= \exp(-\sigma L \tau_{i+1}) \sum_{j=0}^{i} \left[\exp(\sigma L \tau_j) \int_{\tau_j}^{\tau_{j+1}} ds \right]
\end{aligned}$$

$$\leq \exp(-\sigma L \tau_{i+1}) \sum_{j=0}^{i} \left[\int_{\tau_j}^{\tau_{j+1}} \exp(\sigma L s)\, ds \right]$$
$$= \exp(-\sigma L \tau_{i+1}) \int_0^{\tau_{i+1}} \exp(\sigma L s)\, ds$$
$$\leq \frac{1}{\sigma L}$$

Using the above inequalities in conjunction with (7.37), we obtain, for all $i \geq 0$,

$$\left|x(\tau_{i+1})\right| \leq |x_0| \exp(-\sigma \tau_{i+1}) + \frac{1}{\sigma L} \max_{0 \leq j \leq i} \left|v(\tau_j)\right| \tag{7.38}$$

Now, for all $i \geq 0$ and $t \in [\tau_i, \tau_{i+1})$, it holds that

$$\left|x(t)\right| \leq \max\left\{\left|x(\tau_i)\right|, \left|x(\tau_{i+1})\right|\right\} \tag{7.39}$$

Combining (7.38) and (7.39) finishes the proof. □

The proof of the Theorem 7.2 follows from Lemma 7.1, which guarantees that $|x_i(t)| \leq \exp(\sigma r)|x_i(0)| \exp(-\sigma t) + \frac{1}{\sigma L_i} \sup_{0 \leq s \leq t} |f_i(z(s), \dots, x_{i-1}(s))|$, where $\sigma \in (0, \frac{\ln(1+rL)}{rL})$. Using the functions $V_1(z, x) = |z|$, $V_{i+1}(z, x) = |x_i|$ ($i = 1, \dots, n$), and the set $S(t) := \Re^m \times \Re^n$, one can show that hypotheses (SG3), (SG4), and (SG5) hold for system (7.31), (7.33) with gains $\gamma_{i,j}(s)$ ($i, j = 1, \dots, n+1$) satisfying $\gamma_{i,j}(s) \equiv 0$ for $j \geq i$. Therefore, the cyclic small-gain conditions hold, and Theorem 5.2 guarantees URGAS for system (7.31), (7.33).

Example 7.3.2 Consider the infinite-dimensional dynamical system described by PDEs

$$\frac{\partial x}{\partial t}(t, z) + c \frac{\partial x}{\partial z}(t, z) = b\big(x(t, z)\big) x(t, z) \quad z \in (0, 1]$$
$$x(t, 0) = 0 \tag{7.40}$$

with $x(t, z) \in \Re$, $b : \Re \to \Re$ being locally Lipschitz, $c > 0$, and initial condition $x(0, z) = x_0(z)$, where $x_0 \in C^1([0, 1]; \Re)$ with $x_0(0) = \frac{dx_0}{dz}(0) = 0$, under the following hypothesis:

(H) There exists a constant $K \geq 0$ such that $b(x) \leq K$ for all $x \in \Re$.

Using the method of characteristics and hypothesis (H), it can be shown that the infinite-dimensional dynamical system (7.40) admits a unique classical solution $x(t, \cdot) \in C^1([0, 1]; \Re)$ with $x(t, 0) = \frac{\partial x}{\partial z}(t, 0) = 0$ for all $t \geq 0$. Moreover, the zero solution is globally asymptotically stable, since for every $x_0 \in C^1([0, 1]; \Re)$ with $x_0(0) = \frac{dx_0}{dz}(0) = 0$, the unique classical solution $x(t, \cdot) \in C^1([0, 1]; \Re)$ of (7.40) with initial condition $x(0, z) = x_0(z)$ satisfies $x(t, z) = 0$ for all $t \geq c^{-1} z$ (uniform global attractivity).

Using a uniform space grid of $n+1$ points with space discretization step $\Delta z = \frac{1}{n}$, setting $x_i(t) = x(t, i\Delta z)$, $i = 0, 1, \dots, n$, and approximating the spatial derivative by the backward difference scheme $\frac{\partial x}{\partial z}(t, i\Delta z) \approx \frac{x(t,i\Delta z)-x(t,(i-1)\Delta z)}{\Delta z} = \frac{x_i(t)-x_{i-1}(t)}{\Delta z}$ for $i = 1, \dots, n$, we obtain the following set of ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= -\left(\frac{c}{\Delta z} - b(x_1)\right)x_1 \\ \dot{x}_i &= -\left(\frac{c}{\Delta z} - b(x_i)\right)x_i + \frac{c}{\Delta z}x_{i-1} \quad i = 2, \dots, n \end{aligned} \tag{7.41}$$

Clearly system (7.41) has the structure of system (7.27) with $a_i(x_i) = \frac{c}{\Delta z} - b(x_i)$ for $i = 1, \dots, n$. Moreover, if the space discretization step is selected so that

$$K\Delta z < c \tag{7.42}$$

where $K \geq 0$ is the constant involved in Hypothesis (H), then inequalities (7.28) hold as well with $L_i = \frac{c}{\Delta z} - K$ for $i = 1, \dots, n$. Theorem 7.2 allows us to conclude that, for every $h > 0$, the numerical scheme

$$\begin{aligned} x_1(t+h) &= \frac{x_1(t)}{1 + h\left(\frac{c}{\Delta z} - b(x_1(t))\right)} \\ x_i(t+h) &= \frac{x_i(t) + \frac{ch}{\Delta z}x_{i-1}(t)}{1 + h\left(\frac{c}{\Delta z} - b(x_i(t))\right)} \quad i = 2, \dots, n \end{aligned} \tag{7.43}$$

will give the correct qualitative behavior. The reader should notice that for the case $b(x) \equiv 0$, inequality (7.42) is automatically satisfied (with $K = 0$), and the numerical scheme (7.43) is related to the so-called implicit upwind numerical scheme for the advection equation, which is unconditionally stable.

Small-gain arguments can also be used for the system

$$\dot{x}_i = -a_i(x_i)x_i + f_i(x_{-i}) \quad i = 1, \dots, n \tag{7.44}$$

where $x \in \Re^n$, $f_i : \Re^{n-1} \to \Re$, and $a_i : \Re \to \Re$ ($i = 1, \dots, n$) are locally Lipschitz mappings with $f_i(0) = 0$ ($i = 1, \dots, n$). Here $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ for $1 < i < n$ and $x_{-1} = (x_2, \dots, x_n)$, $x_{-n} = (x_1, \dots, x_{n-1})$.

Assume the existence of constants $L_i > 0$ and $G_{i,j} \geq 0$ ($i, j = 1, \dots, n$) such that

$$a_i(x_i) \geq L_i \quad \text{and} \quad \left|f_i(x_{-i})\right| \leq \max_{j \neq i} G_{i,j}|x_j| \quad \text{for all } x \in \Re^n \tag{7.45}$$

Consider the following first-order numerical scheme for (7.44):

$$x_i(t+h) = x_i(t) - ha_i\big(x_i(t)\big)x_i(t+h) + hf_i\big(x_{-i}(t)\big) \quad i = 1, \dots, n \tag{7.46}$$

The resulting hybrid system with constant step-size selection is

$$\begin{aligned} \dot{x}_j(t) &= \frac{-a_j(x_j(\tau_i))}{1 + h_i a_j(x_j(\tau_i))}x_j(\tau_i) + \frac{1}{1 + h_i a_j(x_j(\tau_i))}f_j(x_{-j}(\tau_i)) \quad j = 1, \dots, n \\ \tau_{i+1} &= \tau_i + h_i \\ h_i &= r\exp\big(-u(\tau_i)\big) \\ & x(t) \in \Re^n, u(t) \in [0, +\infty) \end{aligned} \tag{7.47}$$

where $r > 0$.

Using Lemma 7.1 and Theorem 5.2, we are in a position to show the following:

Theorem 7.3 $0 \in \Re^n$ *is URGAS for system* (7.47), *provided that for each* $p = 2, \dots, n$, *it holds that*

$$G_{i_1,i_2} G_{i_2,i_3} \cdots G_{i_p,i_1} < \left(\frac{\ln(1 + r \max(L_1, \dots, L_n))}{r \max(L_1, \dots, L_n)} \right)^p L_{i_1} L_{i_2} \cdots L_{i_p} \tag{7.48}$$

for all $i_j \in \{1, \dots, n\}$, $i_j \neq i_k$ *if* $j \neq k$.

Indeed, Lemma 7.1, in conjunction with (7.45), implies that, for all $i = 1, \dots, n$, the following estimates hold:

$$\left|x_i(t)\right| \leq \exp(\sigma r)\left|x_i(0)\right| \exp(-\sigma t) + \frac{1}{\sigma L_i} \max_{j \neq i} G_{i,j} \sup_{0 \leq s \leq t} \left|x_j(s)\right| \quad \forall t \geq 0$$

with $\sigma = \frac{\ln(1+r\max(L_1,\dots,L_n))}{r\max(L_1,\dots,L_n)}$. Using the functions $V_i(x) = |x_i|$ $(i = 1, \dots, n)$ and the set $S(t) := \Re^n$, one can show that hypotheses (SG3), (SG4), and (SG5) hold for system (7.47) with gains $\gamma_{i,j}(s) := (1+\varepsilon)\frac{G_{i,j} r \max(L_1,\dots,L_n)}{\ln(1+r\max(L_1,\dots,L_n))L_i} s$ for all $\varepsilon > 0$, $i, j = 1, \dots, n$, $i \neq j$, and $\gamma_{i,i}(s) \equiv 0$ for $i = 1, \dots, n$. Therefore, by (7.48) the cyclic small-gain conditions hold for sufficiently small $\varepsilon > 0$, and Theorem 5.2 guarantees UGAS for system (7.47).

7.4 Applications to Economics Problems

In this section, we study the discrete-time model

$$\begin{aligned} x(t+1) &= 1 - f_1\big(x(t)\big) - f_2\big(x(t), y(t), u(t)\big) \\ y(t+1) &= y(t) - f_2\big(x(t), y(t), u(t)\big) \\ &\big(x(t), y(t)\big) \in (0, +\infty) \times [0, c_3],\ u(t) \in \Re \end{aligned} \tag{7.49}$$

where

$$\begin{aligned} f_1(x) &:= r \max\big\{0; \min(c_2; x - c_1)\big\} \\ f_2(x, y, u) &:= \min\big\{y; \max\big\{u; y - c_3; -f_1(x)\big\}\big\} \end{aligned}$$

and $c_i > 0$ $(i = 1, 2, 3)$, $r \in (0, c_2^{-1})$, and $c_1 < 1$ are constants. The discrete-time model (7.49) is an equivalent form of model (1.123) studied in Example 1.7.1 in Chap. 1 and expresses the evolution of the price of a certain commodity in a completely competitive market under the intervention of the government (buffer stocks). Notice that $f_2(x, y, 0) = 0$ for all $(x, y) \in (0, +\infty) \times [0, c_3]$. There exists an equilibrium set for the case $u(t) \equiv 0$ given by

$$(x, y) = (x_{\text{eq}}, y)$$

where $y \in [0, c_3]$ is arbitrary, and

$$x_{\text{eq}} = \begin{cases} \frac{1+rc_1}{1+r} & \text{if } c_1 + c_2 > 1 - c_2 r \\ 1 - c_2 r & \text{if } c_1 + c_2 \leq 1 - c_2 r \end{cases} \tag{7.50}$$

The variable $u(t)$ is directly related, by means of (1.121), to the quantity of the commodity released to the market by the government at period $t+1$ and is the control input of the system (i.e., $u(t)$ can be manipulated in order to achieve a certain control objective).

We next state the tracking control problem for (7.49).

The tracking control problem for (7.49) Let $x^* > 0$ be the desired price. Is there a static feedback law (or price stabilization policy)

$$u(t) = \varphi\big(x(t), y(t)\big) \tag{7.51}$$

such that the solution of the closed-loop system (7.49) with (7.51) satisfies $\lim_{t\to+\infty} x(t) = x^*$ for all initial conditions $(x(0), y(0)) \in (0, +\infty) \times [0, c_3]$?

The following proposition provides an answer to the tracking control problem.

Proposition 7.1 (Necessary Condition for solvability of the tracking control problem for (7.49)) *If the tracking control problem for* (7.49) *is solvable, then the following condition must hold*:

$$x^* = x_{\mathrm{eq}}$$

where x_{eq} *is the equilibrium price defined by* (7.50).

Proof By virtue of (7.49) and the continuity of the function f_1, it follows that the limit $\lim_{t\to+\infty} f_2(x(t), y(t), \varphi(x(t), y(t)))$ exists and is given by

$$\lim_{t\to+\infty} f_2\big(x(t), y(t), \varphi\big(x(t), y(t)\big)\big) = 1 - x^* + f_1\big(x^*\big)$$

The above equality implies that $\lim_{t\to+\infty}(y(t+1) - y(t)) = x^* - 1 - f_1(x^*)$. If $x^* - 1 - f_1(x^*) > 0$, then we obtain $\lim_{t\to+\infty} y(t) = +\infty$, which contradicts the constraint $y(t) \in [0, c_3]$. On the other hand, if $x^* - 1 - f_1(x^*) < 0$, then we obtain $\lim_{t\to+\infty} y(t) = -\infty$, which again contradicts the constraint $y(t) \in [0, c_3]$. Thus, we must necessarily have $x^* - 1 - f_1(x^*) = 0$, and consequently the desired price $x^* > 0$ coincides with x_{eq}, the unique equilibrium price defined by (7.50). The proof is complete. □

Proposition 7.1 indicates a major limitation imposed by the use of buffer stocks: the price dynamics can only have a unique accumulation point, which is no other than the equilibrium price. This important limitation may be used to explain the failure of buffer stock stabilization policies: if the government tries to lead the commodity price to values different from the equilibrium values, the buffer stock stabilization policy (no matter how smart) will fail.

Since the desired price must coincide with the equilibrium price, we may state next the price stabilization problem for (7.49).

The price stabilization problem for (7.49) Is there a static feedback law (or price stabilization policy) given by (7.51) with $\varphi(x_{\mathrm{eq}}, y) = 0$ for all $y \in [0, c_3]$ such that the closed-loop system (7.49) with (7.51) and output $Y = x - x_{\mathrm{eq}}$ is URGAOS?

The price stabilization problem is particularly interesting when $c_1 + c_2 > 1 - c_2 r$ and $r \geq 1$: Example 2.8.2 in Chap. 2 showed us that in any other case system (7.49) with $u \equiv 0$ and output $Y = x - x_{eq}$ is URGAOS.

Solution of the Price Stabilization Problem for (7.49) Set

$$u(t) = 1 - x_{eq} - f_1\bigl(x(t)\bigr) \tag{7.52}$$

or equivalently, in the original coordinates (system (1.120)),

$$G(t) = \min\bigl\{Q(t); \max\bigl\{S_{eq} - g\bigl(P(t)\bigr); Q(t) - Q_{max}\bigr\}\bigr\} \tag{7.53}$$

where S_{eq} is the equilibrium supply that corresponds to the equilibrium price $P_{eq} = \frac{a+c}{d+b}$, and $g(P(t))$ is the estimated commodity supply for the period $t+1$.

The feedback law described by (7.52), (7.53) is particularly simple and attempts to bring the total quantity of the product available in the market at period $t+1$ (given by $S(t+1) + G(t)$) as close as possible to the equilibrium supply S_{eq}. For this reason, we call this price stabilization policy as: "Keep Supply at Equilibrium" (KSE) price stabilization policy.

Proposition 7.2 (Conditions for successful KSE policy) *Consider the price stabilization problem for* (7.49) *under the hypotheses* $c_1 + c_2 > 1 - c_2 r$ *and* $r \geq 1$. *If* $c_3 > R$, *where*

$$R := \max\{R_1; R_2; R_3; R_4\} \tag{7.54}$$

$$\begin{aligned}
R_1 &:= \min\bigl\{\bigl(1 - r^{-1}\bigr)(1 - x_{eq}); c_1 + c_2 r - 1\bigr\} \\
R_2 &:= (r-1)\min\{x_{eq} - c_1; c_1 + c_2 - x_{eq}\} \geq 0 \\
R_3 &:= \min\{c_1 + c_2 r - 1; 1 - c_1 - c_2\} \\
R_4 &:= (r+1)^{-1}\min\bigl\{r(x_{eq} + c_2 r - 1); (r-1)(c_1 + c_2 + c_2 r - 1)\bigr\}
\end{aligned} \tag{7.55}$$

then the closed-loop system (7.49) *with* (7.52) *and output* $Y = x - x_{eq}$ *is URGAOS. Particularly, there exists* $T > 0$ *such that for every initial condition* $(x(0), y(0)) \in (0, +\infty) \times [0, c_3]$, *the solution of the closed-loop system* (7.49) *with* (7.52) *satisfies*

$$x(t) = x_{eq} \quad \forall t \geq T \tag{7.56}$$

We call the quantity aR, where $a > 0$ is the constant involved in (1.114), the "*critical storage capability*." Clearly, Proposition 7.2 guarantees that if the storage capability is larger than the critical storage capability, i.e., $Q_{max} > aR$, where Q_{max} is the maximum storage capability involved in (1.111), then the "Keep Supply at Equilibrium" (KSE) policy is successful. Moreover, the critical storage capability can be computed analytically using (7.54), (7.55). The conclusions of Proposition 7.2 may be used in order to calculate the storage cost needed for a successful buffer stock stabilization policy (since the storage cost is directly related to the storage capability Q_{max}) and determine whether the cost of the buffer stock stabilization policy is high or not. Notice that the result of Proposition 7.2 may be interpreted as a trade off between the efficiency of the price stabilization policy and the storage cost. It should be noted here that the total cost of the buffer stock stabilization policy includes also the purchase cost, i.e., the cost of purchasing quantities of the commodity.

Proposition 7.2 provides the sharpest characterization of the property of global attractivity of the equilibrium point under the feedback law (7.52). Indeed,

- if $0 \le c_3 \le R_3$, then a nontrivial period-2 solution exists:

$$(x_1, y_1) = (1 - c_2 r + c_3, c_3)$$
$$(x_2, y_2) = (1 - c_3, 0),$$

- if $0 < c_3 \le R_2$, then a nontrivial period-2 solution exists:

$$(x_1, y_1) = \big(x_{\mathrm{eq}} - (r-1)^{-1} c_3, c_3\big)$$
$$(x_2, y_2) = \big(x_{\mathrm{eq}} + (r-1)^{-1} c_3, 0\big),$$

- if $\max\{0; c_1 + r c_2 - 1\} \le c_3 \le R_4$, then a nontrivial period-2 solution exists:

$$(x_1, y_1) = (1 - c_2 r + c_3, c_3)$$
$$(x_2, y_2) = \big(1 + r c_1 - r + r^2 c_2 - c_3 - r c_3, 0\big),$$

- if $\max\{0; 1 - c_1 - c_2\} \le c_3 \le R_1$, then a nontrivial period-2 solution exists:

$$(x_1, y_1) = (1 - c_3, 0)$$
$$(x_2, y_2) = (1 + r c_1 - r + c_3 + r c_3, c_3).$$

A final remark for the tracking control problem for (7.49) must be given. Proposition 7.1 showed a major limitation of the buffer stock stabilization policy. However, if government is allowed to change the tax coefficient of the commodity, then the equilibrium price changes and can coincide with the desired price x^*. Then the KSE buffer stock policy may be used to stabilize the price dynamics. Thus, we conclude that the tracking control problem for (7.49) is indeed solvable by making simultaneous use of appropriate tax and buffer stock policies.

The proof of Proposition 7.2 is based on the spirit of Lemma 6.8 in Chap. 6, where the "dead-beat" property was achieved.

Proof of Proposition 7.2 Notice that

$$f_2\big(x, y, 1 - x_{\mathrm{eq}} - f_1(x)\big) = \min\big\{y; \max\big\{y - c_3; 1 - x_{\mathrm{eq}} - f_1(x)\big\}\big\}$$

$$1 - f_1(x) - f_2\big(x, y, 1 - x_{\mathrm{eq}} - f_1(x)\big)$$
$$= \begin{cases} 1 - f_1(x) - y + c_3 & \text{if } y + f_1(x) > 1 - x_{\mathrm{eq}} + c_3 \\ x_{\mathrm{eq}} & \text{if } 1 - x_{\mathrm{eq}} \le y + f_1(x) \le 1 - x_{\mathrm{eq}} + c_3 \\ 1 - f_1(x) - y & \text{if } y + f_1(x) < 1 - x_{\mathrm{eq}} \end{cases}$$

$$y - f_2\big(x, y, 1 - x_{eq} - f_1(x)\big)$$
$$= \begin{cases} c_3 & \text{if } y + f_1(x) > 1 - x_{eq} + c_3 \\ y - 1 + x_{eq} + f_1(x) & \text{if } 1 - x_{eq} \le y + f_1(x) \le 1 - x_{eq} + c_3 \\ 0 & \text{if } y + f_1(x) < 1 - x_{eq} \end{cases}$$
$$f_1(x) = \begin{cases} 0 & \text{if } x \le c_1 \\ -c_1 r + rx & \text{if } c_1 < x < c_1 + c_2 \\ c_2 r & \text{if } x \ge c_1 + c_2 \end{cases}$$

Making use of the above equalities in conjunction with hypotheses $y(t) \in [0, c_3]$, $c_1 + c_2 > 1 - c_2 r$, and $r \ge 1$, it can be shown that

$$x(t+1) = 1 + c_1 r - r x(t) + c_3 - y(t) \quad \text{and} \quad y(t+1) = c_3$$
$$\text{if } \big(x(t), y(t)\big) \in B_1 := \big\{x_{eq} + r^{-1}(c_3 - y) < x < c_1 + c_2\big\} \tag{7.57}$$
$$x(t+1) = 1 - c_2 r + c_3 - y(t) \quad \text{and} \quad y(t+1) = c_3$$
$$\text{if } \big(x(t), y(t)\big) \in B_2 := \{x \ge c_1 + c_2 \text{ and } y > 1 - x_{eq} + c_3 - c_2 r\} \tag{7.58}$$
$$x(t+1) = 1 - y(t) \quad \text{and} \quad y(t+1) = 0$$
$$\text{if } \big(x(t), y(t)\big) \in B_3 := \{0 < x \le c_1 \text{ and } y < 1 - x_{eq}\} \tag{7.59}$$
$$x(t+1) = 1 + c_1 r - r x(t) - y(t) \quad \text{and} \quad y(t+1) = 0$$
$$\text{if } \big(x(t), y(t)\big) \in B_4 := \big\{c_1 < x < x_{eq} - r^{-1} y\big\} \tag{7.60}$$
$$x(t+1) = x_{eq}$$
$$\text{if } \big(x(t), y(t)\big) \in B_5 = (0, +\infty) \times [0, c_3] \Big\backslash \left(\bigcup_{i=1}^{4} B_i\right). \tag{7.61}$$

Moreover, since $c_1 + c_2 > 1 - c_2 r$ which directly implies $c_1 < x_{eq} < c_1 + c_2$ (notice that $c_1 < 1$), the following implications hold:

$$\big(x(t), y(t)\big) \in B_1 \cup B_2 \quad \Rightarrow \quad \big(x(t+1), y(t+1)\big) \in B_3 \cup B_4 \cup B_5 \tag{7.62}$$
$$\big(x(t), y(t)\big) \in B_3 \cup B_4 \quad \Rightarrow \quad \big(x(t+1), y(t+1)\big) \in B_1 \cup B_2 \cup B_5 \tag{7.63}$$
$$\big(x(t_0), y(t_0)\big) \in B_5 \quad \Rightarrow \quad x(t) = x_{eq} \quad \forall t \ge t_0 + 1 \tag{7.64}$$

In order to show (7.56), it suffices to show that for every initial condition $(x(0), y(0)) \in (0, +\infty) \times [0, c_3]$, it holds that $(x(T), y(T)) \in B_5$, where

$$T = 2\left\lceil \frac{1 - x_{eq} - \min\{c_3 r^{-1}; c_1 + c_2 - x_{eq}\}}{\delta} \right\rceil + 6$$

and $\delta > 0$ is to be selected.

The proof will be made by contradiction. Suppose on the contrary that there exists initial condition $(x(0), y(0)) \notin B_5$ such that $(x(t), y(t)) \notin B_5$ for all $t \in [0, T]$. Without loss of generality, we may assume that $(x(0), y(0)) \in B_1 \cup B_2$ (since if $(x(0), y(0)) \in B_3 \cup B_4$, then by (7.63) we would have $(x(1), y(1)) \in B_1 \cup B_2$ and then consider the solution with initial condition $(x(1), y(1)) \in B_1 \cup B_2$, which also satisfies $(x(t), y(t)) \notin B_5$ for all $t \in [0, T-1]$). By virtue of implications (7.62) and (7.63), we must have

$$\begin{aligned} \big(x(t), y(t)\big) &\in B_1 \cup B_2 \quad \text{if } t \text{ is even} \quad \text{and} \\ \big(x(t), y(t)\big) &\in B_3 \cup B_4 \quad \text{if } t \text{ is odd} \end{aligned} \tag{7.65}$$

Moreover, by virtue of (7.57), (7.58), (7.59), and (7.60), we obtain

$$\begin{aligned} y(t) &= 0 \quad \text{if } t \in [1, T] \text{ is even} \quad \text{and} \\ y(t) &= c_3 \quad \text{if } t \in [1, T] \text{ is odd} \end{aligned} \tag{7.66}$$

By virtue of (7.57), (7.58), (7.59), (7.60), (7.65), and (7.66), we must also have

$$x_{\text{eq}} < x_{\text{eq}} + \min\{c_3 r^{-1}; c_1 + c_2 - x_{\text{eq}}\} \le x(t) \quad \text{if } t \in [2, T] \text{ is even.} \tag{7.67}$$

Next, we show that there exists $\delta > 0$ such that

$$x(t+2) \le x(t) - \delta \quad \text{for all even integers } t \in [2, T] \tag{7.68}$$

Thus, we obtain a contradiction, since (7.67), in conjunction with (7.68) and the fact that $x(t) \le 1$ for all $t \in [1, T]$, gives $x_{\text{eq}} + \min\{c_3 r^{-1}; c_1 + c_2 - x_{\text{eq}}\} \le x(2+2k) \le 1 - k\delta$ for all integers $k \ge 1$ with $2k + 2 \le T$, which cannot hold for $k \ge 1 + \delta^{-1}[1 - x_{\text{eq}} - \min\{c_3 r^{-1}; c_1 + c_2 - x_{\text{eq}}\}]$.

Consequently, the rest part of the proof is devoted to the determination of the constant $\delta > 0$ that satisfies (7.68). For all $(x(t), 0) \in B_1 \cup B_2$, it can be shown that

Case I: If $x_{\text{eq}} + r^{-1}(x_{\text{eq}} + c_3 - c_1) \le x(t) < c_1 + c_2$ and $c_3 < 1 - x_{\text{eq}}$, then $x(t+2) = 1 - c_3$.
Case II: If $x_{\text{eq}} + (r+1)r^{-2}c_3 < x(t) < c_1 + \min\{c_2; r^{-1}(1 + c_3 - c_1)\}$, then $x(t+2) = x_{\text{eq}} + r^2(x(t) - x_{\text{eq}}) - (r+1)c_3$.
Case III: If $x(t) \ge c_1 + c_2$ and $c_3 < 1 - x_{\text{eq}}$ and $c_3 \le c_1 + c_2 r - 1$, then $x(t+2) = 1 - c_3$.
Case IV: If $x(t) \ge c_1 + c_2$ and $c_1 + c_2 r - 1 < c_3 < r(r+1)^{-1}(x_{\text{eq}} + c_2 r - 1)$, then $x(t+2) = 1 + c_1 r - r + c_2 r^2 - (r+1)c_3$.
Case V: If none of the above holds, then $x(t+2) = x_{\text{eq}}$.

In Case I (i.e., if $c_3 < 1 - x_{\text{eq}}$ and $c_3 < c_1 + rc_2 - 1$) by virtue of the inequality $c_3 > R_1$, we must necessarily have $c_3 > (1 - r^{-1})(1 - x_{\text{eq}})$, and consequently (7.68) holds with $\delta := (1 + r^{-1})(x_{\text{eq}} + c_3) - (1 + r^{-1}c_1) > 0$.

In Case II (i.e., if $c_3 < r(x_{\text{eq}} - c_1)$ and $c_3 < r^2(r+1)^{-1}(c_1 + c_2 - x_{\text{eq}})$) by virtue of the inequality $c_3 > R_2$, we must necessarily have $c_3 > (r-1)\min\{x_{\text{eq}} - c_1; c_1 + c_2 - x_{\text{eq}}\}$, and therefore (7.68) holds with $\delta = (1+r)[c_3 - (r-1)\min\{c_1 + c_2 - x_{\text{eq}}; c_1 + r^{-1}(1 + c_3 - c_1) - x_{\text{eq}}\}] > 0$.

In Case III (i.e., if $c_3 < 1 - x_{\text{eq}}$ and $c_3 \le c_1 + rc_2 - 1$) by virtue of the inequality $c_3 > R_3$, we must necessarily have $c_3 > 1 - c_1 - c_2$, and consequently (7.68) holds with $\delta := c_1 + c_2 + c_3 - 1 > 0$.

In Case IV (i.e., if $c_1 + c_2 r - 1 < c_3 < r(r+1)^{-1}(x_{\text{eq}} + c_2 r - 1)$) by virtue of the inequality $c_3 > R_4$, we must necessarily have $c_3 > (r-1)(r+1)^{-1}(c_1 + c_2 +$

$c_2 r - 1)$, and consequently (7.68) holds with $\delta := (1 + r)c_3 + (r - 1)(1 - c_1 - c_2 - c_2 r) > 0$.

The constant $\delta > 0$ that satisfies (7.68) can be selected as the minimum of the corresponding constants given for each case. Thus property (7.56) is proved. The fact that the closed-loop system (7.49) with (7.52) and output $Y = x - x_{eq}$ is URGAOS follows from Lemmas 1.3, 1.4 in Chap. 1 and Lemmas 2.1 and 2.2 in Chap. 2.

The proof is complete. □

7.5 Historical and Bibliographical Notes

1. There are many results dealing with the stabilization of chemostat models (see [1, 3, 5, 14, 17, 19–21, 25, 28, 29] also see [26, 27]). Delayed chemostat models have been studied primarily for their dynamic behavior (see [30, 32, 33]). The results in Sect. 7.2 first appeared in [19].
2. The conversion of the problem of the correct dynamic behavior of the numerical approximation to a rigorous feedback stabilization problem was first given in [16]. Lyapunov methods for the solution of the feedback design problem can be used as well (see [18], where many more results are presented).
3. The results in Sect. 7.4 first appeared in [2]. However, the reader should notice that the conclusion of Proposition 7.2 is slightly stronger than the conclusion of Proposition 3.2 in [2].

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Chapter 8
Open Problems

In this chapter, we would like to give a list of open and unanswered problems in Mathematical Control Theory. The solutions of these open problems will be very important for the development of modern nonlinear control theory. Expectedly novel mathematical analysis and synthesis tools need to be developed to address these challenging problems. The interested reader should also consult the book [3] for other significant and important open problems in Mathematical Control Theory.

Open Problem #1 *Under what conditions WIOS implies IOS?*

A qualitative characterization of the IOS property for abstract control systems as discussed in this book has not been available yet. For systems described by ODEs, many qualitative characterizations of the ISS and IOS properties are provided in [21–23]. Moreover, Theorem 4.1 in Chap. 4 gives a complete qualitative characterization of the WIOS property:

"*0-GAOS*" + "*RFC*" + "*the continuity with respect to initial conditions and external inputs*" *implies WIOS*

A similar qualitative characterization for the IOS property in a general context of abstract dynamical systems as discussed in this book will be very important for control designs and applications.

Open Problem #2 *Development of small-gain techniques for dynamical systems described by Partial Differential Equations* (*PDEs*).

Small-gain results have been well studied for finite-dimensional nonlinear systems described by ordinary differential, or difference, equations (see, e.g., [8–10] and references therein). However, as of today, there is little research devoted to the development of small-gain techniques for nonlinear systems described by Partial Differential Equations (PDEs). We believe that the small-gain results provided in the present book (Theorems 5.1 and 5.2 in Chap. 5) will pave the road for the application of small-gain results to systems described by PDEs.

I. Karafyllis, Z.-P. Jiang, *Stability and Stabilization of Nonlinear Systems*,
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Open Problem #3 *Formulas for the Coron–Rosier methodology.*

Theorem 6.1 in Chap. 6 is an existence-type result. Although its proof is constructive, it cannot be easily applied for feedback design purposes. The creation of formulas for the Coron–Rosier approach will be very significant for control purposes, since the Coron–Rosier approach can allow nonconvex control sets and does not require additional properties for the Control Lyapunov Function. The significance of the solution of this open problem is also noted in [5].

Open Problem #4 *When is a nonlinear, time-varying, time-delay system stabilizable?*

We have recently provided a positive answer to the above question when the system only involves state-delay [13]. A complete answer to the question of when the nonlinear time-varying system with both state and input delays is stabilizable remains open and requires deeper investigation. Nonetheless, it should be mentioned that sufficient, but not necessary, conditions for the solution of the stabilization problem with input delays are proposed in the recent work of Krstić [14–16] (also see [11]). To our knowledge, a necessary and sufficient condition for stabilizability is missing even for linear time-varying systems with input delays.

Open Problem #5 *Application of small-gain results for distributed feedback design of large-scale nonlinear systems.*

Large-scale systems are abundant in various fields of science and engineering and have gained increasing attention due to emerging engineering and biomedical applications. Examples of these applications are from smart grids with green and renewable energy sources, modern transportation networks, and biological networks. There has been some success with the use of decentralized control strategy for both linear and nonlinear large-scale systems; see [7, 19] and many references therein. Clearly more remains to be accomplished in this exciting field. We feel that small-gain is a very appropriate tool for addressing some of these modern-day challenges. The small-gain results of the present book (Theorems 5.1 and 5.2 in Chap. 5) make a preliminary step forward toward studying some complex large-scale systems beyond the past literature of decentralized systems and control.

Open Problem #6 *Extension of the discretization approach for autonomous systems.*

The discretization approach for Lyapunov functionals was described in Chap. 2 (Propositions 2.4 and 2.5). However, as remarked in Chap. 2, the discretization approach requires good knowledge of some approximation of the solution map, and its use has been restricted to time-varying systems with special structure (see [1, 17, 18]). An extension of the discretization approach for autonomous systems would be an important contribution in stability theory because such a result would allow the use of positive definite functions with non sign-definite derivative. The required extension of the discretization approach must utilize appropriate differential inequalities in the same spirit as the classical Lyapunov's approach (without requiring knowledge of the solution map or a system with special structure). The recent work in [12] is an attempt in this research direction (see also references therein). However, the problem is still completely "untouched."

Open Problem #7 *Application of feedback design methodologies to other mathematical problems.*

In this book, we have seen the applications of certain tools of modern nonlinear control theory to problems arising from mathematics and economics. Particularly, we have seen

- applications of small-gain results to game theory (see Sect. 5.5 in Chap. 5),
- applications to numerical analysis (see Sect. 7.3).

We believe that feedback design methodologies can be applied with success to other areas of mathematical sciences. Fixed Point Theory (see [6]) and Optimization Theory can be benefited by the application of certain tools of modern nonlinear control theory. Corollary 5.4 in Chap. 5 already shows that small-gain results can have serious consequences in Fixed Point Theory. Further connections between Fixed Point Theory and Stability Theory are provided by the work of Burton (see [4] and references therein) but are in the opposite direction from what we propose, that is, the work of Burton applies results from Fixed Point Theory to Stability Theory.

The efforts for the solution of problems in Game Theory, Numerical Analysis, Fixed Point Theory, and Optimization Theory will necessarily demand the creation of novel results in stability theory and feedback stabilization theory. Therefore, the application of modern nonlinear control theory to other areas of applied mathematics will result to a "knowledge feedback mechanism" between Mathematical Control Theory and other areas in mathematics!

Open Problem #8 *Integral input-to-state stability (for short, iISS) in complex dynamical systems.*

The external stability results of this book are exclusively targeted at extensions of Sontag's ISS property and its variants to a very general context of complex dynamic systems. That is, we want to address a wide class of dynamical systems which may not satisfy the semigroup property, motivated by important examples of hybrid systems, switched systems, and time-delay systems. It remains an open and important, but interesting, question to know how much we could do with the iISS property introduced in [2, 20].

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